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Controllability of Linear Difference Equation in Hilbert Spaces and Applications.

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Abstract

In this paper we present a necessary and sufficient conditions for the exact and approximate controllability of the following linear difference equation

$$z(n+1) = A(n)z(n) + B(n)u(n), \;\; n \in I\!N^*, \;\; z(n) \in Z, \;\; u(n) \in U,$$

where Z, U are Hilbert spaces, $I\!N^* = I\!N \cup \{0\}$, $A \in l^\infty(I\!N, L(Z))$, $B \in l^\infty(I\!N, L(U,Z))$, $u \in l^2(I\!N, U)$. As a particular case we consider the discretization on flow of the following controlled evolution equation

$$z' = Az + Bu, \;\; z \in Z, \;\; u \in U, \;\; t > 0,$$

where Z, U are Hilbert spaces, $B \in L(U,Z)$, $u \in L^2(0,\tau;U)$ and A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in Z, given by:

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \;\; z \in Z, \;\; t \geq 0,$$

according to lemma 1.1. We apply these results to a flow-discretization of the heat equation and the wave equation.

Resumen

En este articulo presentamos condiciones necesarias y suficientes para la controlabilidad exacta y aproximada de la siguiente ecuación en diferencias lineal

$$z(n+1) = A(n)z(n) + B(n)u(n), \;\; n \in I\!N^*, \;\; z(n) \in Z, \;\; u(n) \in U,$$

donde Z, U son espacios de Hilbert, $I\!N^* = I\!N \cup \{0\}$, $A \in l^\infty(I\!N, L(Z))$, $B \in l^\infty(I\!N, L(U,Z))$, $u \in l^2(I\!N, U)$. Como un caso particular consideramos la discretización en el flujo de la siguiente ecuación de evolución controlada

$$z' = Az + Bu, \;\; z \in Z, \;\; u \in U, \;\; t > 0,$$

donde Z, U son espacios de Hilbert , $B \in L(U,Z)$, $u \in L^2(0,\tau;U)$ y A es el generador infinitesimal de un semigrupo fuertemente continuo $\{T(t)\}_{t\geq 0}$ in Z, dado por:

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \;\; z \in Z, \;\; t \geq 0$$

de acuerdo con el lema 1.1. Aplicamos esos resultados a una discretización en el flujo de la ecuación del calor y la ecuación de onda.

key words. difference equations, exact controllability, approximate controllability, heat and wave equation.

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1 Introduction.

One way to obtain Difference Equation in Banach Spaces is making discretization on flow of the evolution equation, this method was used in [3], [5] and [9] to characterize exponential dichotomy of evolution operators and skew product semiflows respectively.

In general, for a controlled evolution equation of the form

$$z' = Az + Bu, \quad z \in Z, \quad u \in U, \quad t > 0, \tag{1.1}$$

where Z, U are Banach spaces, $B \in L(U, Z)$, $u \in L^2(0, \tau; U)$ and A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ in Z, we considere the following discretization on flow:

$$z(n+1) = T(n)z(n) + B(n)u(n), \quad n \in I\!N^*, \tag{1.2}$$

where the control $u = \{u(n)\}_{n \geq 1}$ belong to $l^2(I\!N, U)$.

In particular, we shall work here with those infinitesimal generator A given by the following Lemma from [6].

Lemma 1.1 *Let Z be a Hilbert separable space and $\{A_n\}_{n \geq 1}$, $\{P_n\}_{n \geq 1}$ two families of bounded linear operator in Z, with $\{P_n\}_{n \geq 1}$ a family of complete orthogonal projection such that:*

$$A_n P_n = P_n A_n, n \geq 1.$$

Define the following family of linear operators

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad z \in Z, \quad t \geq 0.$$

Then:

(a) $T(t)$ is a linear and bounded operator if $\|e^{A_n t}\| \leq g(t)$, $n = 1, 2, ..,$ with $g(t) \geq 0$, continuos for $t \geq 0$.

(b) *Under the same condition the above, $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigruop in the Hilbert space Z, whose infinitesimal generator A is given by*

$$Az = \sum_{n=1}^{\infty} A_n P_n z, \quad z \in D(A)$$

with

$$D(A) = \left\{ z \in Z : \sum_{n=1}^{\infty} \|A_n P_n z\|^2 < \infty \right\}.$$

(c) *The spectrum $\sigma(A)$ of A is given by*

$$\sigma(A) = \overline{\bigcup_{n=1}^{\infty} \sigma(\overline{A_n})},$$

where $\overline{A_n} = A_n P_n$.

We shall assume through this paper the following hypothesis:

$$P_j BB^* = BB^* P_j, \ \ j = 1, 2, ... \tag{1.3}$$

Under this condition, in lemma 3.1, we characterize the exact and approximate controllability of the general system (1.2) in terms of the following family of control systems

$$z(n+1) = e^{A_j n} z(n) + B_j u(n), \ \ n \in I\!N^*, \ \ j = 1, 2, ...$$

where $B_j = P_j B$ and $u \in l^2(I\!N, U)$.

