Universidad de los Andes
Facultad de Ciencias
Departamento de Matemática

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I.Dobrakov and T.V. Panchapagesan

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A Simple Proof of the Theorem on Borel Extension and Representability of Operators

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Abstract

Let T be a locally compact Hausdorff space and let $C_o(T)$ be the Banach space of all complex valued continuous functions vanishing at infinity in T, provided with the supremum norm. Let X be a quasicomplete locally convex Hausdorff space. A simple proof of the theorem on regular Borel extension of X-valued σ-additive Baire measures on T is given, which is more natural and direct than the existing ones. Using this result is obtained the integral representation and weak compactness of a continuous linear map $u : C_o(T) \to X$ when $c_o \not\subset X$. The proof of the latter result is direct in the sense that it doesn't use the technique of reduction to compact metrizable case of T unlike the proof of Theorem 5 of Pelczyński [15] or that of the necessity part of Theorem 5.3 of Thomas [18]. Also is given an alternative proof of the sufficiency part of Theorem 5.3 of Thomas [18].

1. INTRODUCTION

Let T be a locally compact Hausdorff space and $C_o(T)$ the Banach space of all complex valued continuous functions vanishing at infinity in T, endowed with the supremum norm.

If X is a Banach space with $c_o \not\subset X$ and S is a compact Hausdorff space, then Pelczyński [15] proved that each continuous linear map $u : C(S) \to X$ admits an integral representation with respect to a σ-additive X-valued Borel measure on T and that u is weakly compact. Later, in 1970, this result was extended by Thomas [18] to continuous linear maps $u : C_o(T) \to X$, where X is a locally convex Hausdorff space (briefly, a lcHs) which is quasicomplete and Σ-complete. While Pelczyński [15] used the results of [1], Thomas [18] used the Grothendieck characterizations of weakly compact operators on $C_o(T)$ as given in [6]. We also note that by Theorem 4 of Tumarkin [19] the Σ-completeness of X is the same as that $c_o \not\subset X$. The proofs of Pelczyński [15] and Thomas [18] are rather indirect in the sense that they use the technique of reduction to compact metrizable case.

Recently, one of the authors gave in [14] an alternative direct proof of the said result of Thomas [18] without reducing to compact metrizable case. However, the proof given there is highly technical involving many deep results from [13,14]. So the aim of the present note is to give a simpler direct proof of the above result. For this we use Lemma 1 and Theorem 2 of Grothendieck [6] (no other result of [6] is used), the first part of Theorem 1 of [14] and the theorem on regular Borel extension of quasicomplete lcHs valued Baire measures on T.

The regular Borel extension theorem for Banach space and complete lcHs valued Baire measures on T are well known since the publications of [4,9] and has also been generalized to group-valued measures by Sion [16]. But, even for the case of Banach space valued measures, the proof given in [9] is indirect and quite involved, presupposing the results from earlier papers of the author. Here we present an alternative proof of the said theorem for quasicomplete lcHs valued Baire measures on T by using Theorem 2 of Dinculeanu and Kluvánek [4] and the lemma in § 68 of Berberian [2]. The reader can observe that the present proof is direct, simple and elegant.

Finally, using the first part of Theorem 1 of [14] and Lemma 1 of Grothendieck [6] we also give an alternative proof of the sufficiency part of Theorem 5.3 of Thomas [18]. In this connection, see Remark 2 in Section 4.

2. PRELIMINARIES

In this section we fix notation and terminology. For the convenience of the reader we also give some definitions and results from the literature.

In the sequel T will denote a locally compact Hausdorff space and $C_o(T)$ the Banach space of all complex valued continuous functions vanishing at infinity in T, endowed with norm $||\cdot||_T$ given by $||f||_T = sup_{t\in T}|f(t)|$.

Let $\mathcal{K}$ (resp. $\mathcal{K}_o$) be the family of all compacts (resp. compact $G_\delta s$) in T. $\mathcal{B}_o(T)$, $\mathcal{B}_c(T)$ and $\mathcal{B}(T)$ are the σ-rings generated by $\mathcal{K}_o$, $\mathcal{K}$ and the class of all open sets in T, respectively. The members of $\mathcal{B}_o(T)$ are called Baire sets and those of $\mathcal{B}_c(T)$ are called σ-Borel sets in T. The members of $\mathcal{B}(T)$ are called Borel sets in T. Since a subset E of T belongs to $\mathcal{B}_c(T)$ if and only if E is a σ-bounded Borel set, the members of $\mathcal{B}_c(T)$ are called σ-Borel sets.

