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"APPLICATION AND CONSTRUCTION

OF RIGID SPACES II"

NOTAS DE MATEMATICA

N° 18

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BY

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1978

ABSTRACT

Theorem A: Given an abstract group G, a subgroup H of G and a metric space X, one can construct a bigger metric space X^* containing X as a closed subspace such that:

i) the homeomorphism group of X^* is isomophic to G and

ii) the isometry group of X^* is isomorphic to H.

Theorem B: The extremally disconnected rigid spaces exist in such an abundance that every topological space is a quotient of one such space.

Theorem C: Let X be any infinite set and let f be any function from X into the set of all cardinal numbers not exceeding the cardinality of X. Then there is a connected (metrizable) topology on X such that for each x in X, the subspace $X\setminus\{x\}$ has exactly $f(x)$ connected components.

Besides proving these theorems several methods of constructing rigid extensions of spaces are discussed as tools for more important results.

INTRODUCTION

This is a continuation of our previous work [9]. The sections and theorems have been so numbered as to reveal this fact. Also, the results of [9] will be freely applied here, with due reference to sections in which they occur. When reference is made to a section or a theorem just by its number, without stating the paper in which it appears, then we shall understand that it is either in the present paper or in its predecessor [9].

The first five sections of this paper deal with extending a given space to a rigid space. This may be considered as a counterpart of [18]. In the last two sections, we give two applications of these constructions inside topology itself.

For any group G, let $\underline{C}(G)$ be the class of all topological spaces having its full homeomorphism group isomorphic to G. It has been proved by J De Groot [4] that $\underline{C}(G)$ is nonempty for each G. Here we improve his result by showing that for each G, the class $\underline{C}(G)$ is large in several senses of the term:

i) $\underline{C}(G)$ is so large that subspaces of its members exhaust all spaces.

ii) $\underline{C}(G)$ is so large that on any infinite set of cardinality $m \geq |G|$, there are 2^{2^m} topologies belonging to it.

iii) $\underline{C}(G)$ is so large that for any infinite connected space X, there is a special type of quotient map from a sum of copies of X onto some member of $\underline{C}(G)$.

It is known that for any metric space, all its isometric self-injections form a subgroup of its homeomorphism group. It is natural to ask whether there are any more relations between the isometry group and the homeomorphism group. Should it be a special kind of subgroup? We prove in the last section that the isometry group is independent of the homeomorphism group, but for the fact that it is a subgroup. The precise statement of a more general result has been given as Theorem A in the abstract above. This result has been already proved in [4] in the very special particular case that G = H and X has only one point.

Many Theorems of this paper speak of the abundance of rigid spaces combined with a special topological property. To mention an example here, let us consider the property of extremal disconnectedness. The existence of an extremally disconnected rigid space has been proved only recentrly [17]. But we show that they are in plenty in the following senses: Let E be the class of all extremally disconnected rigid spaces. Then

(i) Quotients of E exhaust all topological spaces.

(ii) Subspaces of E exhaust all extremally disconnected spaces and include some more.

(iii) On any infinite set there are as many topologies belonging to E, as there are topologies not belonging to E.

Theorem 3.2.2 (See Corollary 3.2.3) answers a question of Nix posed in [16] and Theorem 3.3.3 answers a question of DeGroot and McDowell [5] and improves a result of Lozier [15]. The results of this paper have also applications in the study of automorphism groups of algebraic structures. They will be discussed in a later paper [10].

Two of the results of this paper are among the nine Theorems announced in [8].

Now we summarize our results on rigid extensions. While proving that every topological space possesses a rigid extension we simultaneously ask how nice this extension can be? What topological properties can be preserved? What are bound to be lost? What can be newly gained? Considering rigid extensions with a peculiar property, can we simultaneously have rigidity for a larger class of maps? Can we control the cardinality of rigid extension in the presence of that topological property? We describe here several processes of rigid extensions in order to answer all these questions. All these processes stem out of the C-process described in [9]. Some of the results proved are as below:

If X is a topological space, it always has a rigid extension X^*. This X^* can always be chosen to be connected and locally connected. In addition, X^* can be so chosen that each of the following properties is preserved in the extension process: Separation axioms (i.e. T_1-axiom, Hausdorffness, Urysohn axiom, regularity, complete regularity, normality and complete normality), matrizability, first countability and any coreflective property (that is property preserved by sums and quotients). (see § 3.1., 3.2 and 3.4). On the other hand X^* can be chosen to be totally disconnected or zero-dimensional according as X is (see § 3.3.). In all these cases, if X is infinite, we can chose X^* to have the same cardinality as X (See § 3.1., 3.2 and 3.3). If we are ready to forego this condition of cardinality, we can choose X^* to be compact when X is Tychonoff (See § 3.3). For a consolidated gist of such results, see also Theorem 3.5.6. In § 3.5, we show that X^* can be chosen to be so pathalogical that it has no non-trivial continuous self-maps.

Next, we quote two results of this paper, concerning rigid extensions, which seem interesting. We state them in a form, different from the one in which they are given inside.

1) Give us, any infinite separable space, (possibly with plenty of self-homeomorphisms). We shall then choose a suitable subspace of βN and join the two together in such a way that the final space is rigid (See Theorem 3.3.5 for a precise statement). One remark is relevant here: the subspace of βN that we choose, varies (it has to do so) with the space that you give. The surprising thing is that it may be pre-fixed before hand, if your space is assured to be connected or metrizable.

2) We call a Hausdorff space to be strongly rigid, if every continuous self-map of it is either identity or a constant. J. DeGroot [4] first proved the existence of such a space. Answering a question of Herrlich, we proved in [9] that such spaces can have any infinite cardinality. Here, we show that the quotients of subspaces of such spaces exhaust all topological spaces. It may be observed that quotients alone or subspaces alone cannot do this job.

In this paper we have elaborately discussed more than half a dozen methods of constructing rigid extensions of spaces. We wish to emphasize at present that they are all meant only as tools for certain more significant results: For example, remark 3.1.7 and Corollary 3.2.3 explain where the first three methods are used. The latter method of § 3.2 is used in § 3.6 and § 3.7 while discussing homeomorphism groups; besides it will also be used in a later chapter for a similar problem. The first method of § 3.3 will be used in a succeeding paper to answer a question concerning rigid Boolean algebras; also see Remark 3.3.4 for its significance. The method of § 3.4 has been effectively used in 3.4 to prove Theorem A of the abstract, concerning isometry groups.

3.1. Connected Rigid Extensions

In this section, we give two methods of constructing connected rigid extensions, each having an advantage over the other. (See Remark 3.1). The results of this section will later be used to answer a posed problem.

THEOREM 3.1.1. Every Hausdorff space is homeomorphic to a closed subspace of a connected Hausdorff space rigid for continuous bijections.

Proof. Let X be any Hausdorff space. First, we embed X as a closed subspace of a connected Hausdorff space, with no cut points. This can be done as follows: Let T be the unit circle in the plane, with usual topology, i.e. $T = \{(x,y) \in R X R / x^2+y^2 = 1\}$. For each x in X, take a copy T_x of T and fix a homeomorphism

$h_x : T \to T_x$. Consider $S = X + \Sigma_{x \in X} T_x$. In this discrete topological sum, make the following identifications:

i) Identify each x in X with the point $h_x((1,0))$ of T_x.

ii) Collapse all the points of $\{h_x((-1,0)) \mid x \in X\}$ to a single point.

iii) Similarly collapse the orbit $\{h_x((0,1))/x \in X\}$ to a single point.

Let ϕ be the quotient map thus defined and let Y be the quotient space. The equivalence relation defined by ϕ can be easily checked to be a closed subset of $S \times S$. It follows that Y is a Hausdorff space. It can also be checked easily that the restriction of ϕ to X and to each of the spaces T_x is a homeomorphism onto a closed subspace of Y. Further $Y = \bigcup_{x \in X} \phi(T_x)$. Here, each $\phi(T_x)$ is a connected space with no cut point. Also $\bigcap_{x \in X} \phi(T_x)$ has two points. It follows that Y is a connected space with no cut points.

Our second step is to construct for each cardinal m, a space X_m with a base-point x_m such that

(i) X_m is a connected Hausdorff space.

(ii) x_m is the only cut point of X_m.

(iii) the cut-point-order of x_m in X_m is m.

(If p is a point of a topological space P, then the cut point order of p in P is defined as the cardinality of the set of all connected components of $P\setminus\{p\}$). Such a space X_m can be constructed easily as follows: Take m copies of the circle T, take their disjoint sum, choose one point from each copy and collapse these chosen points to a single point x_m and call the resulting space as X_m. Then it is not hard to check that it satisfies the above requirements.

Thirdly, we fix any point y_o of Y (the space constructed in the first step). We take a C-system $(\{(X_\alpha, x_\alpha)|\alpha\varepsilon(J,0)\}, f)$ (See chapter I) where

(i) (X_o,x_o) coincides with (Y,y_o).

(ii) For each $\alpha \neq 0$, there exists a cardinal m such that (X_α,x_α) is the same as (S_m,x_m) and

(iii) For distinct points of J, the corresponding cardinals satisfying ii) are distinct.

Geometrically this means that the first base space is Y and the other base spaces are from the X_m's and no two distinct base spaces are homeomorphic. Let Z be the C-space so obtained. We shall show that Z is a required extension of X.

Since each base space is a connected Hausdorff space, so is Z (by Propositions 1.3.1 and 1.3.3). Also Y (and hence X also) is homeomorphic to a closed subspace of Z.

