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EXISTENCE OF BOUNDED SOLUTIONS

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Existence of Bounded Solutions of a Second Order System with Dissipation *

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Abstract

In this note, we study the following second order system of ordinary differential equations with dissipation

$$u'' + cu' + dAu + kH(u) = P(t), \quad u \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

where c, d and k are positive constants, $H : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function and $P : R \to \mathbb{R}^n$ is a continuous and bounded function. A is a $n \times n$ matrix whose eigenvalues are positive. Under these conditions, we prove that for some values of c, d and k this system has a bounded solution which is exponentially asymptotically stable. Moreover; if $P(t)$ is almost periodic, then this bounded solution is also almost periodic. These results are applied to the spatial discretization of very well known second order partial differential equations.

Key words. differential equation, bounded solutions, stability.

AMS(MOS) subject classifications. primary: 34; secondary: 45.

1 Introduction

The following second order system of differential equations in $\mathbb{R}^n$ has been studied by Alonso and Ortega in [2]

$$u'' + cu' + Au + \nabla G(u) = P(t), \quad u \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{1.1}$$

where $c > 0$ is a constant, A is a $n \times n$ symetric semidefinite positive constant matrix, P is a continuous function and bounded, and $G \in C^2(\mathbb{R}^n)$. They were interested in the existence of a bounded solution of (1.1) which is exponentially asymptotically stable. For the sake of convenience, we formulate here the main result of that work.

Theorem 1.1 *Let $\lambda_1(A) \geq 0$ be the smallest eigenvalue of the matrix A and assume that there exist non-negative constants a and b such that*

$$aI_n \leq D^2G(\xi) \leq bI_n, \quad \forall \xi \in \mathbb{R}^n, \tag{1.2}$$

with $a + \lambda_1(A) > 0$ and

$$b < a + c^2 + 2c\sqrt{a + \lambda_1(A)}.$$

Then (1.1) has a unique bounded solution which is exponential asymptotically stable.

Moreover; if $P(t)$ is τ-periodic, then such a solution is also τ-periodic.

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For the proof of Theorem 1.1 they used the method of guiding functions and a quadratic Lyapunov function. That result can be applied to a spatial discretization of the following very well known partial differential equations:

Example 1.1 *The Sine-Gordon Equation with Dirichlet boundary conditions*

$$\begin{cases} U_{tt}+cU_t-dU_{xx}+k\sin U = & p(t,x), \quad 0<x<L, \quad t\in I\!R, \\ U(t,0)=U(t,L)=0, & t\in I\!R, \end{cases} \tag{1.3}$$

where c, d and k are positive constants, $p: I\!R\times[0,L]\to I\!R$ is continuous and bounded.

For each $N\in I\!N$ the spatial discretization of this equation is given by

$$\begin{cases} u_i''+cu_i'+d\delta^{-2}(2u_i-u_{i+1}-u_{i-1})+k\sin u_i = & p_i(t), \quad 1\le i\le N, \quad t\in I\!R, \\ u_0=u_{N+1}=0. \end{cases} \tag{1.4}$$

This discrete version of the equation (1.3) can be studied for several reasons. First, they represent a simple scheme that might be used to simulate equation (1.3) nume rically. Second, the partial differential equations are usually derived as continuous approximation of discrete systems. Another reason, could be purely mathematical. This system can be written in the vector form (1.1), where A is the following matrix

$$d\delta^{-2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{1.5}$$

$\delta=L/(N+1)$ and $G(\xi)=-k\sum_{i=1}^N\cos\xi_i$.

Example 1.2 *The suspension bridge model proposed by Lazer and Mckenna(see, [4], [5]).*

$$\begin{cases} U_{tt}+cU_t+dU_{xxxx}+kU^+ = & p(t,x), \quad 0<x<L, \quad t\in I\!R, \\ U(t,0)=U(t,L)=U_{xx}(t,0) & =U_{xx}(t,L)=0, \quad t\in I\!R, \end{cases} \tag{1.6}$$

where c,d and k are positive constants, $p: I\!R\times[0,L]\to I\!R$ is continuous and bounded.