Finally, we apply these results to a discrete version of the heat and wave equation.

2 Preliminaries Results.

In this section we shall present a discrete version of theorem 4.1.7 from [2] for the following general controlled difference equation in Hilbert spaces

$$z(n+1) = A(n)z(n) + B(n)u(n), \ \ n \in I\!N, \ \ z(0) = z_0, \tag{2.4}$$

where $z(n) \in Z$, $u(n) \in U$, where Z, U are Hilbert spaces, $I\!N^* = I\!N \cup \{0\}$, $A \in l^\infty(I\!N, L(Z))$, $B \in l^\infty(I\!N, L(U,Z))$, $u \in l^2(I\!N, U)$.

To this end, we shall give the definition of exact and approximate controllability for the system (2.4).

Consider the set $\Delta = \{(m,n) \in I\!N \times I\!N : m \geq n\}$ and let $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$ be the evolution operator associated to A, i.e., $\Phi(m,n) = A(m-1)\cdots A(n)$ and $\Phi(m,n) = I$, for $m = n$.

Then, the solution of (2.4) is given by the discrete variation constant formula:

$$z(n) = \Phi(n,0)z(0) + \sum_{k=1}^{n} \Phi(n,k)B(k-1)u(k-1), \ \ n \in I\!N. \tag{2.5}$$

Definition 2.1 *(***Exact Controllability***) The system (2.4) is said to be exactly controlable if there is $n_0 \in I\!N$ such that for every z_0, $z_1 \in Z$ there exists $u \in l^2(I\!N, U)$ such that $z(0) = z_0$ and $z(n_0) = z_1$.*

Definition 2.2 *(***Approximate Controllability***) The system (2.4) is said to be approximately controlable if there is $n_0 \in I\!N$ such that for every z_0, $z_1 \in Z$, $\varepsilon > 0$ there exists $u \in l^2(I\!N, U)$ such that $z(0) = z_0$ and $\|z(n_0) - z_1\| < \varepsilon$.*

Definition 2.3 *For the system (2.4) we define the following concepts:*

a) The controllability map (for $n \in I\!N$) is define as follows $B^n : l^2(I\!N, U) \longrightarrow Z$ by

$$B^n u = \sum_{k=1}^{n} \Phi(n,k)B(k-1)u(k-1) \tag{2.6}$$

b) The grammian map (for $n \in I\!N$) is define by $L_{B^n} = B^n B^{n*}$

Proposition 2.1 *The adjoint $B^{n_0 *}$ of the operator B^{n_0} is given by*
$B^{n_0 *} : Z \longrightarrow l^2(I\!N, U)$

$$(B^{n_0 *}z)(k-1) = \begin{cases} B^*(k-1)\Phi^*(n_0,k)z, & k \leq n_0 \\ 0, & k > n_0, \end{cases} \tag{2.7}$$

and

$$L_{B^{n_0}} z = \sum_{k=1}^{n_0} \Phi(n_0,k)B(k-1)B^*(k-1)\Phi^*(n_0,k)z, \ \ z \in Z. \tag{2.8}$$

Proof

$$\begin{aligned}
\langle \mathbb{B}^{n_0} z, z\rangle &= \left\langle \sum_{k=1}^{n_0} \Phi(n_0,k)B(k-1)u(k-1), z \right\rangle_{Z,Z} \\
&= \sum_{k=1}^{n_0} \langle \Phi(n_0,k)B(k-1)u(k-1), z\rangle_{Z,Z} \\
&= \sum_{k=1}^{n_0} \langle u(k-1), B^*(k-1)\Phi^*(n_0,k)z\rangle_{U,U} \\
&= \sum_{k=1}^{n_0} \langle u(k-1), B^*(k-1)\Phi^*(n_0,k)z\rangle_{U,U} + \sum_{k=n_0+1}^{\infty} \langle u(k-1), 0\rangle_{U,U} \\
&= \langle u, \mathbb{B}^{n_0*} z\rangle_{l^2(\mathbb{N},U),l^2(\mathbb{N},U)},
\end{aligned}$$

which prove (2.1). Clearly, (2.8) follows immediately from definition 2.2 and (2.1). □

The following theorem is a discrete version of theorem 4.1.7 from [2].

Theorem 2.1 *(a) The equation (2.4) is exactly controllable for some $n_0 \in \mathbb{N}$ if, and only if, one of the following statements holds:*

(i) $\mathrm{Rang}(\mathbb{B}^{n_0}) = Z$

(ii) There exists $\gamma > 0$ such that

$$\langle L_{\mathbb{B}^{n_0}} z, z\rangle \geq \gamma \|z\|_Z^2, \ \ \forall z \in Z,$$

(iii) There exists $\gamma > 0$ such that

$$\|\mathbb{B}^{n_0*} z\|_{l^2(\mathbb{N},U)} \geq \gamma \|z\|_Z, \ \ \forall z \in Z,$$

(b) The equation (2.4) is approximately controllable for some $n_0 \in \mathbb{N}$ if, and only if, one of the following statements holds:

(i) $Ker(\mathbb{B}^{n_0*}) = \{0\}$.