Next, we show that distinct points of Z have distinct cut-point orders. Let X be any point of Z different from the base point y_o of Y contained in Z. We shall show that $Z\ \{x\}$ has exactly $m + 1$ connected components where m is the cardinal such that X_m is the base space at x. If C is a connected component of $X_m\ \{x\}$, then C^* (as in Notation 1.1.11) is a C-process- space with connected base spaces and hence connected (Proposition 1.3.3); also each C^* is both open and closed in $Z\ \{x\}$. (This follows from Propositions 1.1.3 and 1.1.4 and from the following fact: In any of the base spaces, if any one point is removed, the connected components are open); further the point x is a non-cut-point in the base space in which it is a non-base point and therefore $Z\ \{x\}^*$ is also a C-process-space with connected base spaces and hence $Z\ \{x\}^*$ is connected. Thus $Z\ \{x\}$ is a union of $m + 1$ disjoint connected subsets each of which is both open and closed in $Z\ \{x\}$. Consequently the cut-point-order of x in Z is exactly $m + 1$. It is seen that the cut-point-order of y_o in Z is exactly 1. (it is a non-cut-point). Hence it follows that distinct points of Z have distinct cut-point-orders.

Now we complete the proof of the theorem by showing that such a space has to be rigid for continuous bijections. Let $f : Z \to Z$ be any continuous bijection. Then using the invariance of connectedness under continuous surjections, we see that f cannot increase the cut-point-order at any point. In other words, for each x in Z, the cut-point-order of x must be greater than or equal to that of $f(x)$. If we denote by Λ the set of all cardinals that appear as the cut-point-order of points of Z, then the previous paragraph establishes a bijection between A and Z. When composed with this bijection on both sides f yields a bijection $\tilde{f}$ from A to A with the property $\tilde{f}(m) \leq m$ for each m in A. However this is impossible (sinde A is well-ordered), unless $\tilde{f}$ is the identity map. This implies that f is the identity map.

<u>Remark 8.1.2</u>. It can be proved that the extensions constructed above has some additional rigidity properties:

(i) Every one-to-one continuous map from Z into itself fixes every interior point of its range. Consequently, Z is rigid for continuous injections with open range.

(ii) Z is chatoic; that is no two nonempty disjoint open subspaces of Z are homeomorphic. (See Remark 3.2.3).

<u>Remark 3.1.3</u>. Incidentally we see that the following can be proved by similar methods: Let m be any infinite cardinal and let $n \to a_n$ be any function from the set of all cardinals $\leq m$ into itself. Then the following are equivalent:

(i) There exists a connected Hausdorff space of cardinality m such that for each $n \leq m$, there are exactly a_n points having cut-point-order.

(ii) $m = \sum_{n \leq m} a_n$.

The above assertion follows from the next result also which is a restatement of Theorem C of the abstract.

THEOREM 3.1.3. <u>Let X be an infinite set of cardinality m_o and let Y be the set of all cardinal numbers $\leq m_o$. Let f be any</u>

function from X into Y. Then f can be realized as the cut-point-order function, for a suitable topology on X.

Proof. Let Z be any connected Hausdorff space of cardinality m_o, having no cut points. For each m in Y, we let the space Z_m to be the one obtained by taking $m - 1$ (this is equal to m if m is infinite) copies of Z, choosing one point in each copy and identifying all these chosen points into a single point. Note that in Z_m, there is only one cut-point and that its cut-point-order is exactly $m - 1$. Further $|Z_m|$ is easily seen to be m_o.

Now consider a partition of the given set X into a sequence $X_1, X_2, \ldots, X_n, \ldots$, of subsets of equal cardinality.

We recursively construct spaces $Z^{(1)}, Z^{(2)}, \ldots$ such that each contains its predecessor as a subspace and such that $Z^{(n)}\ Z^{(n-1)}$ has cardinality m for every natural number n. Suppose we have constructed $Z^{(1)}, \ldots, Z^{(r)}$ where r is some natural number. Then we take any fixed bijection b_r from $Z^{(r)}\ Z^{(r-1)}$ to X_r.

If Z is a point of $Z^{(r)}\ Z^{(r-1)}$, we attach to this point, a copy of $Z_{f(b_r(z))}$; the hinging is made at the special point of this space, so that the point z has cut-point-order $f(b_r(z))$ in the new space. When this is done for each point of $Z^{(r)}\ Z^{(r-1)}$, we finally get a bigger space $Z^{(r+1)}$.

By induction, we get an increasing sequence $Z^{(r)}$ of spaces and we take Z^* to be its direct limit. We naturally get a bijection b from Z^* to X which coincides with b_r for each $Z^{(r)}$ $Z^{(r-1)}$. We transfer the topology of Z^* to X via this bijection. Then this is a connected Hausdorff topology on X such that X $\{x\}$ has exactly $f(x)$ connected components, for each x in X,

<u>Remark 3.1.3(b)</u>. In particular, if m_o is such that there are m_o cardinal numbers smaller than m_o(e.g., the first infinite cardinal number), then we can take the special case that f is an one-to-one function. In this case, the topology on X constructed above, has to be rigid for continuous bijections.

Remark 3.1.4. Actually, we have not made the essential use of Hausdorffness. We have shown that every topological space can be embedded in a rigid space.

Remark. It should be admitted that the extension constructed above, is in general, too huge. But in some special cases, this drawback can be over-come. For example, we can prove:

THEOREM 3.1.5 *Let X be any infinite separable Hausdorff space. Then there exists a separable Hausdorff space X^* with the following properties:*

a) *X is homeomorphic to a closed subspace of X^*.*

b) $|X^*| = |X|$.

c) *X^* is rigid for continuous bijections.*

We shall only sketch the proof here. Imitate the proof of

Theorem 3.1.1., with the following changes: Instead of attaching a copy of the circle T to each point of X, attach copies of T, only at the points of a countable dense subset of X. Then the space Y constructed as in Theorem 3.1.1. (but for the change mentioned above), will be a connected separable space with no cut-points. (For, it contains one such dense subspace). Let the spaces X_m be constructed as in Theorem 3.1.1. for each finite cardinal m. For each point of a countable dense subset of Y, attach a copy with m odd. Thus we get space Y_1. If for some positive integer Y_m has been already constructed such that Y_{n-1} Y_n, then to each point of a countable dense subset of Y_n Y_{n-1} attach a copy of X_m where m is an integer divisible by 2^n but not by 2^{n+1}. At each stage, take care that distinct X_m's are attached at distinct points. Let X^{I} be the direct limit of the spaces $Y_1, Y_2, \ldots$ Then it can be proved that X^{*} contains a countable dense subset, distinct points of which have distinct cut-points-orders in X^{*}, such that all points outside this dense subset are non-cut-points. It can be checked that this implies that X^{*} is rigid for continuous bijections.

Now the cardinality of X^{*} is $\max(|X|, \mathbf{c})$. Hence the theorem is proved for spaces of cardinality $\geq c$.

For cardinalities $\leq c$, the same method works, when the circle T is replaced by any countable connected Hausdorff space without cut-points. (e.g. the space of Bing |2|).

Remark. However, the separability of X can be dropped from the hypothesis of Theorem 3.1.5, if we are satisfied with rigidity only for homeomorphisms. This is proved below:

THEOREM 3.1.6. *Every infinite Hausdorff space is homeomorphic to a closed subspace of a connected Hausdorff rigid space of same cardinality.*

Proof. First, we show that for each infinite cardinal m, there exists a family $\underline{A}_m$ of topological spaces with the following properties:

(i) Each member of $\underline{A}_m$ is a connected Hausdorff space of cardinality m with no cut-points,

(ii) No two distinct members of $\underline{A}_m$ are homeomorphic,

(iii) $|\underline{A}_m| \geq 2^m$.

The existence of such a family $\underline{A}_m$ can be proved as follows:

Let m be any infinite cardinal. Then as we have seen in the proof of Theorem 3.1.5 every Hausdorff space of cardinality m can be embedded in a connected Hausdorff space of cardinality m with no cut-points. Evidently, each such space can contain at most 2^m types of subspaces. Hence we will be through if we show that there are 2^{2^m} distinct Hausdorff topological types of cardinality m. But this is easily done, because we know that there are 2^{2^m} types of maximal nondiscrete topologies $(D \ \{p\}$ where p is an element of $\beta D \ D$ where D is a discrete space of cardinality $m)$ of cardinality m.

Our second step will be to construct the required rigid extension. Let X be any infinite Hausdorff space and let $|X| = m$. Then embed X in a connected Hausdorff space Y of cardinality m with no cut-points (Say, as in the proof of Theorem 3.1.5). Then construct by c-process, a space X^* such that:

1) Y is a base-space of X^*,

2) All other base-spaces of X^* are chosen from the family $\underline{A}_m$, and

3) No member of $\underline{A}_m$ is chosen more than once as a base space.

Our third step is to show that in such a space (that is in a C-space in which base spaces are connected Hausdorff spaces with no cut-points), the property of being a abase-space is topological. That is the image of any base base-space under any self-homeomorphism of X must itself be a base-space. This is proved by showing that the base-spaces are precisely the maximal connected subsets with no cut-points.

Let A be any nomempty connected subset of X^* with no cut-points. Let x be any point of A. Let h(x) be defined as in 1.1. (That is h(x) is the base-point of that base-space in which x is a non-base-point). Consider $\{h(x)\}^*$ (For notation, see 1.1). Since $\{h(x)\}^*$ $\{h(x)\}$ is easily seen to be open and closed in X^* $\{h(x)\}$, conclude that A is contained in $\{h(x)\}^*$. Also for each Y in A, Y^* $\{Y\}$ is open and closed in X^* $\{Y\}$ whereas A $\{Y\}$ is connected. Hence A is either contained in Y^* or disjoint with Y^* $\{Y\}$. Now if B is a base-space such that $A \quad B$ has at least two elements, then the above facts imply that A is disjoint with $\{b\}^*$ $\{b\}$ for each b in B, but A is contained in B^*. It follows that A is contained in B itself. Thus every connected subset of X^* with no cut-points, is contained in some base space. Since each base-space is a connected subset with no cut-points in itself, our assertion of the last paragraph has been proved.

Finally, we see as usual that X is homeomorphic to a closed subspace of X^*. We complete the proof by showing that X^* is rigid. If h is any self-homeomorphism of X^*, then by what we have seen just now, h must carry base-spaces onto base-spaces; but on the other hand, no two distinct base spaces are homeomorphic, by our choice of them. Hence it follows that each base-space is left invariant under h. Now if x in X^* is an arbitrary point different from the unique non-cut-point 0, then there are exactly two base-spaces B_1 and B_2 to which x belongs. Further $B_1 \quad B_2 = \{x\}$. Since $h(B_1) = B_1$ and $h(B_2) = B_2$, it follows that $h(x) = x$. Thus h must be the identity map.

