Universidad de los Andes
Facultad de Ciencias
Departamento de Matemática

Applications of a Theorem of Grothendieck to Vector Measures

T. V. Panchapagesan

Notas de Matemática
Serie: Pre-Print
No. 162

Mérida - Venezuela
1997

Applications of a Theorem of Grothendieck to Vector Measures

T. V. Panchapagesan *

June 9, 1997

Abstract

Let $\mathcal{R}$ be a ring of subsets of a nonempty set Ω and $\Sigma(\mathcal{R})$ the Banach space of uniform limits of sequences of $\mathcal{R}$-simple functions in Ω. Let X be a quasicomplete locally convex Hausdorff space (briefly, lcHs). Given a bounded X-valued vector measure m on $\mathcal{R}$, the concepts of m-integrability of functions in $\Sigma(\mathcal{R})$ and of representing measure of a continuous linear mapping $u : \Sigma(\mathcal{R}) \to X$ are introduced. Based on these concepts and a theorem of Grothendieck on the range of the biadjoint u^{**} of $u \in \mathcal{L}(\Sigma(\mathcal{R}), X)$, it is shown that such a mapping u is weakly compact if and only if its representing measure is strongly additive (which is the quasicomplete lcHs version of Theorem VI.1.1 of [3]). The result subsumes the range theorems of Tweddle [12] and Kluvánek [9]. Also is deduced the theorem on extension in [10]. The methods of proof for all these results in vector measures is more natural than the known ones.

Dedicated to the memory of Professor I. Kluvánek

Let X be a quasicomplete locally convex Hausdorff space (briefly, a quasicomplete lcHs). Using James' criterion for weak compactness of a set, Tweddle [12] showed that the closed convex hull of the range of a σ-additive X-valued vector measure defined on a σ-ring of sets is weakly compact. His proof is first given for the case of a σ-algebra, and then, by appealing to the Eberlein theorem ([6, Theorem 8.12.7]), is extended to the case of σ-rings. This result subsumes the Bartle-Dunford-Schwartz theorem [1] on the range of a σ-additive Banach space-valued vector measure defined on a σ-algebra of sets.

Later, an alternative proof of Tweddle's theorem was given by Kluvánek in [9]. His proof is based on the theory of σ-additive X-valued closed vector measures developed in the first part of [9]. He showed that the closed balanced convex hull of the range of a σ-additive X-valued vector measure defined on a σ-algebra of sets is weakly compact, which, as observed in [10], also extends to the case of σ-rings.

The range theorem of Tweddle [12] or of Kluvánek [9], plays a key role in the proof of the theorem on extension given on pp.178-179 of [10], which gives several necessary and sufficient

*Supported by the C.D.C.H.T. project C-586 of the Universidad de los Andes, Mérida, Venezuela, and by the international cooperation project between CONICIT-Venezuela and CNR-Italy. 1991 *Mathematics Subject Classification.* Primary 28B05, 46G10. *Key words and phrases.* σ-additive vector measures, strongly additive vector measures, weakly compact operator.

conditions for an X-valued weakly σ-additive vector measure defined on a ring of sets $\mathcal{R}$ to admit an X-valued σ-additive extension to the σ-ring generated by $\mathcal{R}$.

For a ring of sets $\mathcal{R}$ let $\Sigma(\mathcal{R})$ be the Banach space of all uniform limits of sequences of $\mathcal{R}$-simple functions. Then appealing to the range theorem of Tweddle [12] and the theorem on extension in [10] and using an argument similar to that in the proofs of Theorem I.5.2 and Corollary I.5.3 of [3] regarding the involvement of the Stone representation space of $\mathcal{R}$, one can show that an X-valued vector measure defined on $\mathcal{R}$ is strongly additive if and only if its range is relatively weakly compact. Then the proof of Theorem VI.1.1 of [3] can suitably be modified to show that a continuous linear mapping $T : \Sigma(\mathcal{R}) \to X$ is weakly compact if and only if its representing measure is strongly additive.

The aim of the present note is to give a direct proof of the quasicomplete lcHs-version of Theorem VI.1.1 of [3] for the case of a ring of sets, without appealing to the range theorem of Tweddle [12] and the theorem on extension in [10]. For this we shall use the equivalence of (1) and (3) of Corollary 9.3.2 of [6], which is essentially due to Lemmas 1 and 2 of Grothendieck [7]. Then the range theorems of Tweddle [12] and Kluvánek [9] are immediate. By invoking the equivalence of (1) and (2) of Corollary 9.3.2 of [6] and our principal result, we also deduce the theorem on extension in [10], by providing a new proof to show (x) $\Rightarrow$ (vii) $\Rightarrow$ (i) (see Corollary 2). The reader can observe that our method of proof for all these principal results in vector measures is very natural, elegant and powerful, in contrast to the earlier proofs.

Finally, we also include a generalization of the second part of Corollary VI.1.2 of [3] to quasicomplete lcHs.

For the convenience of the reader, we shall recall some definitions and results from the theory of vector measures and give some lemmas extending the results known for algebras or σ-algebras of sets to rings or σ-rings of sets, respectively.

In the sequel, X denotes a lcHs (over $\mathbb{C}$) with topology τ and $\mathcal{R}$ and $\mathcal{S}$ denote respectively a ring and a σ-ring of subsets of a non empty set Ω. $ca(\mathcal{S})$ is the Banach space of all σ-additive complex measures μ on $\mathcal{S}$ with $||\mu|| = \sup_{E\in\mathcal{S}} var(\mu, E)$ and $ba(\mathcal{R})$ is the Banach space of all complex-valued bounded additive set functions ν on $\mathcal{R}$ with $||\nu|| = \sup_{E\in\mathcal{R}} var(\nu, E)$. Let $\sigma(\mathcal{R})$ be the σ-ring generated by $\mathcal{R}$. Let $ba^+(\mathcal{R}) = \{\nu \in ba(\mathcal{R}) : \nu \geq 0\}$.