DEFINITION 1. Let $\mathcal{S}$ be a σ-ring of sets in T such that $\mathcal{K} \subset \mathcal{S}$ or $\mathcal{K}_o \subset \mathcal{S}$. A complex measure μ on $\mathcal{S}$ is said to be $\mathcal{S}$-regular if, given $E \in \mathcal{S}$ and $\epsilon > 0$, there exists a compact $K \in \mathcal{S}$ and an open set $U \in \mathcal{S}$ with $B \subset E \subset U$ such that $|\mu(B)| < \epsilon$ for every $B \in \mathcal{S}$ with $B \subset U\backslash K$. When $\mathcal{S} = \mathcal{B}(T)$ (resp. $\mathcal{S} = \mathcal{B}_c(T)$, $\mathcal{S} = \mathcal{B}_o(T)$) , we use the terminology Borel (resp. σ-Borel, Baire) regularity in place of $\mathcal{S}$-regularity.

The following proposition is well known. See, for example, Theorem 3.7 of [10] and Theorem 2.4 of [11].

PROPOSITION 1. *Every complex Baire measure μ_o on T is regular and has unique extension μ on $\mathcal{B}(T)$ (resp. μ_c on $\mathcal{B}_c(T)$) such that μ is a Borel (resp. σ-Borel) regular complex measure. Moreover, $\mu|_{\mathcal{B}_c(T)} = \mu_c$. Besides, μ and μ_c are positive and finite if μ_o is so.*

$M(T)$ is the Banach space of all bounded complex Radon measures on T with their domain restricted to $\mathcal{B}(T)$ so that each $\mu \in M(T)$ is a regular (bounded) complex Borel measure on T and has norm $||\cdot||$ given by $||\mu|| = var(\mu, T)$ where the variation of μ is taken with respect to $\mathcal{B}(T)$.

We denote $var(\mu, E)$ by $|\mu|(E)$, for $E \in \mathcal{B}(T)$.

A vector measure is an additive set function defined on a ring of sets with values in a lcHs. In the sequel X denotes a lcHs with topology τ. Γ is the set of all τ-continuous seminorms on X. The dual of X is denoted by X^*.

The strong topology $\beta(X^*, X)$ of X^* is the locally convex topology induced by the seminorms $\{p_B : B$ bounded in $X\}$, where $p_B(x^*) = sup_{x\in B}|x^*(x)|$. X^{**} denotes the dual of $(X^*, \beta(X^*, X))$ and is endowed with the locally convex toplogy τ_e of uniform convergence on equicontinuous subsets of X^*. Note that $(X^*, \beta(X^*, X))$ and (X^{**}, τ_e) are lcHs.

It is well known that the canonical injection $J : X \to X^{**}$ given by $< Jx, x^* >=< x, x^* >$ for all $x \in X$ and $x^* \in X^*$, is linear. On identifying X with $JX \subset X^{**}$, one has $\tau_e|_{JX} = \tau_e|_X = \tau$.

DEFINITION 2. A linear map $u : C_o(T) \to X$ is called a weakly compact operator on $C_o(T)$ if $\{uf : ||f||_T \leq 1\}$ is relatively weakly compact in X.

Let E and F be lcHs and let $u : E \to F$ be a continuous linear map. Then the adjoint u^* and the biadjoint u^{**} of u are well defined linear maps and $u^* : (F, \sigma(F^*, F)) \to (E^*, \sigma(E^*, E))$ and $u^{**} : (E^{**}, \tau_e) \to (F^{**}, \tau_e)$ are continuous (see Corollary to Proposition 1, § 12 ,Chapter 3 of Horváth [8] and Proposition 8.7.27 of Edwards [5]).

The following result (Corollary 9.3.2 of Edwards [5] which is essentially due to Lemma 1 of Grothendieck [6]) plays a key role in Section 4.

