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Notas de Matemática
Serie: Pre-Print
No. 199

Mérida - Venezuela
1999

Notas de Matemática, No. 199
Mérida, 1999.

Generalizations of some Fixed Point Theorems*

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Keywordss: Fixed Point, w-distance, w-contraction.

Abstract

In this paper some generalizations of fixed point theorems are obtained using the w-distance on a metric space.

Introduction.

In 1996, O.Kada-T. Suzuki-W. Takahashi [6] introduced the concept of w-distance on a metric space, gave some examples, properties of w-distance and they improved the Caristi's fixed point [1], Ekeland's ϵ-Varational Principle [5] and the non convex minimization theorem according to Takahashi [14]. Finally, using the concept of w-distance proved a fixed point theorem in a complete metric space. This theorem generalize the fixed point theorems of Subrahmanyan [11], Kannan [7] and Ciric [2].

In the same year Suzuki-Takahashi [13] gave another properties of w-distance and using this notion they proved a fixed point theorem for set valued mappings on complete metric spaces which are related with Nadler's Fixed Point Theorem [9] y Edelstein's Theorem [4]. Finally, they gave a characterization of metric completeness.

In 1977, Suzuki [12] gave another properties of w-distance which generalize some of them [6], he proved several fixed point theorems which are generalizations of the Banach Contraction Principle and Kannan's Fixed Point Theorem and moreover discuss a characterization of metric completeness.

1.- Preliminares.

*This work was supported by C.D.C.H.T.-ULA under proyect C-903-98-05-B. 1991 Mathematical Classification: 47H10, 54E50.

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real numbers and $\mathbb{R}^+ = [0, +\infty)$.

Definition 1.1.
Let (M, d) be a metric space. Then a function $p : M \times M \to [0, +\infty)$ is called a w-distance on M if the following are satisfied:

w_1.- *$p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in M$.*

w_2.- *For any $x \in M$, $p(x, \cdot) : M \to [0, +\infty)$ is lower semicontinuous.*

w_3.- *For any $\epsilon > 0$, exists $\delta = \delta(\epsilon) > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$, for any $x, y, z \in M$.*

The metric d is a w-distance on M. Some other examples of w-distance are given in [6] and [14]. The following results are crucial in the proof of our theorems. The next lemma was proved in [6].

Lemma 1.1.
Let (M, d) be a metric space and let p be a w-distance on M. Let x_n and y_n be sequences in M, let α_n and β_n be sequences in $[0, +\infty)$ converging to 0, and let $x, y, z \in M$. Then the following hold:

1.- *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$ then $y = z$;*

2.- *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then y_n converge to z;*

3.- *If $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$ then x_n is a Cauchy sequence;*

4.- *If $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then x_n is a Cauchy sequence.*

The following result is proved in [6].

Lemma 1.2
Let (M, d) be a metric space, let p be a w-distance on M, and let q be a function from $M \times M$ into $[0, +\infty)$ satisfaying (w_1) and (w_2) in the definition of w-distance. Suppose that $p(x, y) \leq q(x, y)$ for every $x, y \in M$. Then q is also a w-distance on M. In particular, if q satisfies (w_1) and (w_2) in the

definition of w-distance and $d(x,y) \leq q(x,y)$ for every $x,y \in M$, entonces q es una ω-distance on M.

Definition 1.2. *Let $\epsilon \in (0,+\infty)$. A metric space (M,d) is called ϵ-chainable [4] if for every $x,y \in M$ there exists a finite sequence $x_0, x_1, ..., x_n$ and $d(x_i, x_{i+1}) < \epsilon$, $i = 0,1,...,n-1$. Such sequence is called an ϵ-chain in M joining x and y.*

The following result was proved in [13].

Lemma 1.3.
Let $\epsilon \in (0,+\infty)$ and let (M,d) be an ϵ-chainable metric space. Then the function $p : M \times M \to [0,+\infty)$ defined by

$$p(x,y) = \inf\{\sum_{i=0}^{k-1} d(x_i, x_{i+1}) : x_0, ..., x_k \text{ is an } \epsilon - \text{chain joining } x \text{ and } y\}$$

is a w-distance on M.

2.- Fixed Point Theorems.

In [13] we found the following,

Definition 2.1.
Let (M,d) be a space metric and let T be a mapping from M into itself. We say that T is a w-contraction if there is a w-distance p on M and $k \in [0,1)$ such that for every $x,y \in M$,

$$p(Tx,Ty) \leq kp(x,y).$$

In the case of $p = d$, T is called a contraction.
It is clear that if T_1, $T_2 : M \to M$ are w-contractions then $T_1 \cdot T_2 : M \to M$ is also a w-contraction and hence the set of all w-contractions defined from M into itself is a semigroup.