A *vector measure* is an additive set function defined on a ring of sets with values in a lcHs. An X-valued vector measure m on $\mathcal{R}$ is said to be ***strongly additive*** (resp. ***exhausting***) on $\mathcal{R}$ if $\sum_{n=1}^{\infty} m(E_n)$ is τ-convergent (resp. $\lim_n m(E_n) = 0$) for each disjoint sequence (E_n) in $\mathcal{R}$. A family $\{m_i : i \in I\}$ of exhausting X-valued vector measures is said to be ***uniformly exhausting*** if, for each disjoint sequence (E_n) in $\mathcal{R}$, $\lim_n m_i(E_n) = 0$ uniformly in $i \in I$. A family $\{m_i : i \in I\}$ of X-valued strongly additive vector measures on $\mathcal{R}$ is said to be ***uniformly strongly additive*** on $\mathcal{R}$, if, given $\varepsilon > 0$, a τ-continuous seminorm p on X and a disjoint sequence (E_n) in $\mathcal{R}$, there exists n_o such that $\sup_{i\in I} p(\sum_{k=n}^{\infty} m_i(E_k)) < \varepsilon$ for all $n \geq n_o$. A family $(m_\alpha)_{\alpha\in I}$ of X-valued σ-additive vector measures on $\mathcal{R}$ is said to be ***uniformly σ-additive*** on $\mathcal{R}$, if, given $\varepsilon > 0$, a decreasing sequence $E_n \searrow \emptyset$ in $\mathcal{R}$ and a τ-continuous seminorm p on X, there exists n_0 such that $p(m_\alpha(E_n)) < \varepsilon$ for all $n \geq n_0$ and for all $\alpha \in I$. A subset A of $ba(\mathcal{R})$ is said to be ***uniformly μ-continuous*** for some $\mu \in ba^+(\mathcal{R})$, if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup_{\nu\in A} |\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

The following result is well known when $\mathcal{S}$ is a σ-algebra (see, for example, Theorem IV.9.1 of [5]).

Lemma 1. *Let $\mathcal{S}$ be a σ-ring of subsets of a non empty set Ω. A subset A of $ca(\mathcal{S})$ is relatively weakly compact if and only if A is bounded and uniformly σ-additive.*

Proof. By the Eberlein-Šmulian theorem and by the fact that, for each sequence $(\mu_n) \subset ca(\mathcal{S})$, there exists $E \in \mathcal{S}$ such that $var(\mu_n, F) = 0$ for each $F \in \mathcal{S}$ with $F \cap E = \emptyset$ and for each n, we can replace the space $ca(S, \Sigma, \lambda)$ in the proof of Theorem IV.9.1 of [5] by the space $ca(\Omega \cap E, \mathcal{S} \cap E, \lambda)$ of all λ-continuous set functions in $ca(\Omega \cap E, \mathcal{S} \cap E)$. Since $\mathcal{S} \cap E$ is a σ-algebra, the rest of the argument in the proof of Theorem IV.9.1 of [5] holds here to show that the conditions are necessary and sufficient.

Since the Carathéodory-Hahn extension theorem for σ-additive positive measures and Proposition I.1.17 of [3] hold for a ring of sets $\mathcal{R}$, Lemma I.5.1 of [3] holds also for $\mathcal{R}$ and hence we have the following result.

Lemma 2. *Let $\{\mu_i : i \in I\}$ be a family of σ-additive complex measures on $\sigma(\mathcal{R})$. Then $\{\mu_i : i \in I\}$ is uniformly σ-additive on $\sigma(\mathcal{R})$ if and only if the family of the restrictions $\{\mu_i|_{\mathcal{R}} : i \in I\}$ is uniformly strongly additive on $\mathcal{R}$.*

Since a quasicomplete lcHs is sequentially complete the following result is obvious. The reader can also refer to Theorem 4.3 of [4].

Lemma 3. *Let X be a quasicomplete lcHs. Then an X-valued vector measure m (resp. a family $\mathcal{F} = \{m_i : i \in I\}$ of X-valued strongly additive vector measures) on $\mathcal{R}$ is strongly additive (resp. uniformly strongly additive) if and only if m is exhausting (resp. if and only if $\mathcal{F}$ is uniformly exhausting).*

Let S be the Stone representation space of $\mathcal{R}$. Then there exists a ring isomorphism Φ from $\mathcal{R}$ onto the ring $\hat{\mathcal{R}}$ of all compact-open subsets of S. For each $\mu \in ba(\mathcal{R})$, let $\hat{\mu}(\Phi(E)) = \mu(E)$ for each $E \in \mathcal{R}$. Then, for $\mu \in ba(\mathcal{R})$, clearly $\hat{\mu}$ is σ-additive on $\hat{\mathcal{R}}$ and hence has a unique σ-additive extension $\tilde{\mu}$ on $\sigma(\hat{\mathcal{R}})$. Moreover, $||\mu|| = ||\hat{\mu}|| = ||\tilde{\mu}||$ for $\mu \in ba(\mathcal{R})$. These observations and Lemmas 1, 2 and 3 can be used to extend Theorem I.4.6 of Bombal [2], given for algebras of sets, to rings of sets. Thus we have the following result.

Lemma 4. *For a bounded subset A of $ba(\mathcal{R})$ the following statements are equivalent:*

(i) A is relatively weakly compact.

(ii) A is uniformly strongly additive on $\mathcal{R}$.

(iii) A is uniformly exhausting on $\mathcal{R}$.