(We note that in this space X^*, all the points with the exception of a unique non-cut-point, have cut-point-order 2. Hence, for the proof of rigidity, we cannot repeat the arguments of the earlier theorems of this section. The argument of the previous paragraph can however be given to earlier theorems also, but then we will be losing the stronger result, viz. rigidity for continuous bijections).

Remark 3.1.7. We conclude this section with some remarks on the relative merits and demerits of the two methods of rigid extensions discussed here:

1) The first method (that is the one described in Theorem 3.1.1) gives rigidity for a larger class of maps, see also Remark 3.1.2).

2) The first method gives information about the 'distribution of cut-point-orders in a connected space. (See Theorem 3.1.3).

3) The second method (that is, the one described in Theorem 3.1.6) gives rigid extensions, without any increase in cardinality in all cases. In the first method, this was achieved only for separable spaces.

4) In both the methods all separation axioms are preserved by the extension.

5) The first method alone is applicable to answer a question posed in |16| (see 3.1).

6) The second method alone is applicable to obtain the main results of the next chapter and thereby to construct curious intervals in the lattices of topologies.

3.2 Locally Connected Chaotic Extensions

In the last section we constructed connected rigid extensions for arbitrary spaces. The purpose of this section is to improve the results of the previous section, by showing that the connected rigid extension can be chosen to be 'more connected' and 'more rigid' simultaneously. As a by-product we shall answer a question of |16|.

The stronger form of rigidity considered in this section is known as chaoticity. This notion has been considered in |16| and |1| in the following form: A Hausdorff space X is chaotic, if whenever x and Y are distinct points of X, there exist open neighbourhoods V_x and V_Y of x and Y respectively in X, such that no open subset of V_x is homeomorphic to an open subset of V_Y. It will be useful for us to look at this concept in nicer ways:

PROPOSITION 3.2.1. <u>The following are equivalent for a Hausdorff space</u> X:

(i) X <u>is chaotic</u>

(ii) <u>No two distinct open subsets of</u> X <u>are homeomorphic</u>

(iii) <u>No two disjoint open subsets of</u> X <u>are homeomorphic</u>.

The proof is not difficult and hence omitted.

Now, we proceed to the two main results of this section, which improve Theorem 3.1.1 and Theorem 3.1.6 respectively.

<u>THEOREM 3.2.2. Let</u> X <u>be any Hausdorff space. Then there exists a connected locally connected chaotic space</u> X^* <u>such that</u> X <u>is homeomorphic to a closed subspace of</u> X^*.

Further, X^* can be so chosen that the following conditions are satisfied:

(i) if X is infinite and separable, then $|X^*| = |X|$.

(ii) X^* is rigid for continuous bijections.

(iii) If $|X| \geq c$, then X^* satisfies all separation axioms satisfied by X.

(Note that we cannot look for the conclusion of (iii) when $|X| < c$. For, each infinite connected Tychonoff space must have cardinality $\geq c$).

Proof. Obviously it suffices to prove the theorem for infinite spaces. Our first step is to select for each infinite cardinal m, a space S having the following properties:

(i) $|S| \leq m$.

(ii) S is a connected locally connected Hausdorff space.

(iii) There exists a base for S, such that no basic open subset of S has a cut-point.

(iv) If $m \geq c$, then S is completely normal.

(v) S is separable.

The existence of such a spaces is easily proved. For example, for $m \geq c$, we may take S to be any region of the plane, say, open unit disc. If $m \leq c$, we may take S to be any of the countable spaces constructed in |12|.

The second step is to embed the given space X in a space Z satisfying the conditions (i) to (iv). This can be achieved as follows: Let $J = \{(x_1,x_2) \in X \times X \mid x_1 \neq x_2\}$. For each (x_1,x_2) in J, let $S_{(x_1,x_2)}$ be a copy of the space S chosen in the previous paragraph (corresponding to $m = |X|$). Consider the disjoint topological sum $X + \Sigma_{(x_1,x_2) \in J} S_{(x_1,x_2)}$.

For each (x_1,x_2) in J, choose any two distinct points of $S_{(x_1,x_2)}$ and identify them respectively with x_1 and x_2 in X. Let Z be the quotient set thus obtained and let ϕ be the quotient map. Now the topology on Z that will be of our interest is obtained by weakening the quotient topology on Z only at the points of $\phi(X)$. If X belongs to $\phi(X)$, then a basic neighbourhood of x is defined as $\phi(W)$ where W is a union of the following sets:

(a) an open neighbourhood V of t in X, where t is the unique point of X such that $\phi(t) = x$.

(b) all $S_{(t,v)}$'s with v in V.

(c) Some connected open neighbourhood $V_{(x',Y)}$ of x' for each Y in $X\{t\}$ where x' is the unique element of $S_{(t,Y)}$ such that $\phi(x') = x = \phi(t)$.

Then it can be shown that this specifies a topology on Z satisfying our requirements.

Since the cumbersome notations have concealed the idea here, we prefer to give a geometric description of the above construction now. To each ordered pair of distinct points of X, we have attached a copy of S hinged at these points. A neighbourhood of x in X (in this big space) contains the union of the following:

(a) the set of all points that are near x in X

(b) the set of all points that are near x in each copy of S hinged at x and

(c) all the copies of S that are hinged to x at a point near to x.

Finally, we show that the extension X^* of X has the required properties. The local connectedness of X^* follows from the fact that it is a quotient of a sum of copies of Z, S and T. All the requirements stated in the theorem, except chaoticity, can be easily proved to be true in X^*, along the lines of the corresponding proofs in 3.1.

We complete the proof of the theorem by showing that X^* is chaotic. Let us have the notation that if E is a subset of a topological space Y and x is a point of E, then $C_E(x)$ is the cut-point-order of x in E where E receives the relative topology from Y. Let V_1 and V_2 be any two disjoint basic open subsets of X^* and let x_1 and x_2 be points of V_1 and V_2 respectively. Then because of condition (iii) satisfied by S, we can show that

$$C_{V_1}(x_1) = C_{X^*}(x_1) \quad \text{and} \quad C_{V_2}(x_2) = C_{X^*}(x_2).$$

But since distinct elements of X^* have distinct cut-point-orders it follows that $C_{V_1}(x_1) \neq C_{V_2}(x_2)$. Consequently V_1 and V_2 cannot be homeomorphic. It follows that no two disjoint open subsets of X^* can be homeomorphic.

Thus the space X^* satisfies all the properties mentioned in the statement of the theorem, except (i). As remarked in 3.1.5, this X^* is in general too huge. In case X is separable, we can achieve (i) by allowing certain modifications in the construction of X^* as follows:

Let D be a countable dense subset of X. Let $I = \{(x,Y) / x\varepsilon D, Y\varepsilon D, x \neq Y\}$. Construct a space Z_1 exactly as we constructed Z, with the only change that I plays the role of J. Then Z_1 can be checked to be connected locally connected Hausdorff extension of X. Further Z_1 is the union of the space X and a countable number of copies of S. Consequently Z_1 is separable. Now embed this Z_1 as a closed subspace of a separable connected rigid space as described in Theorem 3.1.5. This extension has the required properties.

COROLLARY 3.2.3. Real line can be embedded in a completely normal connected locally connected chaotic space of same cardinality. This answers all parts of the following question of |16|:

(a) Do chaotic spaces exist?

(b) Do chaotic spaces of cardinality c exist?

(c) Do there exist completely normal connected locally connected chaotic spaces?

In fact, we have very strong answers:

(a) Chaotic spaces of arbitrarily large cardinalities exist.

(b) There are 2^{2^c} distinct types of chaotic spaces of cardinality c (i.e.) as many as there topological types of cardinality c).

(c) The family of completely normal connected locally connected chaotic spaces is a class, and not a set.

COROLLARY 3.2.4. A family F of topological spaces is said to be large in the first sense if for each infinite cardinal m, there exist 2^{2^m} distinct topological types of members of F; it is said to be large in the second sense, if the subspaces of its members exhaust all topological spaces.

Let C be the class of all connected locally connected T_2 rigid spaces. Then C is large in both the senses.

When the starting space X is not separable, the chaotic extension X^* constructed above is too huge. So, we ask whether there are locally connected chaotic extensions for arbitrary Hausdorff spaces, without any increase in cardinality. We have an affirmative answer:

THEOREM 3.2.5. Every infinite Hausdorff space is homeomorphic to a closed subspace of a connected locally connected chaotic space of same cardinality.

Proof. First, embed the given space, as a closed subspace, in a connected locally connected Hausdorff space Y of some cardinality having no cut points. This is possible as shown in the first two steps of the proof of Theorem 3.2.2.

Next, if m is the cardinal of the given space, take a family $\underline{B}_m$ such that (i) each member of $\underline{B}_m$ is a connected locally connected Hausdorff space of cardinality m, without cut-points.

(ii) no two distinct members of $\underline{B_m}$ are homeomorphic.

and (iii) $|\underline{B_m}| = m$,

The existence of such a family can be proved exactly in the same way as we proved the existence of the family $\underline{A_m}$ in the proof of Theorem 3.1.6.

Thirdly, we construct a C-space Y^* with the following conditionsL

(a) The first level base space is Y.

(b) All the base spaces are chosen from $\underline{B_m}$

and (c) No two distinct base spaces are homeomorphic.

Next, we weaken the topology of Y^* as follows: Let Y be a general point in Y^*. Then there are exactly two base-spaces, say B_1 and B_2 that contain Y. Of these, exactly one, say B_1, has Y as its base point. Now let V_Y be the family of all sets of the form $(V_1 \quad V_2)^*$ (that is, the set of all points lying above some point of V_1 or V_2) where V_1 and V_2 are subsets of Y^* such that the following hold:

1) $Y \varepsilon V_1 \quad V_2$

2) V_1 is an open subset of B_1; V_2 is an open subset of B_2.

3) V_1 and V_2 are connected.

and 4) The base point of B_2 does not belong to V_2.