(ii) $\langle L_{\mathbb{B}^{n_0}} z, z\rangle > 0$, $z \neq 0$ *in* Z.

(iii) $B^*(k-1)\Phi^*(n_0,k)z = 0, \ \ k \leq n_0, \ \ \Rightarrow z = 0$.

(iv) $\overline{\mathrm{Rang}(\mathbb{B}^{n_0})} = Z$.

Proof

(a) Since $L_{\mathbb{B}^{n_0}} = \mathbb{B}^{n_0}\mathbb{B}^{n_0*}$, we have that

$$\langle L_{\mathbb{B}^{n_0}} z, z\rangle = \langle \mathbb{B}^{n_0}\mathbb{B}^{n_0*} z, z\rangle = \langle \mathbb{B}^{n_0*} z, \mathbb{B}^{n_0*} z\rangle = \|\mathbb{B}^{n_0*} z\|^2, \forall z \in Z, \tag{2.9}$$

which shows the equivalence between (ii) and (iii).

If (ii) holds, then $L_{B^{n_0}}$ is boundedly invertible. Hence,
$\text{Rang}(L_{B^{n_0}}) = D((L_{B^{n_0}})^{-1}) = Z$. The fact that $L_{B^{n_0}} = B^{n_0}B^{n_0*}$ shows that $\text{Rang}(L_{B^{n_0}}) \subset \text{Rang}(B^{n_0})$. Thus $\text{Rang}(B^{n_0}) = Z$, which shows (i).

Let us suppose that $\text{Rang}(B^{n_0}) = Z$. Then, we shall prove that (iii) holds. First, we assume that B^{n_0} is injective; then $(B^{n_0})^{-1} \in L(Z, l^2(I\!N, U))$ and $(B^{n_0*})^{-1} \in L(l^2(I\!N, U), Z)$. Thus, there exists $\beta > 0$ such that

$$\|(B^{n_0*})^{-1}u\|_Z \leq \beta\|u\|_Z, \ \ \forall u \in l^2(I\!N, U),$$

and with $z = (B^{n_0*})^{-1}u$ we obtain

$$\|z\|_Z \leq \beta\|B^{n_0*}z\|_Z,$$

which is equivalent to (iii) considering $\gamma = 1/\beta$.

For the general case, we define the Hilbert space $X = [KerB^{n_0}]^{\perp}$ endowed with the norm defined by $\|u\|_X = \|u\|_{l^2}$.

Then, we define $\widehat{B}^{n_0}u = B^{n_0}u$, $u \in X$, which makes $\widehat{B}^{n_0}$ a bijective map on X, and our above argument applied to $\widehat{B}^{n_0}$ shows that there exist $\beta > 0$ such that for all $z \in Z$

$$\beta\|\widehat{B}^{n_0*}z\|_X \geq \|z\|_Z.$$

From Lemma A.3.30 [2], the Riesz Representation Theorem and Hahn Banach's Theorem we deduce

$$\begin{aligned}
\|\widehat{B}^{n_0*}z\| &= \sup_{\{u\in X:\|u\|\leq 1\}} \langle u, \widehat{B}^{n_0*}z\rangle = \sup_{\{u\in X:\|u\|\leq 1\}} \langle \widehat{B}^{n_0}u, z\rangle \\
&= \sup_{\{u\in X:\|u\|\leq 1\}} \langle B^{n_0}u, z\rangle = \sup_{\{u\in l^2(I\!N,U):\|u\|\leq 1\}} \langle B^{n_0}u, z\rangle = \|B^{n_0*}z\|_{l^2}.
\end{aligned}$$

Hence, we have that

$$\|B^{n_0*}z\|_{l^2} = \|\widehat{B}^{n_0*}z\|_X \geq \frac{1}{\beta}\|z\|_Z.$$

Once more, with $\gamma = 1/\beta$, we have (iii).

Now, we shall prove that the exact controllability of (2.4) implies (i). Suppose that (2.4) is exactly controllable for some n_0. Given $z \in Z$ we can find z_0 and z_1 in Z such that

$$z_1 = \Phi(n_0, 0)z_0 + z. \tag{2.10}$$

Then there exist $u \in l^2(I\!N, U)$ such that $z_0(0) = z_0$ and $z_u(n_0) = z_1$. Thus,

$$z_1 = z_u(n_0) = \Phi(n_0, 0)z_0 + \sum_{k=1}^{n_0} \Phi(n_0, k)B(k-1)u(k-1) \tag{2.11}$$

Substituting (2.10) in (2.11), we obtain

$$\Phi(n_0, 0)z_0 + z = \Phi(n_0, 0)z_0 + \sum_{k=1}^{n_0} \Phi(n_0, k)B(k-1)u(k-1).$$

Then,

$$z = \sum_{k=1}^{n_0} \Phi(n_0, k)B(k-1)u(k-1) = B^{n_0}u.$$

So, $\text{Rang}(B^{n_0}) = Z$.