Also, the discrete version of the equation (1.6) can be written in the vector form (1.1), where A is the following matrix

$$d\delta^{-4}\begin{pmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -1 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix} \tag{1.7}$$

with $\delta=L/(N+1)$ and $G(\xi)=(k/2)\sum_{i=1}^N(\xi_i)^2$.

In this note, we shall study the following second order system of differential equations in $\mathbb{R}^n$

$$u'' + cu' + dAu + kH(u) = P(t), \quad u \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{1.8}$$

where c, d and k are positive constants, $H : \mathbb{R}^n \to \mathbb{R}^n$ is simplely a locally Lipschitz function and $P \in C_b(\mathbb{R}; \mathbb{R}^n)$, the space of continuous and bounded functions. A is a $n \times n$ matrix whose eigenvalues are positive. Here we have changed the function $\nabla G \in C^1(\mathbb{R}; \mathbb{R}^n)$ by the locally Lipschitz function H and dropped the hypothesis (1.2) of Theorem 1.1.

Of course that, the discrete versions of the partial differential equations (1.3) and (1.6) can be written in the form of our system (1.8), where the matrix A is given respectively by (1.5) and (1.7), which have simple and positive eigenvalues.

Under these conditions, we prove that there exist $k > 0$, $d > 0$ and $c > 0$ such that the equation (1.8) has one and only one bounded solution $u(t)$ which is exponentially asymptotically stable. Moreover, is $P(t)$ is almost periodic, then such a solution is also almost periodic(see Theorems 3.1, 3.2 and Lemma 3.1 in section 3).

Our method is very simple, we just rewrite the equation (1.8) as a first orden system of ordinary differential equations and find the exponential bounds for the solutions of the linear part of this system. Next, we use the variation constant formula and some ideas from [10] [11] to find a formula for the bounded solutions of (1.8). From this formula we can prove the exponential stability easily. Finally, our method can be apply in the case that the equation (1.8) is an abstract second order differential equation in a Hilbert space H with A been an unbounded operator with the following properties:

A is a self-adjoint operator with the spectrum $\sigma(A)$ consisting of isolated eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace and

a) there exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvector of A.

b) for all $x \in D(A)$ we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j x,$$

where $< \cdot, \cdot >$ is the inner product in H and

$$E_j x = \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k}.$$

So, $\{E_j\}$ is a family of complete orthogonal projections in H and $x = \sum_{j=1}^{\infty} E_j x, \quad x \in H$.

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At} x = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j x.$$

2 Preliminaries Results

Before we prove the main Theorems of this work, we shall prove some preliminaries results to be use in the next section. The equation (1.8) can be written as a first order system of ordinary differential equations in the space $W = \mathbb{R}^n \times \mathbb{R}^n$ as follow:

$$w' + \mathcal{A}w + k\mathcal{H}(w) = \mathcal{P}(t), \quad w \in W, \quad t \in \mathbb{R}, \tag{2.1}$$

where $v = u'$ and

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \ \mathcal{H} = \begin{pmatrix} 0 \\ H(u) \end{pmatrix}, \ \mathcal{P} = \begin{pmatrix} 0 \\ P(t) \end{pmatrix} \ \text{and} \ \mathcal{A} = \begin{pmatrix} 0 & -I \\ dA & cI \end{pmatrix}. \tag{2.2}$$

In this section, we shall study the linear part of the equation (2.1):

$$w' + \mathcal{A}w = 0, \quad w \in W, \quad t \in \mathbb{R}. \tag{2.3}$$

From now on, we shall suppose that each eigenvalue of the matrix A is positive and has multiplicity γ_j equal to the dimension of the corresponding eigenspace. Therefore, if $0 < \lambda_1 < \lambda_2 < \cdots \lambda_l$ are the eigenvalues of A, we have the following:
a) there exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvector of A in $\mathbb{R}^n$.