Then the following can be checked without much difficulty:

a) If for each Y in Y^* we declare the V_Y's to form the neighbourhood system at Y, then we get a coarser topology on Y^*.

b) This topology is connected, locally connected and Hausdorff.

c) On each base-space, the topology is unchanged.

d) Every open subset contains at least one-base space completely.

e) If V is an open connected subset of Y^*, containing a base-space B completely, then B is a maximal connected subset of V without cut-points.

f) If $W = (V_1 \quad V_2)^*$ is a basic open set (where V_1 and V_2 satisfy conditions 1) to 4) and if $V = W \ (V_1 \quad V_2)$, then every subset of V, which is maximal with respect to being a connected subset of W without cut-points, must be a base-space. (For this, we have to imitate the corresponding proof of 3.1.).

Finally, we use the above facts to show that Y^* with this topology, is chaotic. Suppose V and W are two disjoint - connected open subsets of Y^* and $h : V - W$ is a homeomorphism. Assume that W is a basic open set. Then $W = (W_1 \quad W_2)^*$ where W_1 and W_2 satisfy conditions similar to 1) to 4). It is easily seen that $A = W \ (W_1 \quad W_2)$ has nonempty interior. Therefore $h^{-1}(A)$ must have nonempty interior. Therefore from d) above, we get that there is a base space B $h^{-1}(A)$. Also from e) above, we see that B is a maximal subset of V with respect to being a connected space without cut-points. Hence $h(B)$ must also be a maximal subset of W with respect to this property. But $h(B)$ is contained in W $(W_1$ $W_2)$, It follows from f) above that $h(B)$ must be a base-space contained in W. Since we have assumed V and W to be disjoint, $h(B)$ must be a base-space different from B.

But by our construction, no two distince base-spaces are homeomorphic. This contradiction proves that no open connected subset of Y^* can be homeomorphic to a disjoint basic open subset of Y^*. This is sufficient to prove the chaoticity of Y^*.

Remark 3.2.6. The extension constructed above, has the same cardinality as the original space; but it may fail to be rigid for continuous bijections. On the other hand the extension constructed in Theorem 3.2.1 is rigid for continuous bijections, but at times its cardinality is too large. Note also that in both types of extensions, all the separation axioms can be preserved, provided the original space has cardinality not less than $\mathfrak{c}$.

3.3. Totally Disconnected Rigid Extensions

We have proved the existence of connected locally connected rigid extensions for all spaces. The next natural question is whether each totally disconnected space possesses a totally disconnected rigid extension. Our earlier methods can be of no use here, since they heavily depended on arguments involving cut-point-order. But still, we can prove:

THEOREM 3.3.1: *Let X be any totally disconnected Hausdorff space. Then there exists a totally disconnected rigid Hausdorff space X^* containing X as a closed subspace. Further X^* can be so chosen that, if X is infinite, then X and X^* have the same cardinality.*

The construction of X^* consists in attaching suitable copies of zero-dimensional rigid spaces to the point of X. More precisely, to each x in X, we choose a space Z_x constructed in 2.4., consider $X + \Sigma_{x \varepsilon X} Z_x$, fix a point z_x in Z_x for each x in X, identify x with z_x for each x in X, and call the quotient space as X^*. Let ϕ be the quotient map.

But the Z_x'x must be suitably chosen. Recall that each Z_x was constructed as a C-space whose base-spaces are maximal nondiscrete topological spaces. We demand here that they should satisfy the following conditions also:

(i) Each base-space has the same cardinality as the given space X (if X is infinite).

(ii) If x and Y are distinct points of X and if $D \quad \{p\}$ and $D \quad \{q\}$ are any two base-spaces of Z_x and Z_Y respectively, then p and q (which are points in $\beta D \quad D$) should not be equivalent (in the sense described in 2.4.).

If X^* is constructed out of such a family Z_x of rigid spaces then we shall show that X^* is rigid. It is apparent

from our construction that outside $\phi(X)$, distinct points look differently. However since $\phi(X)$ is homeomorphic to the given space X, it is quite possible that it admits non-trivial homeomorphisms. Our contention is that still X^* has no nontrivial homeomorphisms. For this, we look for a topological property that distinguishes a part of $\phi(X)$ from its complement. We let A to be the set of all those points of X^* which possess extremally disconnected neighbourhoods. Since each Z_x is extremally disconnected (See 2.4), it can be shown that each point of $X^* \cap \phi(X)$ which is isolated in $\phi(X)$, must belong to A. We claim that these points exhaust A. To prove this, let x be a non-isolated point of $\phi(X)$. Let V be any neighbourhood of x in X^*. Since x is in the closure of $Z_x \setminus \{x\}$ it is clear that $W = V \cap (Z_x \setminus \{x\})$ is a nonempty open subset of V. Also Z_x is closed in X^* (here we use the fact that X is a T_1-space). Therefore $\overline{W} \cap \phi(X) \subset Z_x \cap \phi(X) = \{x\}$. On the other hand, since x is in the closure of $Z_x \setminus \{x\}$ and since V is a neighbourhood of x, we have that x belongs to $\overline{W}$. Thus $\overline{W} \cap \phi(X) = \{x\}$. This is not open in $\phi(X)$. Therefore $\overline{W}$ is not open in V. It follows that V is not extremally disconnected. Thus we have shown that if A is as defined, then $X^* \setminus A$ is precisely the set of non-isolated points of $\phi(X)$. Consequently, if $h : X^* \to X^*$ is any homeomorphism, $h(X^* \setminus A) = X^* \setminus A$. Now A is obviously a sum of copies of rigid spaces Z_x's, such that different copies are of distinct types.

Hence using condition (ii) of our choice of base-spaces and recalling the proof of rigidity of the space Z in 2.4., we can show that A is rigid. Therefore h is identity on A. But A is a dense subspace of X^*. Hence it follows that h is identity on the whole of X^*.

Thus X^* is a rigid extension of X. The other assertions of the theorem can be easily verified for this extension.

REMARK 3.3.2. *The above extension can be proved to preserve all the separation axioms. Moreover, if X is zero-dimensional, so is the rigid extension X^* constructed above. In fact, for zero-dimensional spaces, we can prove something more.*

THEOREM 3.3.3 *Every zero-dimensional Hausdorff space is a subspace of a compact rigid zero-dimensional Hausdorff space.*

Proof. Let X be any zer-dimensional Hausdorff space. We any may assume with-out loss of generality that X is infinite. Our method is to construct a zero-dimensional Hausdorff rigid extension of X and the consider a zero-dimensional compactification of it.

Of course, the rigid extension Z^* constructed in Theorem 3.3.1 is zero-dimensional. But however, we are not sure whether any of its zero-dimensional compactifications must be rigid. Therefore we make some modifications in the construction of X^*. This will be constructed in the same way as we did in the proof of Theorem 1.3.1; but instead of the Z_x's we shall consider different spaces T_x's. These T's will be constructed in the same way (i.e.; by C-process) as Z_x's were constructed in 2.4; but the base-spaces will be changed. Recall that the base-spaces for Z_x's were maximal nondiscrete topological spaces. Here, we shall have somethings else as base-spaces.

Let D be a discrete space and let $p \in \beta D \setminus D$. Take two disjoint copies of $D \setminus \{p\}$, take their sum and identify their non-isolated points. The resulting space is denoted by D_p. There will be no confusion, if we denote the unique non-isolated point of D_p again by p.

Now let $|X| = m$ and let $|D| = 2^m$. Then βD has $2^{2^{2^m}}$ points D has only 2^m subsets of cardinality not exceeding m and each subset can have only 2^{2^m} points in its closure. Consequently there are $2^{2^{2^m}}$ points in $\beta D \setminus D$ that are not in the closure of any subset of D having cardinality $\leq m$. Let A be the set of such points. Then consider a family $\underline{F}$ of topological spaces such that:

(i) $|\underline{F}| = 2^m$.

(ii) Each member of $\underline{F}$ is a D_p for some p in A.

and (iii) If D_p and D_q are in $\underline{F}$, then p and q are uncomparable in the sense of 2.4.

With the members of $\underline{F}$ as base-spaces, construct a C-space T.

Now the proof that T is rigid is just an imitation of the proof that Z is rigid (in 2.4.) provided we prove the following: If D_p and D_q are distinct members of $\underline{F}$, then every continuous map from D_p to D_q is locally constant. For this let $f = D_p \to D_q$ be continuous. We have to consider only the case when $f(p) = q$. Look at one branch (call it B) of $D \quad \{p\}$ that is contained in D_p. Then it is clear that p must be in the closure of $f(B \quad \{p\})$. Let B_1 and B_2 be the two branches of D_q. Let $C_1 = B \quad f^{-1}(B_1 \quad \{q\})$ and $C_2 = B \quad f^{-1}(B_2 \quad \{q\})$. Then C_1 and C_2 are disjoint. Therefore p belongs to the closure of at most one of them (since B is extremally disconnected): say $p \in \overline{C}_1$. It follows that $C_1 \quad \{p\}$ is an open subset of B which is mapped onto an open subset of B_1. But on the other hand, for spaces $D \quad \{p\}$; every open subspace is homeomorphic to the whole space. Therefore f gives rise to a continuous function $\tilde{f} : D \quad \{p\} \to D \quad \{q\}$ such that $\tilde{f}^{-1}(q) = p$. This however is impossible by our choice of $\underline{F}$. This contradiction proves that $p \notin \overline{C}_1$. Similarly, p cannot be in $\overline{C}_2$ also. Therefore $D_p \quad (C_1 \quad C_2)$ is a neighbourhood of p in D_p that is entirely mapped to q. Thus f is locally constant.

Now we construct X^* as in the proof of Theorem 1.3.1., with T playing the role of Z.

The proof that X^* is rigid also needs some essential changes. Now no point of X^* has an extremally disconnected neighbourhood. Hence, the argument of Theorem 3.3.1. cannot hold here. It is here that the second condition in our choice of $\underline{F}$ helps. We let S to be the set of all those points x in X^* such that x is in the closure of some subset of X^* $\{x\}$ which has cardinality m. Then it can be shown that S is precisely the set of non-isolated points of $\phi(X)$. Hence every self homeomorphism h of S must fix S. (i.e. $h(S) = S$), On the other hand X^* S can be proved to be rigid by our routine methods. This means that h must be identity on X^* S and hence on the whole of X^*.