Next, we shall show that (i) implies exact controllability of (2.4). Assume that $\text{Rang}(B^{n_0}) = Z$. Consider z in Z such that

$$z = z_1 - \Phi(n_0, 0)z_0, \tag{2.12}$$

with z_0, z_1 in Z. Then there exist a control u such that

$$B^{n_0}u = z. \tag{2.13}$$

Then, substituting (2.12) in (2.13), we obtain

$$z = z_1 - \Phi(n_0, 0)z_0 = \sum_{k=1}^{n_0} \Phi(n_0, k)B(k-1)u(k-1).$$

Hence

$$z_u(n_0) = \Phi(n_0, 0)z_0 + \sum_{k=1}^{n_0} \Phi(n_0, k)B(k-1)u(k-1).$$

So, we have obtained a solution $z_u(\cdot)$ of (2.4) such that $z_u(n_0) = z_1$ and $z_u(0) = z_0$, i.e., (2.4) is exactly controllable. This conclude the prove of part (a).

(b) From proposition 2.1 follows that (i) and (iii) are equivalent, and (2.9) shows that (i) and (ii) are equivalent. We know that $(Ker(B^{n_0*}))^\perp = \overline{\text{Rang}(B^{n_0})}$. From this it follows that: $\overline{\text{Rang}(B^{n_0})} = Z$ iff $(Ker(B^{n_0*}))^\perp = Z$ iff $Ker(B^{n_0*}) = \{0\}$, which shows that (i) and (iv) are equivalent.

Now, suppose that (2.4) is approximately controllable; then for $\varepsilon > 0$, z, z_0, z_1 in Z, such that $z_1 = \Phi(n_0, 0)z_0 + z$, there exist $u \in l^2(I\!N, U)$ with $z_u(0) = z_0$ and $\|z_u(n_0) - z_1\| < \varepsilon$. Thus,

$$z_u(n_0) = \Phi(n_0, 0)z_0 + \sum_{k=1}^{n_0} \Phi(n_0, k)B(k-1)u(k-1).$$

Therefore

$$
\begin{aligned}
\|B^{n_0}u - z\| &= \|\sum_{k=1}^{n_0}\Phi(n_0,k)B(k-1)u(k-1) - z\| \\
&= \|\sum_{k=1}^{n_0}\Phi(n_0,k)B(k-1)u(k-1) + \Phi(n_0,0)z_0 - z_1\| \\
&= \|z_u(n_0) - z_1\| < \varepsilon,
\end{aligned}
$$

which implies (iv).

Assume that $\overline{\text{Rang}(B^{n_0})} = Z$. Let $z \in Z$ such that $z = z_1 - \Phi(n_0, 0)$ with z_0, z_1 in Z. Then, there exist a control u such that $\|B^{n_0}u - z\| < \varepsilon$. Thus,

$$
\|B^{n_0}u + \Phi(n_0,0)z_0 - z_1\| = \|z_u(n_0) - z_1\| < \varepsilon.
$$

Hence, we have obtained a solution $z_u(\cdot)$ of (2.4) such that $z_u(0) = z_0$ and $\|z_u(n_0)-z_1\| < \varepsilon$; this let us conclude that (2.4) is approximately controllable and finish the prove of part (b).

□

Lemma 2.1 *The equation (2.4) is exactly controllable for $n_0 \in I\!N$ if, and only if, $L_{B^{n_0}}$ is invertible. Moreover, in this case $S = B^{n_0*}L_{B_{n_0}}^{-1}$ is a right inverse of B^{n_0} and the control $u \in l^2(I\!N, U)$ steering an initial state z_0 to a final state z_1 is given by:*

$$
u = B^{n_0*}L_{B^{n_0}}^{-1}(z_1 - \Phi(n_0,0)z_0). \tag{2.14}
$$

Proof Suppose the system (2.4) is exactly controlable. Then, from theorem 2.1 part $(a)-(iii)$, there is $\gamma > 0$ such that $\|B^{n_0*}z\| \geq \gamma\|z\|$, for all $z \in Z$, i.e.,

$$
\|B^{n_0*}z\|^2 \geq \gamma^2\|z\|^2, \;\; z \in Z.
$$

i.e.,

$$
\langle B^{n_0}B^{n_0*}z, z\rangle \geq \gamma^2\|z\|^2, \;\; z \in Z.
$$

i.e.,

$$
\langle L_{B^{n_0}}z, z\rangle \geq \gamma^2\|z\|^2, \;\; z \in Z \tag{2.15}
$$