PROPOSITION 2. *Let E and F be lcHs with F quasicomplete. If $u : E \to F$ is linear and continuous, then u maps bounded subsets of E into relatively weakly compact subsets of F if and only if $u^*(A)$ is relatively $\sigma(E^*, E^{**})$-compact for each equicontinuous subset A of F^*.*

The following result is due to Theorem 2 of Grothendieck [6], and is needed in Section 4.

PROPOSITION 3. *A bounded set A in $M(T)$ is relatively weakly compact if and only if, for each disjoint sequence $\{U_n\}_1^\infty$ of open sets in T,*

$$sup_{\mu\in A}|\mu|(U_n) \to 0$$

as $n \to 0$.

For each τ-continuous seminorm p on X, let $p(x) = ||x||_p,\ x \in X$, and let $X_p = (X, ||\cdot||_p)$ be the associated seminormed space. The completion of the quotient normed space $X_p/p^{-1}(0)$ is denoted by $\tilde{X_p}$. Let $\Pi_p : X_p \to X_p/p^{-1}(0) \subset \tilde{X_p}$ be the canonical quotient map.

Let $\mathcal{S}$ be a σ-ring of subsets of a non empty set Ω. Given a vector measure $m : \mathcal{S} \to X$, for each τ-continuous seminorm p on X let $m_p : \mathcal{S} \to \tilde{X_p}$ be given by $m_p(E) = \Pi_p \circ m(E)$ for $E \in \mathcal{S}$. Then

m_p is a Banach space valued vector measure on $\mathcal{S}$. We define the p-semivariation $||m||_p$ of m by

$$||m||_p(E) = ||m_p||(E) \text{ for } E \in \mathcal{S}$$

and

$$||m||_p(\Omega) = ||m_p||(\Omega) = \sup_{E\in\mathcal{S}} ||m_p||(E)$$

where $||m_p||$ is the semivariation of the vector measure m_p. When m is σ-additive, m_p is a Banach space valued σ-additive vectr asure and hence, by a well known theorem on vector measures, $||m||_p(\Omega) = ||m_p||(\Omega) \leq 4\sup_{E\in\mathcal{S}} ||m(E)||_p < \infty$.

An X-valued vector measure m on a σ-ring $\mathcal{S}$ of subsets of Ω is said to be bounded if $\{m(E) : E \in \mathcal{S}\}$ is bounded in X and equivalently, if $||m||_p(\Omega) < \infty$ for each τ-continuous seminorm p on X.

For the theory of integration of bounded $\mathcal{S}$-measurable scalar functions with respect to a bounded X-valued vector measure the reader may refer to [12]. The following result is due to the first part of Theorem 1 of [14] which is analogous to Theorem VI.2.1 of [3] for lcHs-valued continuous linear maps on $C_o(T)$. It plays a key role in Section 4.

PROPOSITION 4. *Let X be a lcHs. Let $u : C_o(T) \to X$ be a continuous linear transformation. Then there exists a unique X^{**}- valued vector measure m on $\mathcal{B}(T)$ satisfying the following properties:*

(i) $x^*(m) \in M(T)$ for each $x^* \in X^*$ and consequently, $m : \mathcal{B}(T) \to X^{**}$ is σ-additive in $\sigma(X^{**}, X^*)$-topology.

(ii) The mapping $x^* \to x^*m$ of X^* into $M(T)$ is weak-weak* continuous. Moreover, $u^*x^* = x^*m,\ x^* \in X^*$.*

(iii) $x^*uf = \int_T f dx^*m$ for each $f \in C_o(T)$ and $x^* \in X^*$.

(iv) The range of m is τ_e-bounded in X^{**}.

(v) $m(E) = u^{**}(\chi_E)$ for $E \in \mathcal{B}(T)$.

DEFINITION 3. Let $u : C_o(T) \to X$ be a continuous linear map. The vector measure m given in Proposition 4 is called the representing measure of u.

3. REGULAR BOREL (RESP. σ-BOREL) EXTENSION OF X-VALUED BAIRE MEASURES

By using Theorem 2 of [4] and the lemma in § 68 of Berberian we give here a simple direct proof of the theorem on regular Borel and σ-Borel extensions of an X-valued Baire measure on T. To this end, we begin with the following definitions.