(iv) There exists $\mu \in ba^+(\mathcal{R})$ such that A is uniformly μ-continuous.

The following result is well known when $\mathcal{S}$ is a σ-algebra (see, for example, Theorem IV.9.1 of [5]).

Lemma 1. *Let $\mathcal{S}$ be a σ-ring of subsets of a non empty set Ω. A subset A of $ca(\mathcal{S})$ is relatively weakly compact if and only if A is bounded and uniformly σ-additive.*

Proof. By the Eberlein-$\check{S}$mulian theorem and by the fact that, for each sequence $(\mu_n) \subset ca(\mathcal{S})$, there exists $E \in \mathcal{S}$ such that $var(\mu_n, F) = 0$ for each $F \in \mathcal{S}$ with $F \cap E = \emptyset$ and for each n, we can replace the space $ca(S, \Sigma, \lambda)$ in the proof of Theorem IV.9.1 of [5] by the space $ca(\Omega \cap E, \mathcal{S} \cap E, \lambda)$ of all λ-continuous set functions in $ca(\Omega \cap E, \mathcal{S} \cap E)$. Since $\mathcal{S} \cap E$ is a σ-algebra, the rest of the argument in the proof of Theorem IV.9.1 of [5] holds here to show that the conditions are necessary and sufficient.

Since the Carathéodory-Hahn extension theorem for σ-additive positive measures and Proposition I.1.17 of [3] hold for a ring of sets $\mathcal{R}$, Lemma I.5.1 of [3] holds also for $\mathcal{R}$ and hence we have the following result.

Lemma 2. *Let $\{\mu_i : i \in I\}$ be a family of σ-additive complex measures on $\sigma(\mathcal{R})$. Then $\{\mu_i : i \in I\}$ is uniformly σ-additive on $\sigma(\mathcal{R})$ if and only if the family of the restrictions $\{\mu_i|_{\mathcal{R}} : i \in I\}$ is uniformly strongly additive on $\mathcal{R}$.*

Since a quasicomplete lcHs is sequentially complete the following result is obvious. The reader can also refer to Theorem 4.3 of [4].

Lemma 3. *Let X be a quasicomplete lcHs. Then an X-valued vector measure m (resp. a family $\mathcal{F} = \{m_i : i \in I\}$ of X-valued strongly additive vector measures) on $\mathcal{R}$ is strongly additive (resp. uniformly strongly additive) if and only if m is exhausting (resp. if and only if $\mathcal{F}$ is uniformly exhausting).*

Let S be the Stone representation space of $\mathcal{R}$. Then there exists a ring isomorphism Φ from $\mathcal{R}$ onto the ring $\hat{\mathcal{R}}$ of all compact-open subsets of S. For each $\mu \in ba(\mathcal{R})$, let $\hat{\mu}(\Phi(E)) = \mu(E)$ for each $E \in \mathcal{R}$. Then, for $\mu \in ba(\mathcal{R})$, clearly $\hat{\mu}$ is σ-additive on $\hat{\mathcal{R}}$ and hence has a unique σ-additive extension $\tilde{\mu}$ on $\sigma(\hat{\mathcal{R}})$. Moreover, $||\mu|| = ||\hat{\mu}|| = ||\tilde{\mu}||$ for $\mu \in ba(\mathcal{R})$. These observations and Lemmas 1, 2 and 3 can be used to extend Theorem I.4.6 of Bombal [2], given for algebras of sets, to rings of sets. Thus we have the following result.

Lemma 4. *For a bounded subset A of $ba(\mathcal{R})$ the following statements are equivalent:*

(i) A is relatively weakly compact.

(ii) A is uniformly strongly additive on $\mathcal{R}$.

(iii) A is uniformly exhausting on $\mathcal{R}$.

(iv) There exists $\mu \in ba^+(\mathcal{R})$ such that A is uniformly μ-continuous.

To give the notion of the integral of bounded scalar functions with respect to a bounded vector measure defined on $\mathcal{R}$, we introduce the following additional notation and terminology.

For each τ-continuous seminorm p on X, let $p(x) = ||x||_p$, $x \in X$, and let $X_p = (X, ||\cdot||_p)$ be the associated seminormed space. The completion of the quotient normed space $X/p^{-1}(0)$ is denoted by $\tilde{X}_p$. Let $\Pi_p : X_p \to X/p^{-1}(0) \subset \tilde{X}_p$ be the canonical quotient map.

Given a vector measure $m : \mathcal{R} \to X$, for each τ-continuous seminorm p on X let $m_p : \mathcal{R} \to \tilde{X}_p$ be given by $m_p(E) = \Pi_p \circ m(E)$ for $E \in \mathcal{R}$. Then m_p is a Banach space-valued vector measure on $\mathcal{R}$.We define the p-semivariation $||m||_p$ of m by

$$||m||_p(E) = ||m_p||(E) \text{ for } E \in \mathcal{R}$$

and

$$||m||_p(\Omega) = ||m_p||(\Omega) = \sup_{E \in \mathcal{R}} ||m_p||(E)$$

where $||m_p||$ is the semivariation of the vector measure $m_p : \mathcal{R} \to \tilde{X}_p$. Obviously, the range of m is bounded in X if and only if $||m||_p(\Omega) < \infty$ for each τ-continuous seminorm p on X. In that case, the vector measure m is said to be *bounded.*

For an $\mathcal{R}$-simple complex function $s = \sum_{i=1}^{r} \lambda_i \chi_{E_i}$, $\lambda_i \neq 0$, $E_i \in \mathcal{S}, E_i \cap E_j = \emptyset$ for $i \neq j, i, j = 1, 2, ..., r$, and for an X-valued bounded vector measure m we define

$$\int_\Omega s dm = \sum_{i=1}^{r} \lambda_i m(E_i).$$

It is easy to verify that $\int_\Omega sdm$ is well defined. If $S(\mathcal{R})$ denotes the normed space of all $\mathcal{R}$-simple complex functions with pointwise addition and scalar multiplication and with norm the supremum norm $||\cdot||_\Omega$, then the map $u : S(\mathcal{R}) \to X$ given by $us = \int_\Omega sdm$ is linear and continuous.