This implies that $L_{B^{n_0}}$ is one to one. Now, we shall prove that $L_{B^{n_0}}$ is surjective. That is to say

$$
\mathcal{R}(L_{B^{n_0}}) = \text{Rang}(L_{B^{n_0}}) = Z.
$$

For the purpose of contradiction, let us assume that $\mathcal{R}(L_{B^{n_0}})$ is strictly contained in Z. On the other hand, using Cauchy Schwarz's inequality and (2.15) we get

$$\|L_{B^{n_0}}z\|_{l^2} \geq \gamma^2\|z\|^2, z \in Z,$$

which implies that $\mathcal{R}(L_{B^{n_0}})$ is closed. Then, from Hahn Banach's Theorem there exist $z_0 \neq 0$ such that

$$\langle L_{B^{n_0}}z, z_0\rangle = 0, \forall z \in Z.$$

In particular, putting $z = z_0$ we get from (2.15) that

$$0 = \langle L_{B^{n_0}}z_0, z_0\rangle \geq \gamma^2\|z_0\|^2.$$

Then $z_0 = 0$, which is a contradiction. Hence, $L_{B^{n_0}}$ is a bijection and from the Open Mapping Theorem, $L^{-1}_{B^{n_0}}$ is a bounded linear operator.

Now suppose $L_{B^{n_0}}$ is invertible. Then, from Theorem (2.1) it is enough to prove that $\mathcal{R}(B^{n_0}) = Z$. For $z \in Z$ we define the control $u_z \in l^2(I\!N, U)$ as follows

$$u_z = Sz = B^{n_0*}L^{-1}_{B^{n_0}}z.$$

Then $B^{n_0}u_z = z$. The rest of the proof follows from here. □

Lemma 2.2 *The equation (2.4) is approximately controllable for* $n_0 \in I\!N$ *if, and only if,* $\overline{\text{Rang}(L_{B^{n_0}})} = Z$.

Proof Suppose the system (2.4) is approximately controlable for some $n_0 \in I\!N^*$. Then, from Theorem 2.1 part $(b) - (ii)$ we have that

$$\langle L_{B_0^n}z, z\rangle > 0, \ \ \forall z \in Z, \ \ z \neq 0. \tag{2.16}$$

For the purpose of contradiction, let us assume that

$$\overline{\text{Rang}(L_{B_0^n})} \subset Z.$$

Then, from Hanh Banach's Theorem there exists $z_0 \neq 0$ such that

$$\langle L_{B^{n_0}}z, z_0\rangle = 0, \ \ \forall z \in Z.$$

In particular, if we put $z = z_0$, then $\langle L_{B^{n_0}}z_0, z_0\rangle = 0$, which contradicts (2.16).

Now, suppose that $\overline{\text{Rang}(L_{B^{n_0}})} = Z$, i.e., $\overline{\text{Rang}(B^{n_0}B^{n_0*})} = Z$, so $\overline{\text{Rang}(B^{n_0})} = Z$. Then, from Theorem 2.1 we have that (2.4) is approximately controllable. □

3 Main Results.

Now, we study the controllability of the system

$$z(n+1) = T(n)z(n) + B(n)u(n), \quad n \in I\!N^*, \quad z(n) \in Z, \quad u(n) \in U, \tag{3.17}$$

where Z, U are Hilbert spaces, $I\!N^* = I\!N \cup \{0\}$, $B \in l^\infty(I\!N, L(U,Z))$, $u \in l^2(I\!N, U)$ and $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup given by:

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z, \quad t \geq 0$$

according to lemma 1.1.

Proposition 3.1 *The evolution operator $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Lambda}$ associated to the equation (3.17), is given by the formula $\Phi(m,n) = T(\Theta(m,n))$, where*

$$\Theta(m,n) = \frac{m^2 - n^2 - m + n}{2} \in I\!N, m \geq n.$$

Proof We know that

$$\Phi(m,n) = T(m-1)T(m-2)\cdots T(n) = T(m-1)T(m-2)\cdots T(m-k),$$

where $m = n + k$. Then,

$$\begin{aligned}
\Phi(m,n) &= T(m-1+m-2+\cdots+m-k) = T\left(km - \sum_{i=1}^{k} i\right) \\
&= T\left(km - \frac{k(k+1)}{2}\right) = T\left(\frac{2km - kk - k}{2}\right) \\
&= T\left(\frac{k(2m-k) - k}{2}\right) = T\left(\frac{k(m+n) - k}{2}\right) \\
&= T\left(\frac{k(m+n-1)}{2}\right) = T\left(\frac{(m-n)(m+n-1)}{2}\right) \\
&= T\left(\frac{m^2 - n^2 - m + n}{2}\right) = T(\Theta(m,n)).
\end{aligned}$$

□

Proposition 3.2 *Under the hypothesis (1.3) the operator*

$$L_{B^{n_0}} z = B^{n_0} B^{n_0 *} z = \sum_{k=1}^{n_0} \Phi(n_0, k) B(k-1) B^*(k-1) \Phi^*(n_0, k) z,$$

can be written as follows

$$L_{B^{n_0}} = \sum_{j=1}^{\infty} L_{B_j^{n_0}} P_j,$$

where

$$L_{B_j^{n_0}} = B_j^{n_0} B_j^{n_0 *} = \sum_{k=1}^{n_0} e^{A_j \Theta(n_0,k)} BB^* e^{A_j^* \Theta(n_0,k)}$$

and $\Theta(n_0, k) = \dfrac{n_0^2 - k^2 - n_0 + k}{2} \in I\!N$.