b) for all $x \in \mathbb{R}^n$ we have

$$Ax = \sum_{j=1}^{l} \lambda_j \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{l} \lambda_j E_j x, \tag{2.4}$$

where $< \cdot, \cdot >$ is the inner product in $\mathbb{R}^n$ and

$$E_j x = \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k}. \tag{2.5}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in R^n and $x = \sum_{j=1}^{l} E_j x, \quad x \in \mathbb{R}^n$.

c) the exponential matrix e^{-At} is given by

$$e^{-At} = \sum_{j=1}^{l} e^{-\lambda_j t} E_j. \tag{2.6}$$

Theorem 2.1 *Suppose that $c \neq 2\sqrt{d\lambda_j}, \quad j = 1, 2, \ldots, l$. Then the exponential matrix $e^{-\mathcal{A}t}$ of the matrix $-\mathcal{A}$ given by (2.2) can be written as follow*

$$e^{-\mathcal{A}t} w = \sum_{j=1}^{l} \left\{ e^{\rho_1(j)t} Q_1(j) w + e^{\rho_2(j)t} Q_2(j) w \right\}, \quad w \in W, \quad t \in \mathbb{R}, \tag{2.7}$$

where

$$\rho(j) = \frac{-c \pm \sqrt{c^2 - 4d\lambda_j}}{2}, \quad j = 1, 2, \ldots, l \tag{2.8}$$

and $\{Q_i(j) : i = 1, 2\}_{j=1}^{l}$ is a complete orthogonal system of projections in W.

Proof Define the following complete orthogonal system of projections in W

$$\hat{E}_j = \begin{pmatrix} E_j & 0 \\ 0 & E_j \end{pmatrix}, \quad j = 1, 2, \ldots, l. \tag{2.9}$$

Then; if we project the equation (2.3) on the space $\operatorname{Ran}(\hat{E}_j)$ (the range of $\hat{E}_j$), we obtain the following family of systems

$$(S_j) \begin{cases} u' - v = 0 \\ v' + cv + d\lambda_j u = 0 \end{cases}, \quad j = 1, 2, \ldots, l.$$

Henceforth, the solution $w(t)$ of the system (2.3) passing through the point w_0 at $t = 0$ is given by

$$w(t) = e^{-\mathcal{A}t} w_0 = \sum_{j=1}^{l} \hat{E}_j w(t) = \sum_{j=1}^{l} w_j(t), \tag{2.10}$$

where $w_j(t)$ is the solution of the system (S_j) such that $w_j(0) = \hat{E}_j w(0) = \hat{E}_j w_0$.

On the other hand, the system (S_j) can be written as follow

$$y' = B_j y, \quad y \in \operatorname{Ran}(\hat{E}_j), \quad j = 1, 2, \ldots, l,$$

where

$$B_j = \begin{pmatrix} 0 & 1 \\ -d\lambda_j & -c \end{pmatrix}.$$

Hence,

$$w_j(t) = e^{B_j t} w_j(0) = e^{B_j t} \hat{E}_j w_0, \quad j = 1, 2, \ldots, l.$$

Clearly that the eigenvalues of the matrix B_j are given by

$$\rho(j) = \frac{-c \pm \sqrt{c^2 - 4d\lambda_j}}{2}, \quad j = 1, 2, \ldots, l.$$

Since $c \neq 2\sqrt{d\lambda_j}$, $j = 1, 2, \ldots, l$, then the ρ_j, s are simples. Thus, there exist a complete system of orthogonal projections $\{P_i(j)\}_{i=1}^{2}$ in $\mathbb{R}^2$ such that

$$e^{B_j t} = e^{\rho_1(j)t} P_1(j) + e^{\rho_2(j)t} P_2(j), \quad j = 1, 2, \ldots, l.$$

Moreover, we can compute these projections

$$P_1(j) = \frac{1}{\sqrt{c^2 - 4d\lambda_j}} (B_j - \rho_2(j) I_{\mathbb{R}^2}), \quad P_2(j) = \frac{-1}{\sqrt{c^2 - 4d\lambda_j}} (B_j - \rho_1(j) I_{\mathbb{R}^2}).$$