Lemma 5. *Let (s_n) and (s_n') be sequences of $\mathcal{R}$-simple complex functions, converging uniformly to a function f in Ω. Suppose $m : \mathcal{R} \to X$ is a bounded vector measure, where X is a quasicomplete lcHs. Then:*

(i) $(\int_\Omega s_n dm)$ is Cauchy in X .

(ii) $\lim_n \int_\Omega s_n dm = \lim_n \int_\Omega s_n' dm \in X$.

Proof. Let p be a τ-continuous seminorm on X and let $\varepsilon > 0$. Since $||m_p||(\Omega) < \infty$, we can choose n_o such that $||s_n - s_\ell||_\Omega < \frac{\varepsilon}{||m_p||(\Omega)}$ for $n, \ell \geq n_o$. Then

$$|| \int_\Omega s_n dm - \int_\Omega s_\ell dm||_p = || \int_\Omega (s_n - s_\ell) dm_p||_p \leq ||s_n - s_\ell||_\Omega ||m_p||(\Omega) < \varepsilon$$

for $n, \ell \geq n_o$. Hence (i) holds.

As X is sequentially complete, by (i) there exist vectors x, x' in X such that $\lim_n \int_\Omega s_n dm = x$ and $\lim_n \int_\Omega s'_n dm = x'$. Then

$$||x - x'||_p \leq || \int_\Omega s_n dm - x||_p + || \int_\Omega s_n dm - \int_\Omega s'_n dm||_p + || \int_\Omega s'_n dm - x'||_p \to 0$$

as $n \to \infty$, since

$$|| \int_\Omega s_n dm - \int_\Omega s'_n dm||_p \leq ||s_n - s'_n||_\Omega ||m_p||(\Omega) \to 0$$

as $n \to \infty$. As the τ-continuous seminorm p is arbitrary, it follows that $x = x'$. Hence (ii) holds.

Definition 1. Let $\Sigma(\mathcal{R})$ be the Banach space of all bounded complex functions which are uniform limits of sequences of $\mathcal{R}$-simple functions, with pointwise addition and scalar multiplication and with norm the supremum norm $||\cdot||_\Omega$. Given $f \in \Sigma(\mathcal{R})$, let the sequence (s_n) of $\mathcal{R}$-simple complex functions converge uniformly to f in Ω. If $m : \mathcal{S} \to X$ is additive and bounded, and if X is a quasicomplete lcHs, then we say that f is *m-integrable* and define

$$\int_\Omega f dm = \lim_n \int_\Omega s_n dm.$$

In the light of Lemma 5, $\int_\Omega f dm$ is well defined for $f \in \Sigma(\mathcal{R})$.

The following result is immediate from Definition 1.

Lemma 6. *Let X be a quasicomplete lcHs. If f and g belong to $\Sigma(\mathcal{R})$, α, β are scalars and $m : \mathcal{R} \to X$ is additive and bounded, then the following hold:*

(i) $\int_\Omega (\alpha f + \beta g) dm = \alpha \int_\Omega f dm + \beta \int_\Omega g dm$.

(ii) $|| \int_\Omega f dm||_p \leq ||f||_\Omega ||m_p||(\Omega) = ||f||_\Omega ||m||_p(\Omega)$ *for each τ-continuous seminorm p on X.*

(iii) For each $x^* \in X^*$, $x^*(\int_\Omega f dm) = \int_\Omega f d(x^* m)$.

Consequently, the map $u : \Sigma(\mathcal{R}) \to X$ given by $uf = \int_\Omega f dm$ is continuous and linear.

The following result can easily be proved by an argument analogous to that on pp.5-6 of [3].

Lemma 7. *Let X be a quasicomplete lcHs and let $ba(\mathcal{R}, X)$ be the vector space of all X-valued bounded vector measures on $\mathcal{R}$. Then there exists a vector space isomorphism Φ from $ba(\mathcal{R}, X)$ onto the vector space of all continuous linear maps $\mathcal{L}(\Sigma(\mathcal{R}), X)$ such that, for each τ-continuous seminorm p on X and $m \in ba(\mathcal{R}, X)$,*

$$\Phi(m)(f) = \int_\Omega f dm, \quad f \in \Sigma(\mathcal{R})$$

and

$$||\Phi(m)||_p = \sup_{f \in \Sigma(\mathcal{R}), ||f||_\Omega \leq 1} ||\Phi(m) f||_p = ||m||_p(\Omega).$$

In particular, the dual of $\Sigma(\mathcal{R})$ *is* $ba(\mathcal{R})$.

Definition 2. For each $u \in \mathcal{L}(\Sigma(\mathcal{R}), X)$, the unique bounded vector measure m with $uf = \int_\Omega f dm$ for $f \in \Sigma(\mathcal{R})$ is called the *representing measure* of the continuous linear map u.

The following result is well known (see, for example, Corollary 4.12 of [4]). However, we shall give a direct proof.