Lemma 3.1 *(a) System (1.2) is exactly controllable if, and only if, there exist* $\gamma > 0$ *such that*

$$\langle L_{B_j^n} P_j z, P_j z\rangle \geq \gamma \|P_j z\|^2, \ \ \forall z \in Z, \quad j = 1, 2, 3, \ldots$$

(b) System (1.2) is approximately controllable if, and only if, each of the following system

$$z(n+1) = e^{A_j n} z(n) + B_j u(n), \ \ z(n) \in \text{Rang}(P_j), \ \ n \in I\!N, \quad j = 1, 2, 3, \ldots \tag{3.18}$$

is approximately controllable.

(c) System (1.2) is approximately controllable if, and only if,

$$\langle L_{B_j^n} P_j z, P_j z\rangle > 0, \ \ \forall z \neq 0 \ \ in \ \ Z, \quad j = 1, 2, 3, \ldots$$

Proof

(a) Suppose that there exist $\gamma > 0$ such that $\langle L_{B_j^n} P_j z, P_j z\rangle \geq \gamma \|P_j z\|^2$. Then

$$\begin{aligned}
\langle L_{B^{n_0}} z, z\rangle &= \left\langle \sum_{j=1}^{\infty} L_{B_j^{n_0}} P_j z, \sum_{j=1}^{\infty} P_j z \right\rangle \\
&= \left\langle \sum_{j=1}^{\infty} \left(\sum_{k=1}^{n_0} e^{A_j \Theta(n_0,k)} BB^* e^{A_j^* \Theta(n_0,k)} \right) P_j z, \sum_{j=1}^{\infty} P_j z \right\rangle \\
&= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \left\langle \sum_{k=1}^{n_0} e^{A_j \Theta(n_0,k)} BB^* e^{A_j^* \Theta(n_0,k)} P_j z, P_m z \right\rangle \\
&= \sum_{j=1}^{\infty} \left\langle \sum_{k=1}^{n_0} e^{A_j \Theta(n_0,k)} BB^* e^{A_j^* \Theta(n_0,k)} P_j z, P_j z \right\rangle \\
&= \sum_{j=1}^{\infty} \langle L_{B_j^{n_0}} P_j z, P_j z\rangle \geq \gamma \sum_{j=1}^{\infty} \|P_j z\|^2 = \gamma \|z\|^2
\end{aligned}$$

So, (1.2) is exactly controllable by Theorem 2.1 part $(a)-(ii)$. Conversely, suppose that (1.2) is exactly controllable , then by Theorem 2.1 part $(a)-(ii)$, there exist $\gamma>0$ such that $\langle L_{B^{n_0}}z, z\rangle \geq \gamma\|z\|^2$. In particular,

$$\langle L_{B_j^{n_0}}P_jz, P_jz\rangle = \left\langle \sum_{i=1}^{\infty} L_{B_i^{n_0}}P_iP_jz, P_jz\right\rangle = \langle L_{B^{n_0}}P_jz, P_jz\rangle \geq \gamma\|P_jz\|^2,$$

which conclude the proof of (a).

(b) Assume that (1.2) is approximately controllable and there exists j such that

$$z(n+1) = e^{A_jn}z(n)B_ju(n), \quad z(n)\in\mathcal{R}(P_j), \quad n\in I\!N$$

is not approximately controllable. Then by theorem 2.1 part $(b)-(iii)$, there exist $z_j\in R(P_j)$, $z_j\neq 0$ such that

$$B_j^*e^{A_j^*n}z_j = 0.$$

Moreover, since (1.2) is approximately controllable, we have

$$B^*T^*(n)z = B^*\Phi^*(n)z = 0 \Rightarrow z = 0.$$

Now, if we put $z = P_jz_j = z_j$, then

$$B^*T^*(n)z = B^*\sum_{k=1}^{\infty} e^{A_k^*\Theta(n,k)}P_kz = B^*e^{A_j^*\Theta(n,k)}P_jz = (B_j)^*e^{A_j^*\Theta(n,k)}z_j = 0,$$

which implies that $z_j = 0$, and this contradicts the assumption. Therefore, (3.18) is approximately controllable for all j.