Therefore, from (2.10) we get that the solution $w(t)$ of the linear equation (2.3) is given by

$$w(t) e^{-\mathcal{A}t} w_0 = \sum_{j=1}^{l} \left\{ e^{\rho_1(j)t} P_1(j) \hat{E}_j w_0 + e^{\rho_2(j)t} P_2(j) \hat{E}_j w_0 \right\}, \quad w_0 \in W, \quad t \in \mathbb{R}.$$

Then, putting $Q_i(j) = P_i(j)\hat{E}_j$ we obtain the result. $\square$

Remark 2.1 *We have abused of notation in the proof of Theorem 2.1, by consider* 2×2 *matrix* $B = (b_{ij})_{2\times 2}$ *as an operator acting in* $\mathbb{R}^n \times \mathbb{R}^n$*; what we really mean is:*

$$\hat{B} = \begin{pmatrix} b_{11}I_{\mathbb{R}^n} & b_{12}I_{\mathbb{R}^n} \\ b_{21}I_{\mathbb{R}^n} & b_{22}I_{\mathbb{R}^n} \end{pmatrix}.$$

Corollary 2.1 *The spectrum* $\sigma(-\mathcal{A})$ *of the matrix* $-\mathcal{A}$ *is given by*

$$\sigma(-\mathcal{A}) = \{\frac{-c\pm\sqrt{c^2 - 4d\lambda_j}}{2}, \quad j = 1, 2, \ldots, l\}.$$

Corollary 2.2 *Under the hypothesis of Theorem 2.1 we have that*

$$\|e^{-\mathcal{A}t}\| \leq e^{-\beta t}, \quad t \geq 0, \tag{2.11}$$

where

$$0 > -\beta = -\beta(c,d) = max\left\{ Re(\rho_j) = Re\left(\frac{-c\pm\sqrt{c^2 - 4d\lambda_j}}{2}\right) : j = 1, 2, \ldots, l. \;\; i = 1, 2.\right\}.$$

Proof From the formula (2.7) we get that

$$\begin{aligned} \|e^{-\mathcal{A}t}w\|^2 &= \sum_{j=1}^{l}\left\{e^{2\mathrm{Re}(\rho_1(j))t}\|Q_1(j)w\|^2 + e^{2\mathrm{Re}(\rho_2(j))t}\|Q_2(j)w\|^2\right\} \\ &\leq \sum_{j=1}^{l} e^{-2\beta t}\left\{\|Q_1(j)w\|^2 + \|Q_2(j)w\|^2\right\} \\ &= e^{-2\beta t}\|w\|^2, \quad w \in W, \quad t \geq 0. \end{aligned}$$

Therefore, $\|e^{-\mathcal{A}t}\| \leq e^{-\beta t}, \;\; t \geq 0.$ □

3 Main Results

In this section we shall prove the main Theorems of this paper, under the hypothesis of Theorem 2.1($c \neq 2\sqrt{d\lambda_j}, \;\; j = 1, 2, \ldots, l$).

The solution of (2.1) passing through the point w_0 at time $t = t_0$ is given by the variation constant formula

$$w(t) = e^{-\mathcal{A}(t-t_0)}w_0 + \int_{t_0}^{t} e^{-\mathcal{A}(t-s)}\left\{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\right\} ds, \quad t \in \mathbb{R}. \tag{3.1}$$

We shall consider $W_b = C_b(\mathbb{R}, W)$ the space of bounded and continuous functions defined in $\mathbb{R}$ taking values in $W = \mathbb{R}^n \times \mathbb{R}^n$. W_b is a Banach space with suprem norm

$$\|w\|_b = \sup\{\|w(t)\|_W : t \in \mathbb{R}\}, \quad w \in W_b.$$

A ball of radio $\rho > 0$ and center zero in this space is given by

$$B_\rho^b = \{w \in W_b : \|w(t)\|_b \leq \rho, \quad t \in \mathbb{R}\}.$$

Lemma 3.1 *Let w be in W_b. Then, w is a solution of (2.1) if and only if w is given by*

$$w(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\} ds, \quad t \in I\!R. \tag{3.2}$$