Lemma 8. *Let m be an X-valued strongly additive vector measure on $\mathcal{R}$. Then m has a bounded range.*

Proof. If m is not a bounded vector measure, then there exists a τ-continuous seminorm p such that $||m||_p(\Omega) = \infty$. Then there exists $E_1 \in \mathcal{R}$ such that $||m(E_1)||_p > 1$. Since $\mathcal{R} \cap E_1$ is an algebra of subsets of E_1, by Corollary I.1.19 of [3], it follows that $\sup\{||m(F)||_p : F \in \mathcal{R}, F \cap E_1 = \emptyset\} = \infty$. Then there exists $E_2 \in \mathcal{R}$ with $E_2 \cap E_1 = \emptyset$ such that $||m(E_2)||_p > 2$. Thus proceeding step by step, and applying Corollary I.1.19 of [3], we can choose a disjoint sequence (E_n) in $\mathcal{R}$ such that $||m(E_n)||_p > n$ for each n. On the other hand, as m is strongly additive on $\mathcal{R}$, $||m(E_n)||_p \to 0$ when $n \to \infty$. This contradiction shows that m is bounded.

Theorem 1. *Let $\mathcal{R}$ be a ring of subsets of a non empty set Ω and let X be a quasicomplete lcHs. Then the following assertions hold:*

(i) If $u : \Sigma(\mathcal{R}) \to X$ is a continuous linear map, then u is weakly compact if and only if its representing measure is strongly additive on $\mathcal{R}$.

(ii) If m is an X-valued strongly additive vector measure on $\mathcal{R}$ and if $m(\mathcal{R})$ denotes the range of m, then the closed balanced convex hull of $m(\mathcal{R})$ is weakly compact and is contained in the τ-closure of the set $H = \{\int_\Omega f dm : f \in \Sigma(\mathcal{R}), ||f||_\Omega \leq 1\}$. Moreover, the closed convex hull of $m(\mathcal{R})$ is the same as the τ-closure of the set $\{uf : f \in \Gamma\}$, where $\Gamma = \{f \in \Sigma(\mathcal{R}) : 0 \leq f(t) \leq 1, t \in \Omega\}$.

(iii) If $m : \mathcal{R} \to X$ is additive, then m is strongly additive if and only if $m(\mathcal{R})$ is relatively weakly compact.

Proof. (i) Let m be the representing measure of u. Then m is bounded by Lemma 7. Let E be an equicontinuous set in X^*. Let $||x||_{p_E} = \sup_{x^* \in E} |x^*(x)|$, $x \in X$, and let $G_E = \{x^* \circ m : x^* \in E\}$. Then clearly the set G_E is bounded in $ba(\mathcal{R})$ if and only if

$$\sup_{A \in \mathcal{R}, x^* \in E} |(x^* \circ m)(A)| = \sup_{A \in \mathcal{R}} ||m(A)||_{p_E} < \infty. \quad (1)$$

Since $u : \Sigma(\mathcal{R}) \to X$ is linear and continuous, its adjoint $u^* : X^* \to (\Sigma(\mathcal{R}))^* = ba(\mathcal{R})$ is a well defined linear map. Then by Lemma 6 we have

$$< u^*x^*, f > = < x^*, uf > = < x^*, \int_\Omega f dm > = \int_\Omega f d(x^* \circ m) = < f, x^* \circ m >$$

for all $f \in \Sigma(\mathcal{R})$ and $x^* \in X^*$. Then, by the Hahn-Banach theorem, $u^*x^* = x^* \circ m$. Thus $G_E = u^*(E)$.

Suppose m is strongly additive on $\mathcal{R}$. As the topology τ is the same as the topology of uniform convergence in equicontinuous subsets of X^*, it follows that, for a given $\varepsilon > 0$ and a disjoint sequence (A_n) in $\mathcal{R}$, there exists n_o such that $|| \sum_{k=n}^{\infty} m(A_k)||_{p_E} < \varepsilon$ for all $n \geq n_o$. In other words, $\sup_{x^* \in E} | \sum_{k=n}^{\infty} (x^* \circ m)(A_k)| < \varepsilon$ for all $n \geq n_o$. Thus G_E is uniformly strongly additive on $\mathcal{R}$. Since m has bounded range in X, (1) implies that G_E is bounded in $ba(\mathcal{R})$. Consequently, by Lemma 4, the set $u^*(E) = G_E$ is relatively weakly compact. Since E is an arbitrary equicontinuous set in X^*, by Corollary 9.3.2 of [6] (which is essentially due to Lemmas 1 and 2 of [7]), we conclude that u is weakly compact.

Conversely, let u be weakly compact. Then for each equicontinuous subset E of X^*, by Corollary 9.3.2 of [6], u^*E is relatively weakly compact in $ba(\mathcal{R})$. Then $u^*(E) = G_E$ is bounded, and moreover, by Lemma 4, the set G_E is uniformly exhausting on $\mathcal{R}$. In other words, given a disjoint sequence (A_n) in $\mathcal{R}$, $\lim_n (x^* \circ m)(A_n) = 0$ uniformly in $x^* \in E$. Thus, $\lim_n ||m(A_n)||_{p_E} = 0$. As E is an arbitrary equicontinuous set in X^*, it follows that $\lim_n m(A_n) = 0$ and hence m is exhausting on $\mathcal{R}$. Then by Lemma 3, m is strongly additive on $\mathcal{R}$.

(ii) Let the vector measure $m : \mathcal{R} \to X$ be strongly additive. Then by Lemma 8, m is bounded and hence is the representing measure of the continuous linear map $u : \Sigma(\mathcal{R}) \to X$ given by $u(f) = \int_\Omega f dm$ for $f \in \Sigma(\mathcal{R})$. Therefore, by (i), u is weakly compact and consequently, $H = \{uf : ||f||_\Omega \leq 1\}$ is a relatively weakly compact balanced convex subset of X. Since H contains the range of m, it follows by the Hahn-Banach theorem that the closed balanced convex hull of $m(\mathcal{R})$ is weakly compact and is contained in the τ-closure of H.