Now let $S(X^*)$ be the maximal zero-dimensional Hausdorff compactification of X^*. Then the points x of X^* in $S(X^*)$ are distinguished by the property that they are in the closure of two disjoint open subsets of $S(X^*)$ $\{x\}$. Hence every self-homeomorphism of $S(X^*)$ must leave X^* invariant, and hence must be identity on X^* (since X^* is rigid) and hence on the whole of $S(X^*)$ (since X^* is dense). This proves that $S(X^*)$ is rigid.

Remark 3.3.4. The above theorem shows that compact rigid zero-dimensional spaces are abundant. Answering a question of G. Birkhoff |3|, the first such example was given by M. Katetov |14|. Answering a question of J. DeGroot and McDowell |5|, W. Lozier |15| proved that their cardinalities can go arbitrarily high. It is clear that theorem 3.3.3 improves all these results.

THEOREM 3.3.5 *If X is an ifinite separable Hausdorff space, then X can be embedded in a rigid separable space X^* of same*

<u>cardinality, such that $\beta X - X$ is homeomorphic to a subspace of $\beta N - N$.</u>

<u>Proof</u>. Our methods yield this result, on observing that every countable extremally disconnected space can be embedded in βN (see |13|).

3.4. Metrizable Rigid Extensions

Hitherto, we have constructed rigid extensions for arbitrary Hausdorff spaces, preserving several pleasing properties such as connectedness, local connectedness, total disconnectedness, - separation axioms, cardinality, etc. But none of these extensions would preserve first-countability, metrizability etc.

For, each of them is built out of C-process; and any such space is no-where first countable. (See 1.2.). One would naturally like to know then whether every metrizable space can be embedded in a rigid metrizable space. The purpose of this section is to give an affirmative answer to this question.

LEMMA 3.4.1. <u>Let X be a set which is a union of metric spaces, say $X = \bigcup_{\alpha \in J} X_\alpha$ where each X_α is provided with a metric d_α. If α and β belong to J, let d_α and d_β coincide on $X_\alpha \cap X_\beta$. Then there exists a metric d on X such that for each α in J, d is equivalent to d_α on X_α.</u>

<u>Proof</u>. If x and Y are two points of X, call them connectible to each other if there exists a finite sequence (called connection) $x = x_0, x_1, \ldots, x_n = Y$ such that every pair of successive terms of this sequence belongs to same X_α for same α in J.

Define:

$d(x,Y) =$ 1 if x and Y are not connectible to each other min $(1,d_1,(x,Y))$ otherwise.

where $d_1(x,Y) = \inf_c \{d_c(x,Y)\}$ where c ranges over all connections from x to Y and $d_c(x,Y)$ is defined as follows: If $c = \{x = x_o, x_1, \ldots, x_n = Y\}$ is one connection such that $x_i, x_{i+1} \varepsilon X_{\alpha_{i+1}}$ $i = 0, 1, \ldots, n-1$, then

$$d_c(x,Y) = d_{\alpha_1}(X_o,x_1) + d_{\alpha_2}(x_1,x_2)+\ldots+d_{\alpha_n}(x_{n-1},x_n).$$

The compatibility of d_α's in the intersections insures that d_c is well defined. To check that d is a metric, it is enough to check that d_1 is a metric. For this it is enough to verify the triangle inequality. This can also be verified with some easy computations.

For each α in J, we observe that the restriction of d to X_α is given by $\min(1,d_\alpha)$. Hence d and d_α are same on X_α.

THEOREM 3.4.2. Every metrizable space is homeomorphic to a closed subspace of a rigid metrizable space.

Proof. Let (X,d) be any metrizable space. First, imitating the proof of Theorem 3.1.1., embed X as a closed subspace of a connected space Y without cut-points. Now forget the topology of Y and look at its underlying set. It is a union of the space X and the spaces in the family $\{T_x \mid x \underset{Y}{\varepsilon} X\}$. When Y is thus represented as a union of metric spaces, we observe that

X $T_x = \{x\}$ for each x in X, and that T_x T_y has exactly three points when $x \neq Y$, and that the two metrics d_x and d_Y are compatible in this intersection. (Hece for each x in X, d_x denotes the usual metric on T_x). Hence by Lemma 3.4., there is metric d' on Y such that $d' = \min(1,d_1)$ on each T_x and $d' = \min(1,d)$ on X. Let J' be the topology on Y induced by d'. Since the original topology on Y was the strongest one that coincided with the given ones on T_x's and X, we get that J' must be weaker than that. Hence (Y,J') is a connected metrizable space without cut-points.

Next, look at the construction of spaces X_m in the proof of Theorem 3.1.1, They are got by attaching m copies of the circle space. Hence Lemma 3.4.1. easily applies and gives a smaller metric topology on X_m. In this metric topology also, it is easily cheched that there is a unique cut-point, the removal of which results in m connected pieces.

Finally, we observe the construction of X^* in 3.1.1. There again its set is a union of the space Y and the spaces X_m for several m's. Further, whenever two of these interest, they intersect in a single point. Therefore, Lemma 3.4.1. again - applies and gives a metric on X^*. It can be checked that the cut-point-order of any point in X^* is unaltered, whether it be with respect to this metric topology, or be with respect to the original quotient topology of X^*. Hence in the metric topology on X^* also, distinct points have distinct cut-point-orders. Hence the proof of the theorem is complete.

<u>Remark 3.4.3.</u> The next natural question in whether the rigid - metrizable extension of an infinite metrizable space can be chosen without increasing the cardinality. Here we observe the following:

(a) The answer is in general 'no'. For, there is no countably infinite rigid metrizable space. (For, we know that every countably infinite perfect metric space is homeomorphic to the space of all rational numbers. See |13| or |18|).

(b) If X is an uncountable separable metric space, then X can be embedded in a rigid separable metric space of same cardinality. This can be proved along the lines of the theorem in Remark 3.1.4., with an application of Lemma 3.4.1. in the proper places.

<u>Remark 3.4.4.</u> It can be proved with a little greater difficulty that every first countable Hausdorff space can be embedded in a first countable Hausdorff rigid space. We prefer to exclude its proof here.

<u>Remark 3.4.5.</u> In fact we can prove that every metrizable space can be embedded in a connected locally connected metrizable space. For this, we have to follow the proof of Theorem 3.2.2. as follows: In the first step, assume m uncountable and choose S to be metrizable, instead of being separable. After the second step, define a metric d on Z as follows: Note that Z is the union of all $S_{(x_1,x_2)}$'s. If s belongs to $S_{(x_1,x_2)}$ and t belongs to $S_{(Y_1,Y_2)}$'s, define:

$$d(s,t) = \min \begin{cases} d(s,x_1) + d(x_1,Y_1) + d(Y_1,t) \\ d(s,x_1) + d(x_1,Y_2) + d(Y_2,t) \\ d(s,x_1) + d(x_1,Y_1) + d(Y_1,Y_2) \\ d(s,x_1) + d(x_1,Y_2) + d(Y_1,Y_2) \\ d(s,x_2) + d(x_2,Y_1) + d(Y_1,t) \\ d(s,x_2) + d(x_2,Y_2) + d(Y_2,t) \\ d(s,x_2) + d(x_2,Y_1) + d(Y_1,Y_2) \\ d(s,x_2) + d(x_2,Y_2) + d(Y_1,Y_2) \end{cases}$$

where the d's in the right side are the metrics given in X or in $S_{(x_1,x_2)}$ or in $S_{(Y_1,Y_2)}$. When s coincides with x_1, we may take it to be on $S_{(x_1,x_2)}$ for any x_2 and we take the infimum of these distances for different values of x_2.

With these and similar conventions for t, we get a metric d on Z. It is a bit time-consuming to check that this induces a connected locally connected topology on Z. Now we proceed with the proof of Theorem 3.2.2. and give a metric for the final space as in Lemma 3.4.1. The space induced by this metric is the required rigid space. We leave the details of the proof.

3.5. Strongly Rigid Extensions

Here, we show that in the presence of total disconnectedness or regularity or functional Hausdorffness, every Hausdorff space possesses a strongly rigid extension. Recall that a Hausdorff space X is sais to be strongly rigid if identity is the only non-constant continuous self-map of X.

THEOREM 3.5.1. <u>Let X be an infinite Hausdorff space. Let X be either regular or functionally Hausdorff. Then X is homeomorphic to a closed subspace of a strongly rigid space X^* such that $|X^*| = |X|$, provied there is a cardinal strictly</u> between $2^{|X|}$ and $2^{2^{|X|}}$.

<u>Proof</u>. Let $|X| = m$. First we embed X as a closed subspace of a connected Hausdorff space X' such that:

(a) $|X'| = m$

and (b) X' is regular or functionally Hausdorff according as X is. This is possible as shown in the proof of Theorem 3.1.1.

Next, we consider the family of strongly rigid spaces of cardinality m, constructed as in Remark 2.5. From that family, we choose a subfamily $\{S_x | x \in X'\}$ of m spaces. We tale Y to be the disjoint topological sum of all these S_x's together with X'. That is we let $Y = X' + \sum_{x \in X'} S_x$.

Now we introduce an equivalence relation on Y by making the following identifications: We note that S_x has two special points $-\infty_x$ and ∞_x. For each x in X, we identify x with the point $-\infty_x$ of S_x. Finally we identify all the points of the set $\{\infty_x | x \in X'\}$ into a single point. Let ϕ be the quotient map induced by the above and let X^* be the quotient of Y under ϕ. Then we claim that X^* is an extension of the required typed.