DEFINITION 4. A σ-additive vector measure $m : \mathcal{B}_o(T) \to X$ (resp. $\mathcal{B}_c(T) \to X$, $\mathcal{B}(T) \to X$) is called an X-valued Baire (resp. σ-Borel, Borel) measure on T.

DEFINITION 5. Let $\mathcal{S}$ be one of $\mathcal{B}_o(T)$, $\mathcal{B}_c(T)$ or $\mathcal{B}(T)$. An X-valued vector measure m on $\mathcal{S}$ is said to be regular if, given $E \in \mathcal{S}$, a seminorm $p \in \Gamma$ and $\epsilon > 0$, there exists a compact $K \in \mathcal{S}$ and an open set $U \in \mathcal{S}$ with $K \subset E \subset U$ such that $||m(B)||_p < \epsilon$ for every $B \in \mathcal{S}$ with $B \subset U \backslash K$. When $\mathcal{S} = \mathcal{B}_o(T)$ (resp. $\mathcal{B}_c(T)$, $\mathcal{B}(T)$) we use the terminology Baire (resp. σ-Borel, Borel) regular.

THEOREM 1. *Let m be an X-valued Baire measure on T and let X be a quasicomplete lcHs. Then there exists a unique X-valued Borel (resp. σ-Borel) regular σ-additive extension $\hat{m}$ (resp. m_c) of m on $\mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$). Moreover, $\hat{m}|_{\mathcal{B}_c(T)} = m_c$.*

Proof. For each $p \in \Gamma$, $m_p : \mathcal{B}_o(T) \to \tilde{X}_p$ is σ-additive. Since the proof of Theorem I.2.4 of [3] holds for σ-rings too, for each $p \in \Gamma$ there exists a finite positive measure μ_p on $\mathcal{B}_o(T)$ such that

$$\lim_{\mu_p(A) \to 0} ||m_p(A)||_p = 0, \quad A \in \mathcal{B}_o(T).$$

By Propisition 1 μ_p has a unique extension $\hat{\mu}_p$(resp. μ_p^c) on $\mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$) such that $\hat{\mu}_p$ (resp. μ_p^c) is a (σ-additive) regular Borel (resp. σ-Borel) finite positive measure. Moreover, $\hat{\mu}_p|_{\mathcal{B}_c(T)} = \mu_p^c$.

For $p \in \Gamma$, let $\rho_p(E, F) = \hat{\mu}_p(E \Delta F)$, for $E, F \in \mathcal{B}(T)$. Then $\rho_p(E, F) = \mu_p^c(E \Delta F)$ for $E, F \in \mathcal{B}_c(T)$. Let $s(\Gamma)$ be the uniform structure defined by the family $\{\rho_p\}_{p \in \Gamma}$ of semidistances on $\mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$) and let Θ (resp. Θ_c) be the topology induced by $s(\Gamma)$ on $\mathcal{B}(T)$ (resp. on $\mathcal{B}_c(T)$). Then clearly, $\Theta|_{\mathcal{B}_c(T)} = \Theta_c$.

AFFIRMATION 1. $\mathcal{B}_o(T)$ is Θ-dense (resp. Θ_c-dense) in $\mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$).

In fact, given $A \in \mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$), $p \in \Gamma$ and $\epsilon > 0$, it suffices to show that there exists $E \in \mathcal{B}_o(T)$ such that $\rho_p(A, E) < \epsilon$. Since $\hat{\mu}_p$ is Borel regular (resp. μ_p^c is σ-Borel regular), there exists a compact K and an open set U (resp. an open set $U \in \mathcal{B}_c(T)$) such that $K \subset A \subset U$ and $\hat{\mu}_p(U \backslash K) < \epsilon$ (resp. $\mu_p^c(U \backslash K) < \epsilon$). As $K \in \mathcal{B}_c(T)$ and $\hat{\mu}_p|_{\mathcal{B}_c(T)} = \mu_p^c$ is σ-Borel regular, by the lemma in § 68 of Berberian [2] there exists $E \in \mathcal{B}_o(T)$ such that $\hat{\mu}_p(K \Delta E) = \mu_p^c(K \Delta E) = 0$. Then $\rho_p(A, E) \leq \hat{\mu}_p(A \Delta K) + \hat{\mu}_p(K \Delta E) \leq \hat{\mu}_p(U \backslash K) < \epsilon$ (resp. $\rho_p(A, E) \leq \mu_p^c(U \backslash K) < \epsilon$). Hence the affirmation holds.