Proof Suppose that w is a solution of (2.1). Then, from the variation constant formula (3.1) and the uniqueness of the solution of (2.1) we get that

$$w(t) = e^{-\mathcal{A}(t-t_0)}w(t_0) + \int_{t_0}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\} ds, \quad t \geq t_0. \tag{3.3}$$

On the other hand, from (2.11) we obtain that

$$\|e^{-\mathcal{A}(t-t_0)}w(t_0)\| \leq e^{-\beta(t-t_0)}\|w(t_0)\|, \quad t \geq t_0,$$

and since $\|w(t)\| \leq M, \quad t \in I\!R$, we get the following estimate

$$\|e^{-\mathcal{A}(t-t_0)}w(t_0)\| \leq Me^{-\beta(t-t_0)}, \quad t \geq t_0,$$

which implies that

$$\lim_{t_0 \to -\infty} \|e^{-\mathcal{A}(t-t_0)}w(t_0)\| = 0.$$

Therefore, passing to the limit in (3.3) when t_0 goes to $-\infty$ we conclude that

$$w(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\} ds, \quad t \in I\!R.$$

Suppose that w is a solution of the integral equation (3.2). Then

$$\begin{aligned} w(t) &= \int_{-\infty}^{0} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\} ds \\ &+ \int_{0}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\} ds. \end{aligned}$$

On the other hand, we have that

$$\|\int_{-\infty}^{0} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\} ds\| \leq \int_{-\infty}^{0} e^{-\beta(t-s)} \{kR_w + L_p\} ds = \frac{kR_w + L_p}{\beta},$$

where R_w and L_p are constants such that

$$\|\mathcal{H}(w(s))\| \leq R_w, \quad \|\mathcal{P}(s)\| \leq L_p, \quad s \in I\!R.$$

Hence, the following improper integral is well defined

$$w_0 = \int_{-\infty}^{0} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\} ds,$$

and

$$w(t) = e^{-\mathcal{A}(t-s)}w_0 + \int_{0}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\} ds.$$

This concludes the proof of the lemma. □

Theorem 3.1 *For $\rho > 0$ and $k > 0$ there exist $d, c > 0$ such that the equation (2.1) has one and only one solution $w(\cdot)$ which belong to the ball B^b_ρ. Moreover, this bounded solution is exponentially asymptotically stable.*

Proof For the existence of such solution, we shall prove that the following operator has a unique fixed point in the ball B^b_ρ, $\ T : B^b_\rho \to B^b_\rho$

$$(Tw)(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\} ds, \quad t \in I\!R. \tag{3.4}$$

Consider $R = \sup_{w \in B_\rho} \|\mathcal{H}(w)\|$ and $L = \sup_{s \in I\!R} \|\mathcal{P}(s)\|$ and put $M = kR + L$. Then, for all $w \in B^b_\rho$ we get

$$\|Tw(t)\| \leq \int_{-\infty}^{t} Me^{-\beta(t-s)} ds = \frac{M}{\beta}.$$

From corollary 2.2, we can choose c and d such that

$$\frac{M}{\beta(c,d)} = \frac{kR + L}{\beta(c,d)} \leq \rho, \tag{3.5}$$

then $Tw \in B^b_\rho$, $\ w \in B^b_\rho$.

Now, we shall prove that T is a contraction mapping. In fact, for w_1, $\ w_2 \in B^b_\rho$ we have that

$$\|Tw_1(t) - Tw_2(t)\| \leq \int_{-\infty}^{t} e^{-\beta(t-s)} kL_\rho \|w_1(s) - w_2(s)\| ds,$$

where L_ρ is the Lipschitz constant of $\mathcal{H}$ in the ball

$$B_{2\rho} = \{w \in W : \|w\| \leq 2\rho\}.$$

So,

$$\|Tw_1 - Tw_2\|_b \leq \frac{kL_\rho}{\beta} \|w_1 - w_2\|_b.$$

Hence; if we choose c and d such that

$$\frac{kL_\rho}{\beta(c,d)} < 1, \tag{3.6}$$

then, T is a contraction mapping. Therefore, T has a unique fixed point w_b in B^b_ρ. i.e.,

$$w_b(t) = (Tw_b)(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w_b(s)) + \mathcal{P}(s)\} ds, \quad t \in I\!R,$$

and from Lemma 3.1 $w_b(t)$ is solution of the equation (2.1).

