Universidad de los Andes

Facultad de Ciencias

Departamento de Matemática

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Bhaskar Srivastava

Notas de Matemática

Serie: Pre-Print

No. 181

Mérida - Venezuela

1999

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Mérida, 1999.

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Abstract

A continued fraction $C(-q,q)$ is defined as a special case of a general continued fraction $F(a,b,c,\lambda,q)$, which we have considered earlier in a separate paper. This continues fraction is also a special case of Ramanujan's continued fraction. In this paper we have found some very interesting q-identities and some identities analogous to identities given by Ramanujan involving $G(-q,q)$ and $H(-q,q)$ and one identity which gives the square of a continued fraction.

1 Introduction

In an earlier paper [4] we consider the continued fraction $C(-q,q)$ and obtained some identities. In this paper we give some more interesting q-identities. The first of these identities gives the square of a continued fraction. We have defined

$$\begin{aligned} C(-q,q) &= \frac{\left(1+\frac{1}{q}\right)q}{1+}\ \frac{q^2}{1+}\ \frac{\left(1+\frac{1}{q^2}\right)q^2}{1+}\ \frac{q^4}{1+\ldots\ldots} \\ &= \frac{\sum_{n=0}^{\infty} q^{n(n-1)/2}(-q)_n/(q)_n}{\sum_{n=0}^{\infty} q^{n(n+1)/2}(-q)_n/(q)_n} \\ &= 1+\frac{(q^2;q^4)_\infty^2}{(q^2;q^4)_\infty(q;q^4)_\infty} \end{aligned}$$

by using summatin formula Slater [3, eqn 8 and 13]. Also [4,p.200,eqn 2.2].

2 Notation

$$\begin{aligned} (a;q^k)_n &= \Pi_{n-1}^{n=0}(1-aq^{kn}); \quad n\geq 1 \\ (a;q^k)_0 &= 1 \\ (a;q^k)_\infty &= \Pi_{j\geq 0}(1-aq^{kj}), \end{aligned}$$

when $k=1$, q^k shall be omitted from the various symbols, in case there is no chance of ambiguity.

3 An interesting q-identity

We shall prove the identity

$$\frac{(1+1/q)q}{1+}\;\frac{q^2}{1+}\;\frac{(1+1/q^2)q^3}{1+}\;\frac{q^4}{1+\ldots\ldots}$$

$$=\frac{(q^2;q^4)^2_\infty}{(q^4;q^4)^2_\infty}\left[\sum_{n=0}^{\infty}q^{4n^2+2n}\,\frac{1+q^{4n+1}}{1-q^{4n+1}}-\sum_{n=0}^{\infty}q^{4n^2+6n+2}\frac{1+q^{4n+3}}{1-q^{4n+3}}\right] \tag{1}$$

The proof of this identity depends on the continued fraction (1.1) and Ramanujan's $_1\Psi_1$-summation [1,p.101],namely

$$_1\Psi_1\mid a;b;q;z\mid=\sum_{n=0}^{\infty}(a)_nZ^n/(b)_n$$

$$=\frac{(b/a)_\infty(az)_\infty(q/az)_\infty(q)_\infty}{(q/a)_\infty(b/az)_\infty(b)_\infty(z)_\infty} \tag{2}$$

We shall first prove a series of identity:

$$\sum_{n=0}^{\infty}\frac{q^{in}}{1-q^{4n+i}}=\sum_{n=0}^{\infty}q^{4n^2+2in}\,\frac{1+q^{4n+i}}{1-q^{4n+i}} \tag{3}$$

$i=1,2,3$

Now

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{in}}{1-q^{4n+i}} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q^{in+4n+im}\\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q^{i(n+m)+4(n+m)+im}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q^{in+4n(m+n+1)+i(m+n+1)}\\ &= \sum_{m=0}^{\infty}\frac{q^{4m}}{1-q^{4m+i}}+\sum_{n=0}^{\infty}\frac{q^{4n+4n+2in+i}}{1-q^{4n+i}}\\ &= \sum_{n=0}^{\infty}q^{4n^2+2in}\,\frac{1+q^{4n+i}}{1-q^{4n+i}}\end{aligned}$$

This proves (3.3).

By replacing q by q^4 and then setting $q = q^i, b = q^{4+i}4 + i, z = q^i$ in (3.2), we have for $i = 1, 3$.

$$\sum_{n=-\infty}^{\infty} \frac{q^{in}}{1 - q^{4n+i}} = \frac{1}{1 - q^i} {}_1\Psi_1(q^{i}, q^{4+i}, q^4, q^i)$$

$$= \frac{(q^4; q^4)_\infty^2 (q^{2i}; q^4)_\infty (q^{4-2i}; q^4)_\infty}{(q^{4i}; q^4)_\infty^2 (q^{i}; q^4)_\infty^2} \tag{4}$$