Let $\tilde{X}$ be the completion of X. Then by Affirmation 1 and by Theorem 2 of Dinculeanu and Kluvánek [4] there exists an additive set function $\hat{m} : \mathcal{B}(T) \to \tilde{X}$ (resp. $m_c : \mathcal{B}_c(T) \to \tilde{X}$) such that $\hat{m}|_{\mathcal{B}_o(T)} = m$ (resp. $m_c|_{\mathcal{B}_o(T)} = m$) and for every $p \in \Gamma$ we have

$$\lim_{\hat{\mu}_p(A) \to 0} ||\hat{m}(A)||_p = 0, \quad A \in \mathcal{B}(T) \quad (1)$$

(resp.

$$\lim_{\mu_p^c(A) \to 0} ||m_c(A)||_p = 0, \quad A \in \mathcal{B}_c(T) \quad (1')).$$

Since m is σ-additive on $\mathcal{B}_o(T)$, m is bounded and hence there exists a τ-bounded closed set H in X such that $m(\mathcal{B}_o(T)) \subset H$. Moreover, given $A \in \mathcal{B}(T)$ (resp. $A \in \mathcal{B}_c(T)$), by Affirmation 1 there eists a net $\{E_\alpha\} \subset \mathcal{B}_o(T)$ such that $E_\alpha \to A$ in Θ and

$$\hat{m}(A) = \lim_\alpha m(E_\alpha) \quad (2)$$

(resp.

$$m_c(A) = \lim_\alpha m(E_\alpha) \quad (2')).$$

As $(m(E_\alpha))$ is τ-Cauchy in X and is contained in the τ-bounded closed set H, it follows from the hypothesis on X that $\hat{m}(A)$ (resp. $m_c(A)$) belongs to H. Hence the range of $\hat{m}$ (resp. m_c) is contained in X. Moreover, by (2) and (2') we also have that $\hat{m}(A) = m_c(A)$ for $A \in \mathcal{B}_c(T)$. Thus $\hat{m}|_{\mathcal{B}_c(T)} = m_c$.

From (1) (resp. (1')) and the fact that $\hat{\mu}_p$ (resp. μ_p^c) is a finite Borel (resp. σ-Borel) regular positive measure, it follows that $\hat{m}$ (resp. m_c) is a σ-additive (X-valued) regular Borel (resp. σ-Borel) vector measure.

If $\hat{m}'$ (resp. m_c') is another X-valued σ-additive regular Borel (resp. σ-Borel) extension of m, then for each $x^* \in X^*$, $x^*\hat{m}'$ and $x^*\hat{m}$ (resp. x^*m_c and x^*m_c') are regular Borel (resp. σ-Borel) complex measures extending x^*m. Then by the uniqueness part of Proposition 1 and by the Hahn-Banach theorem we conclude that $\hat{m} = \hat{m}'$ (resp. $m_c = m_c'$). Thus the extension is unique.

Remark 1. An operator theoretic proof of the above theorem is given in [14]. But the above proof is simple and elementary.

4. MAIN THEOREM

In this section we give an alternative simple measure theoretic proof of Theorem 5.3 of Thomas [18] for which he employs his theory of Radon vector measures and the Grothendieck characterizations of weakly compact operators on $C_o(T)$.

THEOREM 2. *Let $u : C_o(T) \to X$ be a continuous linear map and suppose X is a quasicomplete lcHs with $c_o \not\subset X$. Let m be the representing measure of u and let $m_o = m|_{\mathcal{B}_o(T)}$. Then the following assertions hold.*

(i) m_o has range in X and is σ-additive in τ.

(ii) m is an X-valued σ-additive (in τ) regular Borel measure.

(iii) $uf = \int_T f dm, \quad f \in C_o(T)$.

(iv) m is uniquely determined by (ii) and (iii).