Moreover, by considering Abel's partial sums as in the proof of Theorem VI.1.1 of [3], one can show that uf belongs to the convex hull of $m(\mathcal{R})$ for each $\mathcal{R}$-simple scalar function $f \in \Gamma$. Then it follows that the τ-closure of $\{uf : f \in \Gamma\}$ coincides with the τ-closure of the convex hull of $m(\mathcal{R})$.

(iii) The condition is necessary by (ii). Conversely, let $m(\mathcal{R})$ be relatively weakly compact. Then m is bounded and hence is the representing measure of u where $uf = \int_\Omega f dm$, $f \in \Sigma(\mathcal{R})$. Then by considering Abel's partial sums as in the proof of Theorem VI.1.1 of [3] and appealing to the Krein theorem ([6, Theorem 8.3.1]), it follows that $\{uf : ||f||_\Omega \leq 1\}$ is relatively weakly compact. Hence u is weakly compact and then (i) implies that m is strongly additive.

This completes the proof of the theorem.

Since a σ-additive vector measure on a σ-ring is strongly additive, we have the following corollary which gives the results of Tweddle [12] and Kluvánek [9] on the range of σ-additive vector measures.

Corollary 1. *Let X be a quasicomplete lcHs. If m is a σ-additive X-valued vector measure on a σ-ring S, then the range $m(S)$ and its balanced convex hull are relatively weakly compact.*

As a corollary of Theorem 1, we also deduce the following theorem on extension in [10]. We provide a new operator theoretic proof to show (vii) $\Rightarrow$ (i) , by invoking Lemma 1 of [7]. Though the proof of (x) $\Rightarrow$ (viii) is new, our proof of (i) $\Rightarrow$ (ix) is based on Theorem I.2.4 of [3] whose proof is also valid for σ-rings.

Corollary 2 (Theroem on extension in [10]). *Let X be a quasicomplete lcHs and let m be an X-valued weakly σ-additive vector measure on the ring $\mathcal{R}$.. Then the following statements are equivalent:*

(i) There exists a σ-additive vector measure $\hat{m} : \sigma(\mathcal{R}) \to X$ such that $\hat{m}|_{\mathcal{R}} = m$.

(ii) There is a weakly compact set $Y \subset X$ such that $m(\mathcal{R}) \subset Y$.

(iii) m is bounded and there is a weakly sequentially complete set $Y \subset X$ such that $m(\mathcal{R}) \subset Y$.

(iv) If $E_n \nearrow$ in $\mathcal{R}$, then there exists an element $x \in X$ such that $m(E_n) \to x$ weakly.

(v) If (E_n) is a disjoint sequence in $\mathcal{R}$, then there exists an element $x \in X$ such that $\sum_{n=1}^{\infty} m(E_n)$ converges weakly to x.

(vi) If $E_n \nearrow$ in $\mathcal{R}$, then $\lim_n m(E_n) \in X$ exists.

(vii) m is strongly additive on $\mathcal{R}$.

(viii) m is exhausting on $\mathcal{R}$.

(ix) For every continuous seminorm p on X, there is a bounded non negative σ-additive measure μ_p on $\mathcal{R}$ such that $\mu_p(E) \to 0$, $E \in \mathcal{R}$, implies $||m(E)||_p \to 0$.

(x) For every continuous seminorm p on X, there is a bounded non negative finitely additive set function μ_p on $\mathcal{R}$ such that $\mu_p(E) \to 0$, $E \in \mathcal{R}$, implies $||m(E)||_p \to 0$.

Proof. By the Orlicz-Pettis theorem, m is σ-additive on $\mathcal{R}$. It suffices to show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (vii) $\Rightarrow$ (i) and (x) $\Rightarrow$ (viii). Observing that the proofs of Theorems I.2.1 and I.2.4 of [3] also hold for σ-rings, we have (i) $\Rightarrow$ (ix). (vii) is equivalent to (viii) by Lemma 3. The rest of the equivalences are obvious or are based on the Orlicz-Pettis theorem.

(i) $\Rightarrow$ (ii) Since a σ-additive vector measure on $\sigma(\mathcal{R})$ is also strongly additive, by Theorem 1 $m(\mathcal{R})$ is relatively weakly compact and hence (i) implies (ii).

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (vii) Let (E_n) be a disjoint sequence in $\mathcal{R}$. By hypothesis, $m(\mathcal{R})$ is bounded. Then, for each $x^* \in X^*$, $x^* \circ m$ is a bounded σ-additive complex measure on $\mathcal{R}$ and hence is strongly additive. Thus $\sum_{n=1}^{\infty} |(x^* \circ m)(E_n)| < \infty$. On the other hand, Y is weakly sequentially complete and hence there exists $x \in Y$ such that $\sum_1^{\infty} m(E_n)$ converges weakly to x. Then by the Orlicz-Pettis theorem

(vii) holds.

(vii) $\Rightarrow$ (i) Let Z be the set of all bounded σ-additive complex measures on $\mathcal{R}$. Then Z is a subspace of $ba(\mathcal{R})$. Let $W = ba(\mathcal{R})$. It is well known that each $\mu \in Z$ has a unique σ-additive complex-valued extension $\mu^\wedge$ to $\sigma(\mathcal{R})$ and $||\mu|| = ||\mu^\wedge||$.

AFFIRMATION. Let μ, μ_1 and $\mu_2 \in Z$ and $\alpha \in \boldsymbol{C}$. Then the following hold.

(a) $(\mu_1 + \mu_2)^\wedge = \mu_1{}^\wedge + \mu_2{}^\wedge$.

(b) $(\alpha\mu)^\wedge = \alpha\mu^\wedge$.