Now we introduced the following,

Definition 2.2.
Let (M,d) be a space metric with a w-distance p on M and let $T : M \to M$ be a mapping. Then,

a.- *An element* $x \in M$ *is w-asymptotic regular for* T *if*

$$\lim_{n\to\infty} p(T^n x, T^{n+1} y) = 0; for\ any\ y \in M$$

b.- T *is w-asymptotic regular if all elements* $x \in M$ *are asymptotic regular for* T.

c.- *Two elements* x *and* y *of* M *are w-asymptotic equivalent under* T, *if*

$$\lim_{n\to\infty} p(T^n x, T^n y) = 0.$$

It is clear that this definition extend their respective notions, (see [10]). The following result is generalization of Banach Contraction Principle.

Theorem 2.1.
Let (M, d) *be a complete metric space, let* T *be a mapping from* M *into itself and suppose that* T *is a w-contaction. Then,*

a.- *There exists a unique* $z \in M$ *such that* $Tz = z$.

b.- *The point* z *satisfies* $p(z, z) = 0$.

c.- $\{T^n(x)\}, n \in \mathbb{N}$ *converge to* z *for any* $x \in M$.

d.- $p(T^n x, z) \leq \frac{k^n}{1-k} p(x, Tx)$ *for all* $x \in M$.

e.- T *is w-asymptotic regular.*

f.- *Each two elements* $x, y \in M$ *are asymptotic equivalente under* T.

Proof.
Since T *is a w contraction there exist a w-distance* p *on* M *and* $k \in [0, 1)$ *such that*

$$p(Tx, Ty) \leq kp(x, y), \ \forall x, y \in M.$$

Let $x \in M$ *and define* $x_n = T^n x$, *for any* $n \in \mathbb{N}$. *Then we have, for any* $n \in \mathbb{N}$,

$$p(x_n, x_{n+1}) \leq k^n p(x, Tx) \tag{1}$$

For any m *and* n *with* $m > n$ *we have*

$$p(x_n, x_m) \leq \frac{k^n}{1-k} p(x, Tx) \tag{2}$$

By lemma 1.1, x_n is a Cauchy sequence in M. Since M is complete, x_n converges to some point $z \in M$ so

$$T^n x \to z \tag{3}$$

Since $z_m \to z$ and $p(x_n, \cdot)$ is lower semicontinuous, we have

$$p(x_n, z) \leq \liminf_{m\to\infty} p(x_n, x_m) \leq \frac{k^n}{1-k} p(x, Tx) \tag{4}$$

that is,

$$p(T^n x, z) \leq \frac{k^n}{1-k} p(x, Tx) \tag{5}$$

and lemma 1.1,

$$\lim_{n\to\infty} p(x_n, z) = 0. \tag{6}$$

On the other hand,

$$p(x_n, Tz) = p(Tx_{n-1}, Tz) \leq kp(x_{n-1}, z)$$

so

$$\lim_{n\to\infty} p(x_n, Tz) = 0. \tag{7}$$

From (6), (7) and lemma 1.1 we conclude

$$Tz = z. \tag{8}$$

Further,

$$p(z, z) = p(Tz, Tz) \leq kp(z, z)$$

and hence

$$p(z, z) = 0. \tag{9}$$

If $y = Ty$ then

$$p(z, y) = p(Tz, Ty) \leq kp(z, y)$$

hence

$$p(z, y) = 0 \tag{10}$$

So, from (9), (10) and lemma 1, $z = y$. Therefore, a fixed point of T is unique.
From (1) we have

$$p(T^n x, T^{n+1} x) \leq k^n p(x, Tx).$$

Hence for all $x \in M$

$$\lim_{n \to \infty} p(T^n x, T^{n+1} x) = 0.$$

Thus all elements of M are w-asymptotic regular under T so T is w-asymptotic regular.

Finally, let $x, y \in M$, $x \neq y$ and

$$p(T^n x, T^n y) \leq k^n p(x, y).$$

Thus, for any $x, y \in M$

$$\lim_{n \to \infty} p(T^n x, T^n y) = 0$$

and x and y are w-asymptotic equivalent under T. ■

It is clear that theorem 2.1 is a generalization de Banach Contraction Principle.

Theorem 2.2.
Let (M, d) be a complete metric space and let T be a mapping from M into itself such that T^m is a w-contraction for some $m \in \mathbb{N}$. Then T has a unique fixed point.

Proof.
Since T^m is a w-contraction for some $m \in \mathbb{N}$ there exists a w-distance p on M and $k \in [0, 1)$ such that

$$p(T^m x, T^m y) \leq k p(x, y)$$

for all $x, y \in M$.