Hence

$$\sum_{n=0}^{\infty} q^{4n^2+2n} \frac{1 + q^{4n+1}}{1 - q^{4n+1}} - \sum_{n=0}^{\infty} q^{4n^2+6n+2} \frac{1 + q^{4n+3}}{1 - q^{4n+3}}$$

$$= \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{4n+3}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1 - q^{4n+3}} \qquad \text{, by (3.3)}$$

$$= \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{4n+1}}$$

$$= \frac{(q^4; q^4)_\infty^2 (q^2; q^4)_\infty^2}{(q^3; q^4)_\infty^2 (q; q^4)_\infty^2} \qquad \text{, by (3.4)}$$

$$= \frac{(q^4; q^4)_\infty^2 (q^2; q^4)_\infty^4}{(q^2; q^4)_\infty^2 (q^3; q^4)_\infty^2 (q; q^4)_\infty^2}$$

$$= \frac{(q^4; q^4)_\infty^2}{(q^2; q^4)_\infty^2} \left[\frac{(1 + 1/q)q}{1+} \; \frac{q^2}{1+} \; \frac{(1 + 1/q^2)q^3}{1+} \; \frac{q^4}{1 + \ldots\ldots} \right]$$

This proves (3.1).

4 Some more Identities

Let us define

$$G(-q, q) = \sum_{n=0}^{\infty} q^{n(n-1)/2} (-q)_n / (q)_n$$

$$= (-q, q)_\infty \left[\frac{1}{(q^2; q^4)_\infty} + \frac{(q^2; q^4)_\infty}{(q; q^4)_\infty (q^3; q^4)_\infty} \right], \tag{5}$$

using summation formula Slater [3, eqn. 3] and

$$H(-q,q)=\sum_{n=0}^{\infty} q^{n(n+1)/2}(-q)_n/(q)_n$$

$$=\frac{(-q;q)_\infty}{q^2;q^4} \tag{6}$$

using summation formula Slater [3,eqn. 8].
Obviously

$$C(-q,q)=\frac{G(-q,q)}{1+H(-q;q)} \tag{7}$$

$$G(-q,q)=H(-q,q)+\frac{(-q;q)_\infty(q^2;q^4)_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty} \tag{8}$$

Next we prove a generalization of (3.4), namely

$$\sum_{n=-\infty}^{\infty}\frac{q^{in}}{1-q^{4n+j}}=\frac{(q^4;q^4)^2_\infty(q^{i+j};q^4)_\infty(q^{4-i-j};q^4)_\infty}{(q^j;q^4)_\infty(q^{4-j};q^4)_\infty(q^{4-i};q^4)_\infty} \tag{9}$$

where $0<i\le 3,\ 0<j\le 3$ and $i+j\neq 4$.
The proof is similar to that of (3.4).
With the help of (4.1),(4.2),(4.4) and (4.5) we have the following, which are analogous to the identities of Ramanajan [2, p.197-198]

$$\frac{(q^4;q^4)^2_\infty}{(-q;q)^2_\infty}[H(-q;q)]^2=\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{4n+2}}\quad ,(i=1,j=2 \text{ in } (4.5))(10)$$

$$\frac{(q^4;q^4)^2_\infty}{(-q;q)^2_\infty}[G(-q;q)-H(-q;q)]^2=\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{4n+1}}\quad ,(i=1,j=1 \text{ in } (4.5))(11)$$

$$\frac{(q^4;q^4)^2_\infty}{(-q;q)^2_\infty}[H(-q,q)]^2=\sum_{n=\infty}^{\infty}\frac{q^{2n}}{1-q^{4n+1}},(i=2,j=1\quad \text{in } (4.5))(12)$$

$$\frac{(q^4;q^4)^2_\infty}{(-q;q)^2_\infty}[G(-q,q)-(H(-q,q))]^2=\sum_{n=-\infty}^{\infty}q^{2n(2n+1)}\frac{1+q^{4n+1}}{1-q^{4n+1}}(13)$$

using (4.7) and (4.9) and putting in (4.5) for $i=1$

5 Conclusion

We have defined $C(-q,q)$ taking as a special case of a general continues fraction $F(a,b,c,\lambda,q)$ consider earlier [4,p.199]:

$$F(a,b,c,\lambda,q) = 1+\frac{(1-1*c)(aq+\lambda)}{(a+aq/c)+}\,\frac{bq+\lambda q^2}{1+}\,\frac{(1-1/cq)(aq^2+\lambda q^3)}{(1+aq/c)+}\,\frac{bq^2+\lambda q^4}{1+\ldots\ldots}$$

by taking $a=0, b=0, \lambda=1$ and $c=-q$.
This continues fraction $C(-q,q)$ is also a special case of Ramanujan's work (Entry 9 and 13 in chapter 16 of Ramanujan's Second Note Book Memoirs of the AMS 53(1985) No.315).
The present paper was motivated by Andrews treatment of the Rogers-Ramanujan continues fraction [1] and the technique employed in the proof is a straight forward modification of his technique.

References

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