In fact, let $|\mu_i| = var(\mu_i, \mathcal{R})$ and $|\mu_i{}^\wedge| = var(\mu_i{}^\wedge, \sigma(\mathcal{R}))$, $i = 1, 2$. Given $A \in \sigma(\mathcal{R})$ and $n \in \boldsymbol{N}$, by ej.8, § 13 of [8] there exists $B_n \in \mathcal{R}$ such that $(|\mu_1|^\wedge + |\mu_2|^\wedge)(A\Delta B_n) < \frac{1}{n}$. Then

$$\begin{aligned} |(\mu_1+\mu_2)^\wedge(A) - (\mu_1+\mu_2)(B_n)| &= |(\mu_1+\mu_2)^\wedge(A) - (\mu_1+\mu_2)^\wedge(B_n)| \\ &\le |(\mu_1+\mu_2)^\wedge(A\backslash B_n)| + |(\mu_1+\mu_2)^\wedge(B_n\backslash A)| \\ &\le |(\mu_1+\mu_2)^\wedge|(A\Delta B_n) \\ &\le (|\mu_1|^\wedge + |\mu_2|^\wedge)(A\Delta B_n) < \frac{1}{n}. \end{aligned}$$

Similarly,

$$|\mu_i{}^\wedge(A) - \mu_i(B_n)| \le |\mu_i{}^\wedge(A\backslash B_n)| + |\mu_i{}^\wedge(B_n\backslash A)| \le |\mu_i{}^\wedge|(A\Delta B_n) \le |\mu_i|^\wedge(A\Delta B_n) < \frac{1}{n}$$

for $i = 1, 2$. Hence $\mu_i{}^\wedge(A) = \lim_n \mu_i(B_n)$ for $i = 1, 2$ and consequently,

$$(\mu_1 + \mu_2)^\wedge(A) = \mu_1{}^\wedge(A) + \mu_2{}^\wedge(A)$$

for $A \in \sigma(\mathcal{R})$. Thus (a) holds. Similarly, one can prove (b).

For each $A \in \sigma(\mathcal{R})$, let $z_A(\mu) = \mu^\wedge(A)$, $\mu \in Z$. By Affirmation we have

$$z_A(\alpha\mu_1 + \beta\mu_2) = (\alpha\mu_1 + \beta\mu_2)^\wedge(A) = \alpha\mu_1{}^\wedge(A) + \beta\mu_2{}^\wedge(A) = \alpha z_A(\mu_1) + \beta z_A(\mu_2)$$

and

$$|z_A(\mu)| = |\mu^\wedge(A)| \le ||\mu^\wedge||||\chi_A||_\Omega = ||\mu||||\chi_A||_\Omega$$

for $\mu, \mu_1, \mu_2 \in Z$ and $\alpha, \beta \in \boldsymbol{C}$. Hence $z_A \in Z^*$. As Z is a subspace of W, by the Hahn-Banach theorem there exists a $w_A \in W^*$ such that $||z_A|| = ||w_A||$ and $w_A|_Z = z_A$ for each $A \in \sigma(\mathcal{R})$.

By hypothesis and Lemma 8, m is bounded and weakly σ-additive on $\mathcal{R}$ and hence, for each $x^* \in X^*$, there exists a unique σ-additive extension $(x^* \circ m)^\wedge$ of $(x^* \circ m)$ to $\sigma(\mathcal{R})$. Let $uf = \int_\Omega f dm$, for $f \in \Sigma(\mathcal{R})$. By Theorem 1 and (vii), $u : \Sigma(\mathcal{R}) \to X$ is weakly compact. Hence by the equivalence of (1) and (2) of Corollary 9.3.2 of [6] (which is due to Lemma 1 of Grothendieck [7]) $u^{**} : W^* \to X^{**}$ has range in X, where X^{**} is the dual of $(X^*, \beta(X^*, X))$. Thus, for each $A \in \sigma(\mathcal{R})$, let $u^{**}(w_A) = m_1(A) \in X$. Then as shown in the proof of Theorem 1(i), we have $u^* x^* = x^* \circ m$ and

$$x^* m_1(A) = < u^{**}(w_A), x^* > = < w_A, x^* \circ m > = < z_A, x^* \circ m > = (x^* \circ m)^\wedge(A) \quad (2)$$

for $x^* \in X^*$ and for $A \in \sigma(\mathcal{R})$. Now, let $\{A_n\}_1^\infty$ be a disjoint sequence in $\sigma(\mathcal{R})$ with $A = \bigcup_1^\infty A_n$. Then by (2) we have

$$x^* m_1(A) = (x^* \circ m)^\wedge(A) = \sum_1^\infty (x^* \circ m)^\wedge(A_n) = \sum_1^\infty x^* m_1(A_n)$$

for each $x^* \in X^*$. Thus by the Orlicz-Pettis theorem we conclude that

$$m_1(A) = \sum_1^\infty m_1(A_n)$$

and hence m_1 is σ-additive on $\sigma(\mathcal{R})$. Moreover, for $A \in \mathcal{R}$, we have

$$x^* m_1(A) = (x^* \circ m)^\wedge(A) = (x^* \circ m)(A) = x^* m(A)$$

for all $x^* \in X^*$. Then, by the Hahn-Banach theorem, $m_1|_{\mathcal{R}} = m$. Thus (i) holds.

(x) $\Rightarrow$ (viii) First we observe that for each continuous seminorm p on X, μ_p is exhausting . Then $\lim_n \mu_p(A_n) = 0$ for any disjoint sequence (A_n) in $\mathcal{R}$ and hence $\lim_n ||m(A_n)||_p = 0$. Since p is arbitrary, it follows that $\lim_n m(A_n) = 0$ and hence the vector measure m is exhausting on $\mathcal{R}$. Thus (viii) holds.