By theorem 2.1, there exists a unique $z \in M$ such that $T^m z = z$ for some $m \in \mathbb{N}$ and

$$Tz = T(T^m z) = T^m(Tz)$$

it follows that $z = Tz$. ■

Remark.
In case that $p = d$ we have the Chu-Diaz's theorem [3].

The next results are generalizations of Maia's theorem [8] and [10].

Theorem 2.3.
Let M be a non empty set and d, ρ two metrics on M and $T : M \to M$ a mapping. Suppose that,

a.- $p(x,y) \leq q(x,y)$ for any $x, y \in M$, where p and q are w-distance on M defined from d and ρ respectively.

b.- (M,d) is a complete metric space.

c.- $T : (M,\rho) \to (M,\rho)$ satisfies

$$q(Tx,Ty) \leq kq(x,y)$$

for any $x, y \in M$ and $0 \leq k < 1$.

Then there exists $z \in M$ such that $z = Tz$. Further the point z satisfies $q(z,z) = 0$ and hence $p(z,z) = 0$.

Proof.
Let $x \in M$. From (c) the sequence $x_n = T^n x, n \in \mathbb{N}$ is a Cauchy sequence in (M,ρ) and by (b) it converges to a point $z \in M$. The rest of the proof is similar to that of theorem 2.1. ■

Now using the lemma 1.2 we get a generalization of theorem 2.3.

Theorem 2.4.
Let (M,d) be a metric space, $p : M \times M \to [0,+\infty)$ a w-distance on M and $T : M \to M$ a mapping. Suppose that,

a.- Let q be a function from $M \times M$ into $[0,+\infty)$ satisfaying (w_1) and (w_2) in the definition of w-distance such that

$$p(x,y) \leq q(x,y)$$

for all $x, y \in M$.

b.- (M,d) is a complete metric space.

c.- $T : M \to M$ satisfies

$$q(Tx,Ty) \leq kq(x,y)$$

for all $x, y \in M$ and $0 \leq k < 1$.

Then there exists $z \in M$ such that $Tz = z$. Moreover, the point z satisfies $q(z,z) = 0$ and $p(z,z) = 0$.

Proof. *By (a) and lemma 1.2 we have that q is a w-distance on M. The remain of the proof is equal to theorem 2.3.* ■

In [4] Edelstein introduced the following,

Definition 2.3.

Let (M,d) be a metric space. A mapping $T : M \to M$ is called (ϵ, k)-uniformly locally contractive if there exist a $\epsilon > 0$ and k with $0 \leq k < 1$ such that

$$d(x,y) < \epsilon \Rightarrow d(Tx,Ty) < kd(x,y)$$

for each $x, y \in M$.

The following theorem gives a generalization of the Eelstein's fixed point theorem on an ϵ-chainable metric spaces.

Theorem 2.5.

Let $\epsilon \in (0,+\infty)$ and let (M,d) be a complete an ϵ-chainable metric space. Suppose that a mapping $T : M \to M$ is (ϵ,k)-uniformly contractive. Then T has unique fixed point .

Proof.

Define a function p from $M \times M$ into $[0,+\infty)$ as follows:

$$p(x,y) = \inf\{\sum_{i=0}^{k-1} d(x_i,x_{i+1}) : \{x_0,...,x_n\} \text{ is a } \epsilon - \text{chain linking } x \text{ and } y\}.$$

From lemma 1.3, p is a w-distance on M. We prove that T satisfies the following condition:

$$p(Tx,Ty) \leq kp(x,y),$$

for any $x, y \in M$ and $0 \leq k < 1$.

Given $x, y \in M$ and any ϵ-chain $\{x_0,...,x_n\}$ with $x_0 = x$ and $x_n = y$, we have $d(x_i,x_{i+1}) < \epsilon$, $i = 0,1,...,n-1$, and hence

$$d(Tx_i,Tx_{i+1}) \leq kd(x_i,x_{i+1}) < k\epsilon \leq \epsilon, i = 0,1,...,n-1,$$

so $Tx_0, Tx_1, ..., Tx_n$ is a ϵ-chain joining the points Tx and Ty and

$$p(Tx,Ty) \leq \sum_{i=0}^{n-1} d(Tx_0,Tx_{i+1}) \leq k\sum_{i=0}^{n-1} d(x_0,x_{i+1}).$$

$\{x_0, ..., x_n\}$ *being an arbitrary* ϵ*-chain, we have* $p(Tx, Ty) \leq kp(x, y)$. *Hence by theorem 2.1(a),* T *has a unique fixed point* $z \in M$, *i.e.,* $Tz = z$. ■

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