If (3.18) is approximately controllable for all j, then, by Theorem 2.1 part $(b)-(ii)$,

$$\langle L_{B_j^{n_0}}P_jz, P_jz\rangle > 0, \quad z\neq 0.$$

So,

$$\begin{aligned}
\langle L_{B^{n_0}}z, z\rangle &= \left\langle \sum_{j=1}^{\infty} L_{B_j^{n_0}}P_jz, \sum_{j=1}^{\infty}P_jz\right\rangle \\
&= \left\langle \sum_{j=1}^{\infty}\left(\sum_{k=1}^{n_0} e^{A_j\Theta(n_0,k)}BB^*e^{A_j^*\Theta(n_0,k)}\right)P_jz, \sum_{j=1}^{\infty}P_jz\right\rangle \\
&= \sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\left\langle \sum_{k=1}^{n_0} e^{A_j\Theta(n_0,k)}BB^*e^{A_j^*\Theta(n_0,k)}P_jz, P_mz\right\rangle \\
&= \sum_{j=1}^{\infty}\left\langle \sum_{k=1}^{n_0} e^{A_j\Theta(n_0,k)}BB^*e^{A_j^*\Theta(n_0,k)}P_jz, P_jz\right\rangle \\
&= \sum_{j=1}^{\infty}\langle L_{B_j^{n_0}}P_jz, P_jz\rangle > 0, \quad z\neq 0
\end{aligned}$$

Hence, (1.2) is approximately controllable and (b) is proved.

(c) follows immediately from (b) and Theorem 2.1 part (b). ∎

4 Applications

Now, as an application of the main results of this paper we shall consider two important examples, a flow-discretization of the controlled heat equation and the controlled wave equation.

Example 4.1 *Heat Equation*

Considere the heat equation

$$\begin{cases} y_t = y_{xx} + u(t,x) \\ y(0,x) = y_0(x) \\ y_x(t,0) = y_x(t,1) = 0 \end{cases} \tag{4.19}$$

The system (4.19) can be written as an abstract equation in the space $Z = L^2[0,1]$

$$\begin{cases} z' = -Az + Bu(t), z \in Z \\ z(0) = z_0 \end{cases} \tag{4.20}$$

where $B = I$, the control function u belong to $L^2[0,r,Z]$ and the operator A is given by $A\phi = -\phi_{xx}$ with domain $D(A) = H^2 \cap H_0^1$, and has the following spectral decomposition.

a) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} j^2\pi^2 \langle z, \phi_j \rangle \phi_j,$$

where $\phi_j(x) = \sin(j\pi x)$.

b) $-A$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z, \quad z \in Z, \quad t \geq 0, \tag{4.21}$$

where $E_j z = \langle \phi_j, z \rangle$ and $\lambda_j = j^2\pi^2$.

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and

$$z = \sum_{j=1}^{\infty} E_j z, \quad z \in Z.$$

Now, the discretization of (4.20) on flow is given by

$$\begin{cases} z(n+1) = T(n)z(n) + B(n)u(n), z \in Z \\ z(0) = z_0 \end{cases} \tag{4.22}$$

In this case, $T^*(t) = T(t)$ and $B = I$. We shall see that (4.22) is exactly controllable. In fact, in this case we have that:

$$B^{n_0} : l^2(I\!N, U) \longrightarrow Z, \quad B^{n_0}u = \sum_{k=1}^{n_0} T(\Theta(n_0,k))u(k-1)$$

and

$$L_{B^{n_0}} : Z \longrightarrow Z, \quad L_{B^{n_0}} = B^{n_0}B^{n_0*} = \sum_{j=1}^{\infty} L_{B_j^{n_0}} E_j z,$$

where $L_{B_j^{n_0}} = \sum_{k=1}^{n_0} e^{-2\lambda_j \Theta(n_0,k)}$.

Now, we shall prove the existence of $\gamma > 0$ such that

$$\langle L_{B_j^{n_0}} E_j z, E_j z\rangle \geq \gamma \|E_j z\|^2.$$

This is equivalent to the existence of $\gamma > 0$ such that

$$\left[\sum_{k=1}^{n_0} e^{-2\lambda_j \Theta(n_0,k)} - \gamma\right] \|E_j z\|^2 \geq 0,$$

which is obviously true for $0 < \gamma < 1$ since $e^{-2\lambda_j \Theta(n_0,n_0)} = 1$.

Then, for such γ we have

$$\langle L_{B^{n_0}} z, z\rangle = \langle \sum_{j=1}^{\infty} L_{B_j^{n_0}} E_j z, E_j z\rangle = \sum_{j=1}^{\infty} \langle L_{B_j^{n_0}} E_j z, E_j z\rangle \geq \gamma \sum_{j=1}^{\infty} \|E_j z\|^2 = \gamma \|z\|^2.$$

Thus, $\langle L_{B^{n_0}} z, z\rangle \geq \gamma \|z\|^2$, $z \in Z$. Therefore, applying Theorem 2.1 part $(a)-(ii)$ we obtain that (4.22) is exactly controllable. □

Example 4.2 *Wave Equation*

Considere the wave equation

$$\begin{cases} y_{tt} = y_{xx} + u(t,x) \\ y(t,0) = y(t,1) = 0 \\ y(0,x) = y_0, y_t(0,x) = y_1(x) \end{cases} \tag{4.23}$$

The system (4.23) can be written as an abstract second order equation in the Hilbert space $X = L^2[0,1]$ as follows:

$$\begin{cases} y'' = -Ay + u(t) \\ y(0) = y_0, y'(0) = y_1 \end{cases} \tag{4.24}$$

where the operator A is given by $A\phi = -\phi_{xx}$ with domain $D(A) = H^2 \cap H_0^1$, and has the following spectral decomposition.