(v) u is a weakly compact operator.

Conversely, if X is a quasicomplete lcHs such that each continuous linear map $u : C_o(T) \to X$ is weakly compact for every locally compact Hausdorff space T, then $c_o \not\subset X$.

In other words, a quasicomplete lcHs X contains no copy of c_o (or equivalently, is Σ-complete in the sense of Thomas [18, Definition 5.2] due to Theorem 4 of Tumarkin [19]) if and only if each continuous linear map $u : C_o(T) \to X$ is weakly compact for every locally compact Hausdorff space T.

Proof. Let $c_o \not\subset X$ and let $u : C_o(T) \to X$ be a continuous linear map. By Proposition 4 there exists a unique X^{**}-valued vector measure m on $\mathcal{B}(T)$ such that

$$x^*uf = \int_T f d(x^*m), \quad f \in C_o(T) \quad (3)$$

for each $x^* \in X^*$, $x^*m \in M(T)$ and the mapping $x^* \to x^*m$ is weak*-weak* continuous.

Let $C \in \mathcal{K}_o$. Then by Theorem 55.B of Halmos [7] there exists a decreasing sequence (f_n) in $C_o(T)$ such that $f_n \searrow \chi_C$ pointwise in T. Then by (3) and by the Lebesgue dominated convergence theorem

$$x^*m(C) = \lim_n \int_T f_n d(x^*m) = \lim_n x^*uf_n \quad (4)$$

for each $x^* \in X^*$.

Let $uf_n = x_n$. For $x^* \in X^*$, $x^*m \in M(T)$ and hence there exist finite positive measures $\mu_{x^*,j}$ on $\mathcal{B}(T)$, $j = 1, 2, 3, 4$, such that

$$x^*m = (\mu_{x^*,1} - \mu_{x^*,2}) + i(\mu_{x^*,3} - \mu_{x^*,4}).$$

Again by (3) and by the Lebesgue dominated convergence theorem we have

$$\begin{aligned} \Sigma_{n=1}^{\infty}|(x^*(x_n - x_{n+1})| &= \Sigma_{n=1}^{\infty}|\int_T (f_n - f_{n+1})d(x^*m)| \\ &\le \Sigma_{j=1}^{4}(\Sigma_{n=1}^{\infty}\int_T (f_n - f_{n+1})d\mu_{x^*,j}) \\ &\le \Sigma_{j=1}^{4}\int_T f_1 d\mu_{x^*,j} + \mu_{x^*,j}(C) \\ &< \infty. \end{aligned}$$

Hence

$$|x^*(x_1)| + \Sigma_{n=1}^{\infty}|x^*(x_n - x_{n+1})| < \infty$$

for each $x^* \in X^*$. Since $c_o \not\subset X$, by Theorem 4 of Tumarkin [19] the formal series $x_1 + \Sigma_{n=1}^{\infty}(x_n - x_{n+1})$ converges unconditionally in the topology τ to some vector $x_o \in X$. In other words, $\lim_n x_n = x_o$. Then by (4) we have

$$x^*(x_o) = \lim_n x^*(x_n) = \lim_n x^*uf_n = \lim_n \int_T f_n d(x^*m) = x^*m(C)$$

for each $x^* \in X^*$. Since $m(C) \in X^{**}$, it follows that $m(C) = x_o \in X$. Thus we have proved that $m(\mathcal{K}_o) \subset X$.

Now let $\Sigma = \{E \in \mathcal{B}_o(T) : m(E) \in X\}$. As $\mathcal{K}_o$ is contained in Σ, it follows that the ring $\mathcal{R}(\mathcal{K}_o)$ generated by $\mathcal{K}_o$ is also contained in Σ. Let (E_n) be a monotone sequence in Σ with $E = \lim_n E_n$. When $E_n \nearrow$, put $F_n = E_n - E_{n-1}$ with $E_o = \emptyset$ and $n \in \mathbb{N}$. When $E_n \searrow$, put $F_n = E_n - E_{n+1}$ for $n \in \mathbb{N}$. Clearly, $m(F_n) \in X$ for all n. Then $E = \cup_1^\infty F_n$ when $E_n \nearrow$ and $E_1 \backslash E = \cup_1^\infty F_n$ when $E_n \searrow$. Since x^*m is σ-additive on $\mathcal{B}(T)$, we have

$$x^*m(E) = \Sigma_1^\infty x^*m(F_n) \quad \text{if } E_n \nearrow$$

and

$$x^*m(E_1) - x^*m(E) = \Sigma_1^\infty x^*m(F_n) \quad \text{if } E_n \searrow .$$