Remark 1. If X is a Banach space, then Theorem 1 and Corollaries 1 and 2 can be proved by appealing to Theorems VI.4.2 and VI.4.8 of [5] instead of Corollary 9.3.2 of [6].

Remark 2. Let X be a quasicomplete lcHs and m an X-valued σ-additive vector measure defined on a σ-ring $\mathcal{S}$. Then in view of the theorem on weak compactness on p.184 of [10] and Theorem 1 above, m is closed in the sense of [10] if and only if $\{uf : f \in \Sigma(\mathcal{S}), 0 \le f(t) \le 1, t \in \Omega\}$ is closed in X, where $uf = \int_\Omega f dm$.

The following corollary generalizes the second part of Corollary VI.1.2 of [3] to quasicomplete lcHs.

Corollary 3. *Let X be a quasicomplete lcHs. If X contains no copy of c_o and $\mathcal{R}$ is an arbitrary ring of sets, then every continuous linear operator $u : \Sigma(\mathcal{R}) \to X$ is weakly compact. Let $\mathcal{F}(\mathbb{N})$ be the ring of all finite subsets of the set of all positive integers $\mathbb{N}$. If every continuous linear operator $u : \Sigma(\mathcal{F}(\mathbb{N})) \to X$ is weakly compact, then X contains no copy of c_o. Consequently, X contains no copy of c_o if and only if, for every ring of sets $\mathcal{R}$, each continuous linear operator $u : \Sigma(\mathcal{R}) \to X$ is weakly compact.*

Proof. Suppose X contains no copy of c_o. Let $u : \Sigma(\mathcal{R}) \to X$ be a continuous linear operator with the representing measure m. Let (E_n) be a disjoint sequence in $\mathcal{R}$. Since m is a bounded vector measure, for each $x^* \in X^*$, $x^* \circ m$ is a bounded scalar valued additive set function and hence $x^* \circ m$ is strongly additive on $\mathcal{R}$. Thus $\sum_{n=1}^{\infty}(x^* \circ m)(E_n)$ is unconditionally convergent and hence $\sum_{n=1}^{\infty}|(x^* \circ m)(E_n)| < \infty$ for each $x^* \in X^*$. Since X contains no copy of c_o, by Theorem 4 of Tumarkin [11], it follows that $\sum_{n=1}^{\infty} m(E_n)$ is convergent in X. Thus m is strongly additive on $\mathcal{R}$ and hence u is weakly compact by Theorem 1.

Now, suppose that every continuous linear operator $u : \Sigma(\mathcal{F}(\mathbb{N})) \to X$ is weakly compact. Let (x_n) be a sequence of vectors in X such that $\sum_{n=1}^{\infty}|x^*(x_n)| < \infty$ for each $x^* \in X^*$. Let us define $m(E) = \sum_{n\in E} x_n$ for each $E \in \mathcal{F}(\mathbb{N})$. Clearly, m is an X-valued vector measure on $\mathcal{F}(\mathbb{N})$ and $\sup_{E\in\mathcal{F}(\mathbb{N})}|(x^* \circ m)(E)| \leq \sum_{n=1}^{\infty}|x^*(x_n)| < \infty$ for each $x^* \in X^*$. Thus the range of m is weakly bounded and hence is bounded. Then the map $u : \Sigma(\mathcal{F}(\mathbb{N})) \to X$, given by $uf = \int_{\Omega} f dm$, is a continuous linear operator by Lemma 6. Consequently, by hypothesis, u is weakly compact. Then, by Theorem 1, its representing measure m is strongly additive, and hence, $\sum_{n=1}^{\infty} m(\{n\}) = \sum_{n=1}^{\infty} x_n$ is unconditionally convergent in X. Now invoking Theorem 4 of Tumarkin [11], we conclude that X contains no copy of c_o.

REFERENCES

1. R. G. Bartle, N. Dunford, and J. T. Schwartz, *Weak compactness and vector measures*, Canad. J. Math.,7, 1955, 289-305.

2. F. Bombal, *Medidas Vectoriales y Espacios de Funciones Continuas*, Lecture notes, Facultad de Matemáticas, Univ. Complutense, Madrid,1984.

3. J. Diestel and J. J. Uhl, *Vector Measures*, Survey, No.15, Amer. Math. Soc., Providence, R.I., 1977.

4. L. Drewnowski, *Topological rings of sets, continuous set functions, integration*, II, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.20, 277-286.

5. N.Dunford and J.T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience, New York, 1958.

6. R. E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Rinehart and Winston, New York, 1965.

7. A. Grothendieck, *Sur les applications linéares faiblement compactes d'espaces du type C(K)*, Canad. J. Math. 5, 1953, 129-173.

8. P.R. Halmos, *Measure Theory*, D. Van Nostrand, New York, 1950.

9. I. Kluvánek, *The range of a vector-valued measure*, Math. Systems Theory, 7, 1973, 44-54.

10. I. Kluvánek, *The extension and closure of vector measure*, Vector and Operator Valued Measures and Applications, (Proc. Sympos., Snowbird Resort, Alta, Utah, 1972), Academic Press, New York, 1973, 175-190.

11. Ju. B. Tumarkin, *On locally convex spaces with basis*, Dokl. Akad. Nauk SSSR, 11, 1970, 1672-1675.

12. I. Tweddle, *Weak compactness in locally convex spaces*, Glasgow Math. J. 9, 1968, 123-127.

Departamento de Matemáticas, Facultad de Ciencias, Universidad de los Andes, Mérida, Venezuela.
E-mail address: panchapa@ciens.ula.ve