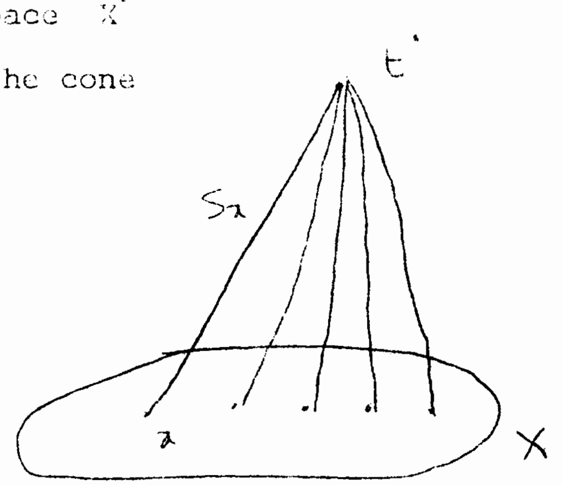
It is a routine verification that the restriction of ϕ to the subspace X' or to any S_x is a homeomorphism onto some - closed subspace of X^*. Hence we can talk (without room for confusion) of the spaces X' and S_x's as subspaces of X^*. Let t be the special point of X^*, namely, $t = \phi(\infty_x)$ for some (and therefore for each) x in X'.

With these notations the space X^* can be geometrically viewed as the cone for which

(i) t is the vertex.

(ii) X' is the base

and (iii) for each x in x', S_x is the line joining x and t.



(Such a view is intended only for easy understanding, and not in a rigorous way).

Since X' is either T_3 or functionally Hausdorff, it is clear that Y is a Hausdorff space and so is $Y \times Y$. Now the equivalence relation induced by ϕ can be checked to be a closed subset of $Y \times Y$. This implies that the space X^* is Hausdorff.

Therefore the only thing that remains to be proved is the triviality of all continuous self-maps of X^*. This will be proved through four steps:

Let $f : X^* - X^*$ be any continuous self-map. Let F be the set of all fixed points of f.

<u>Step 1</u>. We shall first show that $f(X^{*}\ \ F)\ \ X'\ \ \{t, f(t)\}$. Let Y belong to $X^{*}\ (F\ \ X'\ \ \{t\})$. Then there exists x in X' such that $Y \varepsilon S_x\ \ \{-\infty_x, \infty_x\}$. Also $f(Y) \neq Y$. We claim that $f(Y) \varepsilon X'\ \ \{t, f(t)\}$. Suppose $f(Y) \notin X'\ \ \{t\}$. Then we shall show that $f(Y) = f(t)$. Now by the special property of the spaces S_x, it is true that $A = S_x\ \ (f^{-1}(f(Y))\ \ \{f(Y)\})$ is open in S_x. But on the other hand it is closed in $S_x\ \{f(Y)\}$, by the continuity of f. Thus A is both open and closed in $S_x\ \ \{f(Y)\}$.

But $(S_x\ \ \{f(Y)\})\ \ \{t\}$ is connected. This implies that t is in the closure of A. Since A is contained in the closed set $f^{-1}(f(Y))$, it follows that $f(t) = Y$. Thus we have

$$f(X^{*}\ \ (F\ \ X'\ \ \{t\}))\ \ X'\ \ \{t, f(t)\} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \qquad 1$$

Now if z is in X' and if $f(z) \notin X'\ \ \{t\}$, then choose a sequence $z_1, z_2, \ldots$ in S_z converging to z. (This is possible, since z is the point $-\infty_z$ in S_z). Now $f(z_1), f(z_2), \ldots$ must converge to $f(z)$. But no non-trivial sequence can converge to any point outside $X'\ \ \{t\}$. Therefore the above sequence must be eventually constant. Since this is true for each sequence converging to z and since S_z is first countable at z, it follows that $S_z\ \ f^{-1}(f(z))$ is a neighbourhood of z in S_z and hence infinite. Choose a point p in it distinct from z, t and $f(z)$. Then $p \varepsilon X^{*}\ \ (F\ \ X'\ \ \{t\})$ and hence by 2, $f(p) \varepsilon X'\ \ \{t, f(t)\}$. But $f(p) = f(z)$. Thus we have proved that $f(X')$ X' $\{t, f(t)\}$ ________________________________ 2

Putting 1 and 2 together we get that $f(X^{*}$ $F)$ X' $\{t,f(t)\}$.

Step 2: Now fix x in X' and look at $B = S_x$ $(X'$ $\{t,f(t)\})$. It is the union of two disjoint closed sets, namely B F and B $f^{-1}(X'$ $\{t, f(t)\})$. This follows from Step 1. Consequently B F is both open and closed in B. If it is empty, then $f(B)$ X' $\{t, f(t)\}$ and therefore so is $f(S_x)$, by 2 of Step 1. If it is nonempty, t is in its closure, since B $\{t\}$ is connected, and therefore $f(t) = t$. Similarly $f(x) = x$. If B F is also nonempty, then B F is the disjoint union of B $f^{-1}(t)$ and B $f^{-1}(X')$ and each is both open and closed in B. If B $f^{-1}(X')$ is nonempty, then $f(t)$ must belong to X', which is not true. Therefore $f(B$ $F)$ $\{t\}$ and this implies that $f(x) = t$ which is again a contradiction. Therefore B F is empty and therefore every point of S_x is fixed. Thus we have shown that either $f(S_x)$ X' $\{t, f(t)\}$ or S_x F. In the first case, since S_x is connected, either $f(S_x)$ X' or $f(S_x)$ $\{t\}$ or $f(S_x)$ $\{f(t)\}$. Thus one of the following three must hold for each x in X':

1) f is constant on S_x

2) f is identity on S_x

3) $f(S_x)$ X'.

Step 3. Suppose there is at least one x in x' such that f is constant on S_x. If this constant value is not t, then $f(t) \neq t$ and hence f cannot be identity on any other S_Y. Therefore for each Y in X', f is either constant on S_Y with value $f(t)$, or $f(S_Y)$ X'.

If for each Y in X', we have $f(S_Y) = \{f(t)\}$ then f is constant on the whole of X^*. In the latter case, there exists Y in X' such that $f(S_Y)$ X', and in particular $f(t) \in X'$. It follows from Step 2 above, that $f(X^*$ $X')$ X'.

If on the other hand, there is no x in X' such that f is constant on S_x, then f is identity on some S_x's and for the others $f(S_x)$ X'. Only one of these can hold since t belongs to each S_x. If f is identity on each S_x, then f is identity on X^*. If not, $f(S_x)$ X' for each x and therefore $f(X^*$ $X')$ X'.

Thus we have shown that f is either the identity map or a constant map or $f(X^*$ $X')$ X'.

<u>Step 4</u>. We shall next show that if $f(X^*$ $X')$ X' then f must be a constant map. It is only here that we use the fact that X' is either T_3 or functionally Hausdorff. Choose any point x in X', and any point Y in S_x $\{x,t\}$. We claim that $f(Y) = f(t)$. If not, $f(Y)$ and $f(t)$ are distinct elements of t.

If X' is T_3, then we can construct a sequence of open neighbourhoods V_1 V_2 ... of $f(t)$ such that $f(Y)$ is not in the closure of any V_n and such that $\overline{V}_n$ V_{n+1} for each

$n = 1, 2, \ldots$. The same thing is true if X' is only functionally Hausdorff. We have only to choose a real function g such that $gf(Y) \neq gf(t)$. If $2d = |gf(Y) - gf(t)|$, we let $V_n = g^{-1}$ $(gf(t) - \delta_n, gf(t) + \delta_n)$ where $\delta_n = d(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n})$. We consider the function f restricted to S_x.

Then $f^{-1}(V_1), f^{-1}(V_2), \ldots$ is a sequence of neighbourhoods of ∞_x such that for each integer n, $\overline{f^{-1}(V_n)} \quad f^{-1}(V_{n+1})$. But in our space S_x, for such a sequence in S_x, every point will be in the closure of $f^{-1}(V_n)$ for some n. Therefore there exists an integer n such that $Y \in \overline{f^{-1}(V_n)}$ and hence $f(Y)$ belongs to $\overline{V}_n$ which is a contradiction.

Therefore $f(Y) = f(t)$ for every Y in $S_x \quad \{x,t\}$ and for every x in X'. It follows that f is constant on X^*.

<u>Remark 3.5.2.</u> Following Jones and Stone (6) we say that a space X is a T_w-space if for each pair of distinct elements x, Y, in X, there exists a sequence $V_1, V_2, \ldots$ of neighbourhoods of x such that $\overline{V}_n \quad V_{n+1}$ for each $n = 1, 2, \ldots$ and such that Y is not in any V_n.

Then it is easily seen that every regular Hausdorff space and every functionally Hausdorff space is a T_w-space.

Some more arguments in our Theorem 3.5.1., involving no new ideas, yield the following more general result:

Let X be a Hausdorff space such that every connected component of X is a T_w-space. Then the conclusion of the above theorem holds.

In particular, every totally disconnected Hausdorff space and every T_w-space can be embedded in a strongly rigid space.

Remark 3.5.3. The step 4 of the proof of Theorem 3.5.1 shows that if f is a continuous map from S_X into a Hausdorff regular space, then f is constant. It is an immediate consequence that the associated regular space of S_X in the sense of |20| (i.e.) the largest regular topology of S_X cannot be Hausdorff. In other words, no coarser topology can be T_3.

Remark 3.5.4. We have not decided whether every Hausdorff space can be embedded in a strongly rigid space. The above construction and a different argument in the last step of the proof yields an affirmative answer if the space has no nontrivial convergent sequence.

Remark 3.5.5. As in the earlier sections, one may ask whether every space can be embedded in a strongly rigid space, without losing the separation axioms. But here, the answer is negative. It is seen easily that the real line cannot be embedded in any Tychonoff strongly rigid space.

THEOREM 3.5.6. Let X be any infinite space. Then X is homeomorphic to a closed subspace of a connected rigid space X^*. Further, this X^* can be chosen to satisfy any two of the following three properties:

(1) $|X^*| = |X|$.

(2) X^* satisfies all separation axioms satisfied by X.

(3) X^* is rigid for continuous bijections.

In case X is separable and $|X| \geq c$, all the three conditions can be had simultaneously.

Proof. For (1) and (2), see Theorem 3.1.6.

For (2) and (3), see Theorem 3.1.1.