To prove that $w_b(t)$ is exponentially asymptotilly stable, we shall consider any other solution $w(t)$ of the equation (2.1) such that $\|w(0) - w_b(0)\| < \frac{\rho}{2}$. Then, $\|w(0)\| < 2\rho$. As long as $\|w(t)\|$ remains less than 2ρ we get the following estimate:

$$\begin{aligned} \|w(t) - w_b(t)\| &\leq \|e^{-\mathcal{A}t}(w(0) - w_b(0)) + \int_{0}^{t} e^{-\mathcal{A}(t-s)} \{k\mathcal{H}(w(s)) - k\mathcal{H}(w_b(s))\} ds\| \\ &\leq e^{-\beta t}\|(w(0) - w_b(0))\| + \int_{0}^{t} e^{-\beta(t-s)} kL_\rho \|w(s) - w_b(s)\| ds. \end{aligned}$$

Theorem 3.2 *Consider a function H such that $H(0) = 0$ and*

$$\|H(U_1) - H(U_2)\| \leq L\|U_1 - U_2\|, \quad U_1, U_2 \in I\!R^n. \tag{3.7}$$

Suppose $\rho > 0$ big enough such that

$$0 < L_p = \sup_{s \in I\!R} \|P(s)\| < (\beta(c,d) - kL)\rho. \tag{3.8}$$

Then the equation (2.1) has one and only one solution $w_b(t)$ which belong to the ball B^b_ρ in W_b.

Moreover, this bounded solution is the only bounded solution of the equation (2.1) and is exponentially stable in the large.

Proof For the existence of such solution, we shall prove that the following operator has a unique fixed point in the ball B^b_ρ, $T : B^b_\rho \to B^b_\rho$

$$(Tw)(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\} \, ds, \quad t \in I\!R.$$

In fact, for $w \in B^b_\rho$ we have

$$\|Tw(t)\| \leq \int_{-\infty}^{t} e^{-\beta(t-s)} \{k\|w(s)\| + L_p\} \leq \frac{(kL)\rho + L_p}{\beta}.$$

The condition (3.8) implies that

$$kL\rho + L_p < \beta\rho \iff \frac{kL\rho + L_p}{\beta} < \rho.$$

Therefore, $Tw \in B^b_\rho$ for all $w \in B^b_\rho$.

Now, we shall see that T is a contraction mapping. In fact, for all $w_1, w_2 \in B^b_\rho$ we have that

$$\|Tw_1(t) - Tw_2(t)\| \leq \int_{-\infty}^{t} e^{-\beta(t-s)} kL\|w_1(s) - w_2(s)\| ds \leq \frac{kL}{\beta}\|w_1 - w_2\|_b, \quad t \in I\!R.$$

Hence,

$$\|w_1 - Tw_2\|_b \leq \frac{kL}{\beta}\|Tw_1 - w_2\|_b, \quad w_1, w_2 \in B^b_\rho.$$

The condition (3.8) implies that

$$0 < \beta - kL \iff kL < \beta \iff \frac{kL}{\beta} < 1.$$

Therefore, T has a unique fixed point w_b in B^b_ρ

$$w_b(t) = (Tw_b)(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w_b(s)) + \mathcal{P}(s)\} \, ds, \quad t \in I\!R,$$

Then,

$$e^{\beta t}\|w(t) - w_b(t)\| \le \|(w(0) - w_b(0))\| + \int_0^t e^{\beta s} kL_\rho \|w(s) - w_b(s)\| ds.$$

Hence, applying the Gronwall's inequality we obtain

$$\|w(t) - w_b(t)\| \le e^{(kL_\rho - \beta)t}\|(w(0) - w_b(0))\|, \quad 0 \le t \le t_1.$$

