Revista Notas de Matemática
Vol.7(2), No. 320, 2011, pp.195-201
http://www.saber.ula.ve/notasdematematica
Pre-Prints
Departamento de Matemáticas
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New representations of focal curves in the special $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

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Abstract

In this paper, we study matrix representations of focal curves in terms of biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We construct new parametric equations of focal curves in terms of matrix representations in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

key words. Focal curve, Biharmonic curve, Matrices, Para-Sasakian manifold.

1 Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \text{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \text{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study matrix representations of focal curves in terms of biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We construct new parametric equations of focal curves in terms of matrix representations in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

2 Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi = 0, \quad \eta(\xi) = 1, g(X, \xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y on M, [2].

Definition 2.1. A para-Sasakian manifold M is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

for all vector fields X, Y, Z, W orthogonal to ξ [2].

Definition 2.2. A para-Sasakian manifold M is said to be ϕ-symmetric if

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

for all vector fields X, Y, Z, W on M.

Definition 2.3. A para-Sasakian manifold M is said to be ϕ-Ricci symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)(Y)) = 0,$$

for all vector fields X and Y on M and $S(X, Y) = g(QX, Y)$.

If X, Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

We consider the three-dimensional manifold

$$\mathbb{P} = \left\{ \left(x^1, x^2, x^3\right) \in \mathbb{R}^3 : \left(x^1, x^2, x^3\right) \neq (0,0,0) \right\},$$

where $\left(x^1, x^2, x^3\right)$ are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$\mathbf{e}_1 = e^{x^1} \frac{\partial}{\partial x^2}, \quad \mathbf{e}_2 = e^{x^1} \left(\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3} \right), \quad \mathbf{e}_3 = -\frac{\partial}{\partial x^1} \tag{2.4}$$

are linearly independent at each point of $\mathbb{P}$. Let g be the Riemannian metric defined by

$$\begin{aligned} g\left(\mathbf{e}_1, \mathbf{e}_1\right) &= g\left(\mathbf{e}_2, \mathbf{e}_2\right) = g\left(\mathbf{e}_3, \mathbf{e}_3\right) = 1, \\ g\left(\mathbf{e}_1, \mathbf{e}_2\right) &= g\left(\mathbf{e}_2, \mathbf{e}_3\right) = g\left(\mathbf{e}_1, \mathbf{e}_3\right) = 0. \end{aligned} \tag{2.5}$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, \mathbf{e}_3) \text{ for any } Z \in \chi(\mathbb{P}).$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1) = \mathbf{e}_2, \ \phi(\mathbf{e}_2) = \mathbf{e}_1, \ \phi(\mathbf{e}_3) = 0. \tag{2.6}$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3) = 1, \tag{2.7}$$

$$\phi^2(Z) = Z - \ \eta(Z)\mathbf{e}_3, \tag{2.8}$$

$$g\left(\phi Z, \phi W\right) = g\left(Z, W\right) - \eta(Z)\eta(W), \tag{2.9}$$

for any $Z, W \in \chi(\mathbb{P})$. Thus for $\mathbf{e}_3 = \xi$, (ϕ, ξ, η, g) defines an almost para-contact metric structure on $\mathbb{P}$.

Let ∇ be the Levi-Civita connection with respect to g. Then, we have

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_2.$$

3 Matrix Representation of Focal Curves in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

For a unit speed curve γ, the curve consisting of the centers of the osculating spheres of γ is called the parametrized focal curve of γ. The hyperplanes normal to γ at a point consist of the set of centers of all spheres tangent to γ at that point. Hence the center of the osculating spheres at that point lies in such a normal plane. Therefore, denoting the focal curve by C_γ, we can write

$$C_\gamma(s) = (\gamma + c_1\mathbf{N} + c_2\mathbf{B})(s), \tag{3.1}$$

where the coefficients c_1, c_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Further, the focal curvatures c_1, c_2 are defined by

$$c_1 = \frac{1}{\kappa},\ c_2 = \frac{c_1'}{\tau},\ \kappa \neq 0,\ \tau \neq 0. \tag{3.2}$$

Lemma 3.1. *Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed biharmonic curve and C_γ its focal curve on $\mathbb{P}$. Then,*

$$c_1 = \frac{1}{\kappa} = \text{constant and } c_2 = 0. \tag{3.3}$$

Proof. Using (2.3) and (3.2), we get (3.3).