For (3) and (1), see Theorem 3.5.1.

For the last assertion, see Theorem 3.1.5.

3.6. Extensions With a Given Homeomorphism Group.

The purpose of this section is to improve the results of earlier sections, by showing that any space has an extension with a pre-assigned group of homeomorphisms. For any space X, let $H(X)$ denote the group of all self-homeomorphisms of X.

THEOREM 3.6.1. Let X be any Hausdorff space. Let G be any group. Then there exists a Hausdorff space X^* such that

(a) X is homeomorphic to a closed subspace of X^*.

and (b) $H(X^*)$ is isomorphic to G.

Further this X^* can be chosen to be connected and locally connected and to have all the separation axioms satisfied by X.

Proof. Note that for $G = \{e\}$, this theorem has been already proved. (See Theorem 3.2.2.). The deduction of the general case from this will involve an idea, the nucleus of which can also be found in a paper of J. DeGroot |4|.

Step 1. Let e be the identity element of G. Take a family

$\{X_a \mid a \in G\}$ of Hausdorff rigid spaces such that the following conditions hold:

(i) X is homeomorphic to a closed subspace of X_e,

(ii) If a and b are distinct elements of G, then X_a and X_b are not homeomorphic,

and (iii) for each a in G, X_a is a connected locally connected space, constructed as in the proof of Theorem 3.2.5, satisfying all the separation axioms possessed by X,

and (iv) X is contained in the first level of X_e.

Then we note that in X_a, there is a unique non-cut point x_a, every other point being a cut point of order two.

Now fix an arbitrary a in G. Choose any point s_a of level one in it. Omit all points of X_a that lie strictly above s_a. We get a connected locally connected Hausdorff space Y_a having exactly two non-cut-points x_a and s_a. Note that Y_a is also rigid. (Proofs, though non-trivial, involve no new ideas. Hence they are omitted).

Thus Y_a is constructed for each a in G.

Step 2. Consider the disjoint topological sum $\sum_{a \in G} Y_a$ and identify all the points of $\{x_a \mid a \in G\}$ to a single point Y_o. Let Y be the quotient space thus obtained and let ϕ be the quotient map. They Y is a connected Hausdorff space.

Step.3. Give discrete topology to G and consider the product space $Y \times G$. Identify the point (s_a, b) of $Y \times G$ with the point (y_o, ab). When this is done for each pair (a,b) of points in G, let ψ be the quotient map and X^* the quotient space. We are going to show that this space X^* has all the required properties.

<u>Step 4</u>. It can be easily checked that if a is fixed, then the restriction of ϕ to Y_a is a homeomorphism onto a closed subspace of Y. If b is also fixed, then the restriction of ψ to $Y X\{b\}$ is a homeomorphism onto a closed subspace of X^*. Thus each Y_a, and in particular Y_e and hence the given space X, is homeomorphic to a closed subspace of X^*.

It can also be verified by straightforward methods that X^* satisfies all the separation axioms satisfied by X.

Now each $Y X\{b\}$ is connected. Further the point (Y_o, e) has been identified with $(s_{b^{-1}}, b)$ and hence is a point of $Y X b$. Thus X^* is the union of the family $\{Y X\{b\}\ b \in G\}$ of connected subsets. Where (Y_o, e) belongs to every member of this family. Hence X^* is connected.

Also Y is a quotient of a sum of locally connected spaces and hence is locally connected. Therefore so is YXG and so is its quotient X^*.

The only thing that remains to prove is that the group of all self-homeomorphisms of X^* is isomorphic to G. This will be proved in the remaining four steps. For the sake of convenience, if S is a subset of $Y X G$, we do not distinguish S and $\psi(S)$.

<u>Step 5</u>. As a step towards proving that $H(X^*) \simeq G$, here we find all the non-cut-points of X^*. If $a \varepsilon G$, consider the family $\{Y\ X\{b\}\ \ \{(Y_o,a)\} \mid b \ \varepsilon\ G - \{a\}\}$. Since the removal of one or two points of $\{x_a, s_a\}$ in Y_a does not make it disconnected, we see that each member of the above family is connected. Further, if b is any point different from a, then (Y_o,b) belongs to each of them. Also, their union is $X^*\ \ \{(Y_o,a)\}$. These together prove that (Y_o,a) is a non-cut-point of X^*. We claim that every non-cut-point must be of this type. Let (Y,a) be any point of X^* where $Y \neq Y_o$. Then there exists a unique b in G such that Y belongs to Y_b. (Here, we do not distinguish between Y_b and $\phi(Y_b)\ X\{a\}$). This point Y is different from the points x_b and s_b. Hence $Y_b\ \ \{y\}$ is disconnected. Note that in Y_b, if any of its cut-points is removed, the two non-cut-points belong to the same connected component. Then the other component contained in $\phi(Y_b)\ X\ \{a\}$ is open and closed in X^*. Consequently (Y,a) is a cut point of X^*. Thus we have shown that the points of $\{Y_o\}\ X\ G$ are precisely the non-cut-points of X^*.

<u>Step 6</u>. Another assertion that we shall need in our proof is that the sets of the form $\phi(Y_b)\ X\ \{a\}$ have a 'topological. description. More precisely we shall prove the following: Let $A\ \ X^*$. Then $A = \phi(Y_b)\ X\ \{a\}$ for some a,b in G with $b \neq e$, if and only if A is a maximal subset of X^* with respect to the following properties:

(a) there exist exactly two points g_1 and g_2 in G such that (Y_o, g_1) and (Y_o, g_2) belong to A.

(b) The subsets A B where B is any subset of the two elements set $C = \{(Y_o, g_1), (Y_o, g_2)\}$ are connected.

It is obvious that the sets $\phi(Y_b) \times \{a\}$ are maximal subsets having these properties. Conversely, let A be any maximal subset with respect to (a) and (b). Now X^* C is easily seen to fall into three components, consequently A C must be contained in one of them, by condition (b). Now condition (a) insures that this component must be either

$$\phi(Y_{g_1 g_2^{-1}}) \times \{g_2\} \quad C \quad \text{or} \quad \phi(Y_{g_2 g^{-1}}) \times \{g_1\} \quad C.$$

Hence by the maximality of A, we get that

$$A = \phi(Y_{g_1 g_2^{-1}}) \times \{g_2\} \quad \text{or} \quad A = \phi\{(Y_{g_2 g^{-1}}) \times \{g_1\}.$$

Similar considerations prove that the sets of the form $\phi(Y_e) \times \{g\}$ are characterized as maximal closed subsets with respect to the following properties:

(a') There exists a unique point of the form (Y_o, g) in A.

(b') The sets A and A $\{(Y_o, g)\}$ are connected.

(c') A is not contained in any set of the form $\phi(Y_{g_1}) \times \{g_2\}$ where $g_1 \neq e$.

Note that (c') is a topological condition, by the previous paragraph.

<u>Step 7</u>. Now let $h : X^* \to X^*$ be any homeomorphism. Look at $h(Y_o, e)$. By Step 5, there exists a point a in G such that

$h(Y_o,e) = (Y_o,a)$. Now consider $\phi(Y_b) X \{e\}$. This contains the point $(s_b,e) = (Y_o,b)$. Again by step 5, there exists c in G such that $h(Y_o,b) = (Y_o,c)$. Let $A = h(\phi (Y_b X \{e\})$. Then A contains the point (Y_o,c). Again, by step 5, A does not contain any other point of $\{Y_o\} X G$. Therefore by step 6, it follows that A is either $\phi(Y_{ac}-1) X \{c\}$ or $\phi(Y_{ca}-1) X \{a\}$. Therefore $b = ac^{-1}$ or $b = ca^{-1}$. That is, either $a = bc$ or $c = ba$. Suppose $a = bc$. In this case, we have a homeomorphism $Y_b \to \phi(Y_b) X \{e\} \to \phi(Y_{ac}-1) X \{e\} \to Y_{ac}-1 = Y_b$. This takes the point x_b as follows: $x_b \to (Y_o,e) \to (Y_o,a) = (s_b,e) \to s_b$. This contradicts the rigidity of Y_b. Therefore the case $a = bc$ cannot arise. Thus $c = ba$. In this case again, the rigidity of Y_b shows that $h(\phi(x),e) = h (\phi(x),a)$ for every x in Y_b. Similar arguments show that $h(Y_o,g) = (Y_o,ga)$ for every g in G and $h(Y,g) = (Y,ga)$ for every Y in Y.

Now for each a in G, let $T_a : X^* \to X^*$ be defined by the rule $T_a(Y,g) = (Y,ga)$ for each (Y,g) in X^*. Then what we have shown above is that if $h : X^*$ is a homeomorphism, then $h = T_a$ for some a in G. On the other hand, we shall now show that each T_a is a homeomorphism. View T_a as a map from $Y X G$ to $Y x G$. It is the map identity $X : g \to ga$ and hence is a homeomorphism of $Y x G$. Also $T_a(Y_o,g) = (Y_o,ga)$ and $T_a(s_{g_1}-1, g_1^{-1}g) = (s_{g_1},g_1^{-1}g\ a) = (Y_o,ga)$. This means that T_a is compatible with ψ and hence induces our map T_a on X^*. Since ψ is a quotient map, it follows that T_a is a homeomorphism of X^*.

Also, we check that $T_a \circ T_b = T_{ab}$ for every a, b in G. These facts prove that $\{T_a \mid a \in G\}$ is the set of all self-homeomorphism of X^* and the map $T_a \to a$ is an isomorphism from $H(X^*)$ to G.

Remark 3.6.2. Let $m = |X| . |G| . {}_o$. Then in the above construction it is possible to have $|X_a| \leq m$ for each a in G. This would imply that $|X^*| \leq |Y| . |G| \leq m. |G| . |G| = m$. Thus we can choose the extension X^* such that $|X^*| \leq {}_o |X| . |G|$.