Then in both the cases we have $\Sigma_1^\infty |x^*m(F_n)| < \infty$ for each $x^* \in X^*$. As $c_o \not\subset X$, by Theorem 4 of Tumarkin [19] we conclude that the formal series $\Sigma_1^\infty m(F_n)$ is unconditionally convergent to some vector in X in the topology τ. Then it follows in both the cases that there exists a vector $w_o \in X$ such that $\lim_n m(E_n) = w_o$ (in the topology τ). Since x^*m is σ-additive and complex valued, we have

$$x^*m(E) = \lim_n x^*m(E_n) = x^*w_o$$

for all $x^* \in X^*$. As $m(E) \in X^{**}$, we conclude that $m(E) = w_o$. This shows that $E \in \Sigma$ and hence Σ is a monotone class. Now by Theorem 6.B of Halmos [7] it follows that $\Sigma = \mathcal{B}_o(T)$ and so $m(\mathcal{B}_o(T)) \subset X$. Now let $m_o = m|_{\mathcal{B}_o(T)}$. Then the range of m_o is contained in X. Since x^*m is σ-additive, we conclude by the Orlicz-Pettis theorem that m_o is σ-additve in the toplogy τ of X. This proves (i).

As m_o is an X-valued Baire measure on T, by Theorem 1 there exists a unique X-valued σ-additive regular Borel measure $\hat{m}_o$ on T such that $\hat{m}_o|_{\mathcal{B}_o(T)} = m_o$. By Theorem 51.B of Halmos [7], each $f \in C_o(T)$ is $\mathcal{B}_o(T)$-measurable and bounded. Consequently, f is m_o-integrable in the sense of Definition 1 of [12] and

$$\int_T f dm_o \in X, \quad f \in C_o(T). \quad (5)$$

Then by (3) and (5) and by Lemma 6(iii) of [12], we have

$$x^* \int_T f dm_o = \int_T f d(x^*m_o) = \int_T f d(x^*m) = x^*uf$$

and

$$\int_T f d(x^*m_o) = \int_T f d(x^*\hat{m}_o)$$

for $x^* \in X^*$ and $f \in C_o(T)$. Thus the bounded linear functional x^*u on $C_o(T)$ is represented by the regular complex Borel measures x^*m and $x^*\hat{m}_o$ and consequently, by the uniqueness part of the Riesz representation theorem we conclude that $x^*m = x^*\hat{m}_o$. Since this holds for all $x^* \in X^*$, $\hat{m}_o$ is X-valued and m is X^{**}-valued, it follows that $m = \hat{m}_o$. Thus m is X-valued, σ-additive and

Borel regular. Thus (ii) holds.

Since $m_o = m|_{\mathcal{B}_o(T)}$, then by (5) we have $\int_T f dm = \int_T f dm_o \in X$ for $f \in C_o(T)$. Consequently, by Proposition 4(iii), by Lemma 6(iii) of [12] and by the Hahn-Banach theorem we conclude that

$$uf = \int_T f dm, \quad f \in C_o(T).$$

Thus (iii) holds.

If $\tilde{m} : \mathcal{B}(T) \to X$ satisfies (ii) and (iii), then x^*m and $x^*\tilde{m} \in M(T)$ and represent the bounded linear functional x^*u on $C_o(T)$. Hence $x^*m = x^*\tilde{m}$ for each $x^* \in X^*$. Then by the Hahn-Banach theorem we conclude that $m = \tilde{m}$. Thus (iv) holds.