From (3.6) we get that $kL_\rho - \beta < 0$ and therefore

$$\|w(t) - w_b(t)\| \le \|(w(0) - w_b(0))\| < \frac{\rho}{2}, \quad 0 \le t \le t_1.$$

If $\|w(t)\| < 2\rho$ on $[0, t_1)$ with t_1 been the maximal time with this property, then either $t_1 = \infty$ or $\|w(t_1)\| = 2\rho$. But the second case contradicts this computation, therefore the solution $w(t)$ remains in the ball $B_{2\rho}$ of center zero and radio ρ in W for $t \ge 0$.

Hence,

$$\|w(t) - w_b(t)\| \le e^{(kL_\rho - \beta)t}\|(w(0) - w_b(0))\|, \quad t \ge 0.$$

This concludes the proof of the theorem. □

Remark 3.1 *The discrete version (1.4) of the partial differential equation (1.3) can be interpreted as a model for the motion of a system of n linealy coupled pendulums with linear damping and external forces acting on the system; where c is the coefficient of friction and k refers to the length of the pendulums. In the same way, in the equation (1.6) k represents the spring constant of the restoring force due to the cables. In these examples, we can say more about the bonded solutions. In fact, we shall prove that this bounded solution is global and exponentially asymptotically stable in large.*

From Lemma 3.1, w_b is a bounded solution of the equation (2.1). Since condition (3.8) holds for any $\rho > 0$ big enough independent of $kL < \beta(c,d)$, then w_b is the unique bounded solution of the equation (2.1).

To prove that $w_b(t)$ is exponentially stable in the large, we shall consider any other solution $w(t)$ of (2.1) and consider the following estimate

$$\begin{aligned}\|w(t) - w_b(t)\| &\leq \|e^{-\mathcal{A}t}(w(0) - w_b(0)) + \int_0^t e^{-\mathcal{A}(t-s)}\{k\mathcal{H}(w(s)) - k\mathcal{H}(w_b(s))\}ds\| \\ &\leq e^{-\beta t}\|(w(0) - w_b(0))\| + \int_0^t e^{-\beta(t-s)}kL\|w(s) - w_b(s)\|ds.\end{aligned}$$

Then,

$$e^{\beta t}\|w(t) - w_b(t)\| \leq \|(w(0) - w_b(0))\| + \int_0^t e^{\beta s}kL\|w(s) - w_b(s)\|ds.$$

Hence, applying the Gronwall's inequality we obtain

$$\|w(t) - w_b(t)\| \leq e^{(kL_\rho - \beta)t}\|(w(0) - w_b(0))\|, \quad t \geq 0.$$

From (3.8) we get that $kL - \beta < 0$ and therefore $w_b(t)$ is exponentially stable in the large. □

Corollary 3.1 *The bounded solution $w_b(\cdot, P)$ given by Theorem 3.2 depends continuously on $P \in C_b(\mathbb{R}, \mathbb{R}^n)$.*

Proof Let $P_1, P_2 \in C_b(\mathbb{R}, \mathbb{R}^n)$ and $w_b(\cdot, P_1), w_b(\cdot, P_2)$ be the bounded functions given by Theorem 3.2. Then

$$\begin{aligned}w_b(\cdot, P_1) - w_b(\cdot, P_2) &= \int_{-\infty}^t e^{-\mathcal{A}(t-s)}k\{\mathcal{H}(w_b(s, P_2)) - \mathcal{H}(w_b(s, P_1))\}ds \\ &+ \int_{-\infty}^t e^{-\mathcal{A}(t-s)}k\{\mathcal{P}_1(s) - \mathcal{P}_1(s)\}ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\|w_b(\cdot, P_1) - w_b(\cdot, P_2)\|_b &\leq \frac{kL}{\beta}\|w_b(\cdot, P_1) - w_b(\cdot, P_2)\|_b \\ &+ \frac{1}{\beta}\|P_1 - P_2\|_b.\end{aligned}$$

Hence,

$$\|w_b(\cdot, P_1) - w_b(\cdot, P_2)\|_b \leq \frac{1}{\beta - kL}\|P_1 - P_2\|_b.$$

□

We conclude this work with the following lemma about almost periodicity of the bounded solutions of the equation (2.1).