Lemma 3.2. *Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed biharmonic curve and C_γ its focal curve on $\mathbb{P}$. Then,*

$$C_\gamma(s) = (\gamma + c_1\mathbf{N})(s). \tag{3.4}$$

Theorem 3.3. *(see [13]) Let $\gamma : I \longrightarrow \mathbb{P}$ be a non-geodesic biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then,*

$$\begin{aligned}\mathcal{A}\gamma &= \sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}(-\cos\varphi, \sin\varphi e^{-s\cos\varphi + C_1}\left(\sin\left[\Bbbk s + C\right] + \cos\left[\Bbbk s + C\right]\right), \\ &\quad \sin\varphi e^{-s\cos\varphi + C_1}\sin\left[\Bbbk s + C\right]),\end{aligned}$$

$$\begin{aligned}\mathcal{A}^2\gamma &= \frac{\left(\sqrt{\mathfrak{trace}(\mathcal{A}^4)}\right)}{\kappa}(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2, \\ &\quad e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(\Bbbk\sin\varphi\sin\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\cos\left[\Bbbk s + C\right]\right) \\ &\quad + e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right), \\ &\quad - e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right)),\end{aligned} \tag{3.5}$$

where C, $\overline{C}_1, \overline{C}_2$ *are constants of integration and* $\Bbbk = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$.

Now, let us define a special form of focal curve.

Lemma 3.4. *Let $\gamma : I \longrightarrow \mathbb{P}$ be a regular curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then,*

$$C_\gamma(s) = \gamma(s) + c_1 \frac{\mathcal{A}^2\gamma}{\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}}. \tag{3.6}$$

Proof. Using Theorem 3.3, we immediately obtain

$$\mathbf{N} = \left(\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}\right)^{-1} \mathcal{A}^2\gamma.$$

According to the definition of focal curve we have (3.6).

Theorem 3.5. *(see [13]) Let $\gamma : I \longrightarrow \mathbb{P}$ be a non-geodesic biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then the new curvatures of this curve are*

$$\kappa = -\frac{\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}}{\mathfrak{trace}\left(\mathcal{A}^2\right)}, \tag{3.7}$$

$$\tau = \Re\Big[\frac{-\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^4 \sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^{\frac{3}{4}}}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6}\Big],$$

where $\Re = \Big[\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}\Big]^{-\frac{1}{2}}$.

Using above Lemma and Theorem we have following result:

Theorem 3.6. *Let $\gamma : I \longrightarrow \mathbb{P}$ be a biharmonic curve parametrized by arc length. If C_γ is a focal curve of γ, then the parametric equations of C_γ are*

$$
\begin{aligned}
\tilde{x}^1(s) &= -\frac{\left(\mathfrak{trace}\left(\mathcal{A}^2\right)\right)^2}{\mathfrak{trace}(\mathcal{A}^4)}\left(-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2\right)-\cos\varphi s+C_1,\\
\tilde{x}^2(s) &= -\frac{\left(\mathfrak{trace}\left(\mathcal{A}^2\right)\right)^2\sin^3\varphi}{\mathfrak{trace}(\mathcal{A}^4)-(\mathfrak{trace}(\mathcal{A}^2))^2\sin^4\varphi}e^{-s\cos\varphi+C_1}\left(\Pi\cos\left[\Pi s+C\right]+\left[-\Pi+\cos\varphi\right]\sin\left[\Pi s+C\right]\right)\\
&\quad -\frac{\left(\mathfrak{trace}\left(\mathcal{A}^2\right)\right)^2}{\mathfrak{trace}(\mathcal{A}^4)}e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}\left(\Pi\sin\varphi\sin\left[\Pi s+C\right]+\cos\varphi\sin\varphi\cos\left[\Pi s+C\right]\right)\\
&\quad -\frac{\left(\mathfrak{trace}\left(\mathcal{A}^2\right)\right)^2}{\mathfrak{trace}(\mathcal{A}^4)}e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}\left(-\Pi\sin\varphi\cos\left[\Pi s+C\right]+\cos\varphi\sin\varphi\sin\left[\Pi s+C\right]\right)+C_2,\\
\tilde{x}^3(s) &= -\frac{\left(\mathfrak{trace}\left(\mathcal{A}^2\right)\right)^2\sin^3\varphi}{\mathfrak{trace}(\mathcal{A}^4)-(\mathfrak{trace}(\mathcal{A}^2))^2\sin^4\varphi}e^{-s\cos\varphi+C_1}\left(-\cos\varphi\cos\left[\Pi s+C\right]+\left[\Pi s+C\right]\sin\left[\Pi s+C\right]\right)\\
&\quad +\frac{\left(\mathfrak{trace}\left(\mathcal{A}^2\right)\right)^2}{\mathfrak{trace}(\mathcal{A}^4)}e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}\left(-\Pi\sin\varphi\cos\left[\Pi s+C\right]+\cos\varphi\sin\varphi\sin\left[\Pi s+C\right]\right)+C_3,
\end{aligned}
\tag{3.8}
$$

where $C,\overline{C}_1,\overline{C}_2,C_1,C_2,C_3$ *are constants of integration and* $\Pi=\left[\frac{\sqrt{\mathfrak{trace}(\mathcal{A}^4)-(\mathfrak{trace}(\mathcal{A}^2))^2\sin^2\varphi}}{(\mathfrak{trace}(\mathcal{A}^2))\sin\varphi}\right]$.

Proof. Assume that γ be a spacelike biharmonic curve and C_γ its focal curve on $\mathbb{P}$. Substituting equation (3.5) into Lemma 3.4, and by using the Mathematica program we have above system. This completes the proof of the theorem.

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