Let (U_n) be a disjoint sequence of open sets in T and let A be an equicontinuous subset of X^*. Recall that the topology τ is the same as the topology of uniform convergence on equicontinuous subsets of X^*. Thus, if $U = \cup_1^\infty U_n$, then (ii) implies that $||m(U_n)||_{p_A} \to 0$ as $n \to \infty$, where $p_A(x) = \sup_{x^* \in A} |x^*(x)|$. In other words, $\lim_n x^* \circ m(U_n) = 0$ uniformly in $x^* \in A$. Since $u^*A = \{x^* \circ m : x^* \in A\}$ by (ii) of Proposition 4, and since m has bounded range in X by (ii), we have

$$\begin{aligned} \sup\{||\mu|| : \mu \in u^*A\} &= \sup_{x^* \in A} |x^* \circ m|(T) \leq 4 \sup_{x^* \in A, B \in \mathcal{B}(T)} |(x^* \circ m)(B)| \\ &= 4 \sup_{B \in \mathcal{B}(T)} ||(m(B)||_{p_A} < \infty. \end{aligned}$$

Thus u^*A is bounded in $M(T)$. Consequently, by Proposition 3, u^*A is relatively weakly compact in $M(T)$ and then by Proposition 2 we conclude that u is a weakly compact operator. Thus (v) holds.

To prove the converse, let ω be the set $\mathbb{N}$ endowed with the discrete topology. Then ω is a locally compact Hausdorff space. Let (x_n) be a sequence in X such that $\Sigma_1^\infty |x^*(x_n)| < \infty$ for each $x^* \in X^*$. For each $n \in \mathbb{N}$, let $u(\chi_{\{n\}}) = x_n$ and let u be extended linearly on the set S of all $\mathcal{P}(\mathbb{N})$-simple functions. By the hypothesis on (x_n), the set $\{uf : f \in S, ||f||_{\mathbb{N}} \leq 1\}$ is weakly bounded and hence τ-bounded. Then by Theorem 1.32 of Rudin [16], u is continuous. Since X is sequentially complete and S is norm dense in $C_o(\omega)$, u has a unique continuous linear extension to the whole of $C_o(\omega)$ and let us denote the extension too by u. Let m be the representing measure of u. By hypothesis, u is weakly compact and hence by Lemma 1 of Grothendieck [6] u^{**} has range in X and hence, by Proposition 4(v) we have $m(E) = u^{**}(\chi_E) \in X$ for all $E \subset \mathbb{N}$. Then by Proposition 4(i) and by the Orlicz-Pettis theorem we conclude that m is σ-additive in the toplogy τ of X and hence $\Sigma_1^\infty x_n = \Sigma_1^\infty u(\chi_{\{n\}}) = \Sigma_1^\infty u^{**}(\chi_{\{n\}}) = \Sigma_1^\infty m(\{n\}) = m(\mathbb{N}) \in X$. Thus the series $\Sigma_1^\infty x_n$ is unconditonally convergent in X. Now Theorem 4 of Tumarkin [19] implies that $c_o \not\subset X$.

Remark 2. For Banach spaces X containing no copy of c_o and compact Hausdorff spaces T, using [1] Pelczyński established in [15] the weak compactness of continuous linear maps $u : C(T) \to X$. Later, using the theory of Radon vector measures and the Grothendieck characterizations of weakly compact operators as given in Theorem 6 of [6], Thomas [18] extended this result to quasicomplete lcHs X and locally compact Hausdorff spaces T. Both the proofs use the technique of reduction to compact metrizable case. Moreover, Thomas [18] also characterized the quasicomplete lcHs containing no copy of c_o in a form equivalent to the above theorem in terms of bounded Radon vector measures. A direct proof for the first part of the above theorem, without employing the reduction technique, has been given in the proof of Theorem 13 of the recent paper [14] of one of the authors, but is highly technical and based on the results of [13] and those in Section 5 of [14]. The advantage of the present proof of the first part is that it is not only direct, but also is elementary and completely measure theoretic in contrast to the said proof of [14]. Moreover, it is also noted that the final arguments of the second part as given here are much simpler than those in the proof of Theorem 13 of [14].

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Mathematical Institute, Slovak Acdemy of Sciences, 49, Štefanikova, Bratislava, Slovakia.

Departamento de Matemáticas. Facultad de Ciencias. Universidad de Los Andes. Mérida 5101. Venezuela
e-mail address: panchapa@ciens.ula.ve