For all $x \in D(A)$ we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j \langle x, \phi_j \rangle \phi_j = \sum_{j=1}^{\infty} \lambda_j E_j x,$$

where $\lambda_j = j^2\pi^2$, $\phi_j(x) = \sin(j\pi x)$, $\langle \cdot, \cdot \rangle$ is the inner product in X and $E_j x = \langle x, \phi_j \rangle \phi_j$.

So, $\{E_j\}$ is a family of complete orthogonal projections in X and $x = \sum_{j=1}^{\infty} E_j x$, $x \in X$.

Using the change of variables $y' = v$. the second order equation (4.24) can be written as a first order system of ordinary differential equations in the Hilbert space $Z = X^{1/2} \times X$ as

$$\begin{cases} z' = \mathcal{A}z + Bu(t), z \in Z \\ z(0) = z_0 \end{cases} \tag{4.25}$$

where

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \ \mathcal{A} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \tag{4.26}$$

$\mathcal{A}$ is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times X$ and $u \in L^2(0, \tau, X) = U$. The proof of the following theorem follows from Theorem 3.1 (see, [7]) by putting $c = 0$ and $d = 1$.

Theorem 4.1 *The operator $\mathcal{A}$ given by (4.26), is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \in \mathbb{R}}$ given by*

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, z \in Z, t \geq 0, \tag{4.27}$$

where $\{P_j\}_{j \geq 1}$ is a complete family of orthogonal projections in the Hilbert space Z given by

$$P_j = diag[E_j, E_j], j \geq 1 \tag{4.28}$$

and

$$A_j = \widetilde{B}_j P_j, \quad \widetilde{B}_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j & 0 \end{bmatrix}, j \geq 1. \tag{4.29}$$

Now, the discretization of (4.25) on flow is given by

$$\begin{cases} z(n+1) = T(n)z(n) + B(n)u(n), & z \in Z \\ z(0) = z_0 \end{cases} \tag{4.30}$$

where

$$B : U \longrightarrow Z, \quad Bu = \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

We want to show that (4.30) is approximately controllable. In this case, we have

$$B^{n_0} : l^2(I\!N, U) \longrightarrow Z, \quad B^{n_0}u = \sum_{k=1}^{n_0} T(\Theta(n_0,k))Bu(k-1)$$

and

$$L_{B^{n_0}} : Z \longrightarrow Z, \quad L_{B^{n_0}} = B^{n_0}B^{n_0*}$$

Since

$$BB^* = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

we have that

$$P_jBB^* = BB^*P_j, \quad j = 1, 2, 3, \ldots. \tag{4.31}$$

On the other hand, we have that $T^*(t) = T(-t)$. Then

$$\begin{aligned} L_{B^{n_0}}z &= \sum_{k=1}^{n_0} T(\Theta(n_0,k))BB^*T^*(\Theta(n_0,k))z \\ &= \sum_{k=1}^{n_0}\sum_{j=1}^{\infty} e^{A_j\Theta(n_0,k)}P_jBB^*\sum_{i=1}^{\infty} e^{-A_j\Theta(n_0,k)}P_jz \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{n_0} e^{A_j\Theta(n_0,k)}BB^*e^{-A_j\Theta(n_0,k)}P_jz \\ &= \sum_{j=1}^{\infty} L_{B_j^{n_0}}P_jz. \end{aligned}$$

where $L_{B_j^{n_0}} = B_j^{n_0}B_j^{n_0*} = \sum_{k=1}^{n_0} e^{A_j\Theta(n_0,k)}BB^*e^{-A_j\Theta(n_0,k)}$.

Hence, $L_{B^{n_0}} = \sum_{j=1}^{\infty} L_{B_j^{n_0}}$.

Let $z = [z_1, z_2]^T$ in Z. It is not difficult to verify that

$$L_{B_j^{n_0}}P_jz = \sum_{k=1}^{n_0} n_0[0, E_jz_2]^T.$$

Then

$$\langle L_{B_j^{n_0}} P_j z, P_j z \rangle = \langle n_0 [0, E_j z_2]^T, [E_j z_1, E_j z_2]^T \rangle = n_0 \|E_j z_2\|^2 > 0, \ \ \forall j$$

Hence, using (4.31), we have for $z \neq 0$ in Z that

$$\langle L_{B^{n_0}} z, z \rangle = \langle \sum_{j=1}^{\infty} L_{B_j^{n_0}} P_j z, \sum_{j=1}^{\infty} P_j z \rangle = \sum_{j=1}^{\infty} \langle L_{B_j^{n_0}} P_j z, P_j z \rangle = n_0 \sum_{j=1}^{\infty} \|E_j z_2\|^2 = n_0 \|z_2\|^2 > 0.$$

In consequence, by Lemma 3.1 part (c), the equation (4.30) is approximately controllable. □

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