Lemma 3.2 *If $P(t)$ is almost periodic, then the unique bounded solution of the system (2.1) given by Theorems 3.1 and 3.2 is also almost periodic.*

Proof To prove this lemma, we shall use Theorema 1 in the Appendix of [6] which said that. A function $f \in C(\mathbb{R}; \mathbb{R}^N)$ is almost periodic (a.p) if and only if the Hull $H(f)$ of f is compact in the topology of uniform convergence.

Where $H(f)$ is the closure of the set of translates of f under the topology of uniform convergence

$$H(f) = \overline{\{f_\tau : \tau \in \mathbb{R}\}}, \qquad f_\tau(t) = f(t+\tau), t \in \mathbb{R}.$$

Since the limit of a uniformly convergent sequence of a.p. functions is a.p., then the set A_ρ of a.p. functions in the ball B_ρ^b is closed, where ρ is given by Theorem 3.1 or 3.2.

Claim. The contraction mapping T given in Theorems 3.1 and 3.2 leaves A_ρ invariant. In fact; if $w \in A_\rho$, then $f(t) = -k\mathcal{H}(w(t)) + \mathcal{P}(t)$ is also an a.p. function. Now, consider the function

$$\begin{aligned} F(t) = (Tw)(t) &= \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\}\, ds \\ &= \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} f(s) ds, \quad t \in \mathbb{R}. \end{aligned}$$

Then, it is enough to establish that $H(F)$ is compact in the topology of uniform convergence. Let $\{F_{\tau_k}\}$ be any sequence in $H(F)$. Since f is a.p. we can select from $\{f_{\tau_k}\}$ a Cauchy subsequence $\{f_{\tau_{k_j}}\}$, and we have that

$$\begin{aligned} F_{\tau_{k_j}}(t) = F(t+\tau_{k_j}) &= \int_{-\infty}^{t+\tau_{k_j}} e^{-\mathcal{A}(t+\tau_{k_j}-s)} f(s) ds \\ &= \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} f(s+\tau_{k_j}) ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|F_{\tau_{k_j}}(t) - F_{\tau_{k_i}}(t)\| &\le \int_{-\infty}^{t} \|e^{-\mathcal{A}(t+-s)}\| \|f(s+\tau_{k_j}) - f(s+\tau_{k_i})\| ds \\ &\le \|f_{\tau_{k_j}} - f_{\tau_{k_i}}\|_b \int_{-\infty}^{t} e^{-\beta(t-s)} ds = \frac{1}{\beta} \|f_{\tau_{k_j}} - f_{\tau_{k_i}}\|_b. \end{aligned}$$

Therefore, $\{F_{\tau_{k_j}}\}$ is a Cauchy sequence. So, $H(F)$ is compact in the topology of uniform convergence, F is a.p. and $TA_\rho \subset A_\rho$.

Now, the unique fixed point of T in the ball B_ρ^b lies in A_ρ. Hence, the unique bounded solution $w_b(t)$ of the equation (2.1) given in Theorems 3.1 and 3.2 is also almost periodic. □

Although the following corollary follows from the fact that every periodic function is a.p., we shall give here a direct proof.

Corollary 3.2 *If $P(t)$ is periodic, then the unique bounded solution of the system (2.1) given by Theorems 3.1 and 3.2 is also periodic.*

Proof Suppose that $\mathcal{P}$ is periodic of period τ and let w_b be the unique solution of (2.1) in the ball B_ρ^b. Then, $w(t) = w_b(t+\tau)$ is also a solution of the equation (2.1) lying in the ball B_ρ^b, and by the uniqueness of the fixed point of the contraction mapping T in this ball, we conclude that $w_b(t) = w_b(t+\tau), \quad t \in \mathbb{R}$. □

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