Remark 3.6.4. In the process of finding extensions X^* of X such that $H(X^*) \simeq$ the given group G, we have shown that we can preserve connectedness, local connectedness and separation axioms. Similarly one can deal successfully with metrizability and first countability; any coreflective property is obviously preserved. We would like the to consider the question whether on the other extreme, zero-dimensionality or total disconnectedness can be preserved. 3.3. gives an affirmative answer, in the special case that G is a singleton. But in general, the answer is easily seen to be in the negative. See also |4|.

COROLLARY 3.6.5. (J. DeGroot |4|): *Let A be the class of all connected locally connected metrizable spaces. Let G be any group. Then there exists X in A such that H(X) is* *isomorphic to G. We can augment the above result by the following one.*

THEOREM 3.6.6. *Let A be any nonempty family of topological spaces closed under the formation of sums and discrete-quotient images (i.e. images under a quotient map, such that the pre-image of every singleton is discrete). Let G be any group. Then*

(i) *If A contains at least one non-discrete T_1-space, then there exists a connect T_1-space X in A such that $H(X) \sim G$.*

(ii) *If A contains at least one connected Hausdorff space, there is a connected Hausdorff space X in A such that $H(X) \sim G$.*

Proof. Let us start with the assumption of (i). Let Y be a non-discrete T_1-space belonging to A. Let its cardinal be m. Let m' be the least cardinal such that there is a non-closed subset A of Y with cardinality m'. Obviously, m' is infinite since Y is a T_1-space. Now let P be the set of all permutations of Y and let for each α in P, Y_α be a copy of Y, kept pairwise disjoint. Let $Y_1 = \sum_{\alpha \varepsilon P} Y_\alpha$. Let $h_\alpha : Y - Y_\alpha$ be fixed homeomorphism for each α in P. Now a point Y_α of Y_α is declared to be related to Y in Y if $Y_\alpha = h_\alpha(\alpha(Y))$. Let π be the equivalence relation generated out of this rule and let Z be the quotient space and ϕ the quotient map thus obtained. Then clearly ϕ is a discrete-quotient map. A subset A of Z is closed in Z if and only if $\alpha(\phi^{-1}(A) \quad Y)$ is closed in Y for each α in P.

This happens if and only if $|A| < m'$. This at once implies that Z is a connected T_1-space. Thus we have shown that A contains a connected T_1-space.

Hereafter the proof of (i) and (ii) go along the same lines. Let Z be the above connected T_1-space in case (i) and any connected Hausdorff member of A in case (ii). Then Z can be embedded in a connected T_i ($i = 1,2$, respectively) - space Z' of A without cut-points (to each pair of points of Z attach a copy of Z. Get Z_1. To each pair of points of Z_1, attach a copy of Z. Get Z_2 and so on. Then by induction, we get a direct limit system $Z_1 \to Z_2 \to \ldots \to Z_n \to \ldots$ and let Z' be the direct limit. Then it can be proved that Z' is a discrete-quotient of a sum of copies of Z and that it is a connected T_i-space without cut-points).

Then copy the proof of Theorem 3.1.1, with copies of Z' in place of X,T, etc. Then we get a rigid space Z'^*. By considering different sets of cardinal numbers, we can manufacture several such Z^* with different cardinalities. Now procceding along the lines of proof of Theorem 3.6.1., we get a space Z^* for which $H(Z^*) \simeq G$.

Now observe that throughout our process, we have employed only discrete-quotient images of disjoint topological sums of copies of Z. Hence Z^* belongs to A. It can be checked that Z^* is T_1 or T_2 according as Z is.

COROLLARY 3.6.7. *Let* G *be any group. Let* $C(G)$ *be the class of all topological spaces* X *such that* $H(X) \cong G$. *Then*

(i) $C(G)$ *is so large that the subspaces of its members exhaust all topological spaces.*

(ii) $C(G)$ *is so large that every space is a quotient of some member of it.*

(iii) $C(G)$ *is so large that on any infinite set of cardinality* $m \geq |G|$, *there are* 2^{2^m} *types of members of* $C(G)$.

(iv) $C(G)$ *is so large that if* X *is any connected space, there is a connected space* Y *in* $C(G)$ *such that* Y *is a discrete-quotient of a sum of copies of* X.

Proof.

(i) Follows from Theorem 3.6.1 on observing that it remains true when the word 'Hausdorff' is dropped throughout.

(ii) Our proof of this includes a proof of Theorem B of the abstract. Let βD be any infinite discrete space and let D be its Stone-cech compactification. Then, we can build by C-process as described in 2.4, a family of spaces $\{X_p \mid p \in \beta D \; D\}$, with a shrewd choice of the base-spaces.

(a) Each X_p is extremally disconnected rigid space constructed by C-process, with $D \; \{p\}$ as the first level base space.

(b) If p and q are distinct elements of $\beta D \; D$, then no open subspace of X_p is homeomorphic to any open subspace of X .

If we let X to be the sum of all these X_p's, then X can be easily seen to be a rigid space. Since for each p, we have that $D \; \{p\}$ is a quotient of X_p (note that the projection onto the first level base space is a quotient map in every C-process-space), we get a quotient map from X onto $\Sigma_{p \in \beta D \; D} D \; \{p\}$. Let us denote this space by Y. Now observing that every filter is an intersection of ultrafilters, we see that every topology on the set of cardinality $|D|$, is the lattice meet of a class of topologies homeomorphic to some $D \; \{p\}$. This leads one to conclude that every topological space of cardinality $|D|$ is a quotient of Y (which has been proved to be a quotient of X). Thus we have shown that every space is a quotient of an extremally disconnected regular rigid space.

Now, if P is any topological space and G is any group, choose two spaces X and Y and a map f such that

(a) X is a zero-dimensional regular T_2 rigid space.

(b) $f : X - P$ is a quotient map

(c) Z is a connected regular space such that $H(Z) \simeq G$.

|Such a choice of X and f is possible, as seen in the previous paragraph; such a choice of Z is possible as seen in Theorem 3.6.1|. Now let $X + Z$ be the sum of X and Z and let $g : X + Z - P$ be defined by the rule $g|_X \equiv f$ and $g|_Z$ is constant. Then one can check that g is a quotient map. Now any self-homeomorphism of $X + Z$ must leave Z invariant, since Z is the only non-trivial component of $X + Z$. It follows that $H(X+Z) \simeq G$. Thus P is a quotient of some member of $\underline{C}(G)$

(iii) Follows from the following facts:

(a) Every space of cardinality $\geq \aleph_0 |G|$ can be embedded in a space of same cardinality belonging to $\underline{C}(G)$.

(b) A space of cardinality m can have at most 2^m types of subspaces; there are 2^{2^m} types of spaces with cardinal m.

3.7. Groups of Isometries

By an isometry of a metric space, we mean a distance-preserving self-bijection. We know that for every metric space X, all isometries of X from a subgroup of the group $H(X)$ of all auto-homeomorphisms of X. We prove in this section that but for this, there is no other dependence between the isometry group and the homeomorphism group.

Remark 3.7.1. Let X, X_α, d_α etc. be as in Lemma 3.4.1. Let further any two points of X be connectible and let d_1 be defined as in the proof of Lemma 3.4.1. Then d_1 is a metric on X.

THEOREM 3.7.2. *Let G be any group and H any subgroup of G. Let X be any metric space. Then there is a metric space Y such that the following hold:*

(i) *$H(Y)$ is isomorphic to G.*

(ii) *The group of all isometries of Y is isomorphic to H.*

(iii) *X is isometric to a closed subspace of Y.*

(iv) *Y is connected and locally connected.*

<u>Proof</u>. We first outline the construction of Y. Take a family $\{X_\alpha\}\ \alpha \in G\}$ of rigid spaces such that the following hold:

(a) Each X_α is a connected locally connected metric space constructed as in the proof of Remark 3.4.5.

(b) X is isometric to a closed subspace of X_e contained in the first level, where e is the identity of G.

(c) If a and b are distinct elements of G, then X_α and X_b are not homeomorphic.

Now fix a in G, choose any point s_a of level one in it, omit all points of X_a that lie strictly above s_a and let Y_a be the resulting space. Let d_a be the metric in Y_a. Let $\tilde{d}_a$ be the equivalent metric defined by $\tilde{d}_a(x,Y) = 2d_a(x,Y)$ for every pair (x,Y) of elements in Y_a.

Let Z_a be the metric space defined by

$$Z_a = \begin{cases} (Y_a, d_a) & \text{if } a \in H \\ (Y_a, \tilde{d}_a) & \text{if } a \in G \quad H. \end{cases}$$

Consider the disjoint sum (set theoretic co-product) $\sum_{a \in G} Z_a$ and identify set theoretically all the points of $\{x_a \mid a \in G\}$ to a single point Y_o. Observe that there is a canonical embedding of Y_a in the quotient set thus obtained. Give a metric on this resulting set as in Remark 3.7.1. Let Z be the metric space thus obtained.

Now consider the set $Z \times G$. Identify the point (G_a, b) of $Z \times G$ with the point (Y_o, ab). When this is done for each pair (a,b) of points in G, we get a quotient set. For each a in G, the metric space Z is naturally embedded as $Z \times \{a\}$ in this set. Give a metric to this whole quotient set, as in Remark 3.7.1.

Let Y be the resulting metric space.

Then we claim that Y has the required properties. Along the lines of proof of Theorem 3.6.1, one can prove that for each a in G, the map T_a (defined in that proof) is a homeomorphism of Y and that all homeomorphisms arise in this way. This proves (i). It is easy to prove (iii).

Next we observe that not every T_a is an isometry. This is because our definition of the metric in Z_a is not uniform; it is defined in a special way if a is not in H. Consequently we see that T_a is an isometry if and only if it leaves the set $Z \times H$ (modulo the mentioned identifications) invariant. This happens if and only if a is such that whenever x is in H, xa is also in H. This happens if and only if a is in H. Thus the isometries of Y are precisely the T_a's for a in H. This proves (ii).

Finally, we state without proof, the following result.

THEOREM 3.7.3. *If in the statement of Theorem 3.7.2, we replace (iii) by the assertion that X is a quotient of Y, then this new statement is also true.*

Thus the class of all metric spaces with a pre-assigned homeomorphism group and a pre-assigned subgroup of it as the isometry group, is a large one.

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