NOTAS DE MATEMATICA

Nº 14

"FIXED RINGS OF AUTOMORPHISMS OF $K[x,y]$"

POR

R. MARKANDA J. PASCUAL

DEPARTAMENTO DE MATEMATICA

FACULTAD DE CIENCIAS

UNIVERSIDAD DE LOS ANDES

MERIDA - VENEZUELA

1977

FIXED RINGS OF AUTOMORPHISMS OF $K[x,y]$

1. INTRODUCCION. In this paper we answer some of the open questions raised by Fraser and Mader in [3]. Let $R = K[x,y]$ be the polynomial ring in 2-variables x and y, where K is an algebraically closed field of characteristic 0. Let $M = (x,y)$ be a maximal ideal of R and set $A = \{\alpha$ in $\mathrm{Aut}_K(R)$; both $x\alpha - x$ and $y\alpha - y$ are in $M^2\}$. Given α in A, we define $R^\alpha \doteq \{f$ in $R : f\alpha = f\}$ to be the fixed ring of α. It is shown in [Theorem 4.4.3] that if $R^\alpha \neq K$ then there exists f_α in R - K such that $R^\alpha = K[f_\alpha]$. Then the authors ask - whether there exists α in A such that $R^\alpha = K$. Furthermore, which polynomials f_α can occur as the generators of the fixed ring R^α.

Every α in $\mathrm{Aut}_K(R)$ induces a polynomial map $\bar{\alpha} : K^2 \to K^2$ defined by $(a,b)\,\bar{\alpha} = (x\alpha(a,b),\ y\alpha(a,b))$. For a non constant polynomial f in R, we define $V(f) = \{(a,b)$ in $K^2 : f(a,b) = 0\}$. It is shown in 5.8 of [2] that for α in A we have $f\alpha = f$ if and only if $V(f)\,\bar{\alpha} \subseteq V(f)$. The algebraic curve V(f) is left pointwise fixes if and only if f divides (g.c.d. $(x\alpha-x,\ y\alpha-y))^n$ for some $n \geq 1$. Now let $R^\alpha = K[f] \neq K$. Then, by (5.12) of [3], there exists $n \geq 1$ such that $\bar{\alpha}^n$ fixes $V(f)$ pointwise. Does it follow that $\bar{\alpha}$ fixes $V(f)$ pointwise?.

In the Notices of the A.M.S, vol. 24 (1977) page A-319, David Shannon has announced the following,

Theorem 1.1. Let $\alpha \neq 1$ be in A. Then $R^\alpha \neq K$ if and only if α is conjugate to γ, where γ is of the type $x\gamma = x + f(y)$, $y\gamma = y$ or vice-versa.

Using this theorem, in Theorem 2.1, we get a characterization of generators of fixed rings. Then, in Theorem 2.3, we show that $R^\alpha \neq K$ if and only if g.c.d. $(x\alpha - x,\ y\alpha - y) \neq 1$ and deduce the existence of an infinite subset S of A such that $R^\alpha = K$ for every α in S. In theorem 2.5, we show that if $R^\alpha = K[f] \neq K$ then $\bar{\alpha}$ fixes $V(f)$ pointwise.

2. MAIN RESULTS. We start by proving a result of general interest.

Proposition. Let α, β in A such that $R^\alpha \cap R^\beta \neq K$. Then $R^\alpha = R^\beta = R^{\alpha\beta} \neq K$.

Proof. By Theorem 4.4 of [3], $R^\alpha = K[f_\alpha]$ and

$R^{\beta} = K[f_{\beta}]$ for some non-constant polynomials f_{α} and f_{β} in R. Let g be a non-constant polynomial in $R^{\alpha} \cap R^{\beta}$. Then g in R^{α} implies that $g = a_0 + a_1 f_{\alpha} + \ldots + a_n f_{\alpha}^n$, where a_i are in K, for $0 \leq i \leq n$. Now $g\beta = g$ gives

$$a_0 + a_1 f_{\alpha}\beta + a_2(f_{\alpha}\beta)^2 + \ldots + a_n(f_{\alpha}\beta)^n$$

$$= a_0 + a_1 f_{\alpha} + a_2 f_{\alpha}^2 + \ldots + a_n f_{\alpha}^n$$

Using lemma 4.1 of [3], we get $f_{\alpha}\beta = f_{\alpha}$ thus $R^{\alpha} \subseteq R^{\beta}$. Similarly $R^{\beta} \subseteq R^{\alpha}$ and whence $R^{\alpha} = R^{\beta}$.

As an inmmediate application of Theorem 1.1, we get -

Theorem 2.1. Let $\alpha \neq 1$ be in A such that $R^{\alpha} = K[f_{\alpha}] \neq K$. Then $f_{\alpha} = x\beta$ or $y\beta$ for some β in $Aut_K(R)$. Conversely, - given β in $Aut_K(R)$, $x\beta$ and $y\beta$ occur as generators of certain fixed rings.

Now we give below the characterisation of the K-automorphisms of $K[x,y]$.

Theorem 2.2. [Th. 1.5,1]. The group $Aut_K(R)$ of the K-automorphisms $R = K[x,y]$ is generated by primitive - polynomials of the following type:

1) $x\alpha = x$, $y\alpha = cy$ or vice-versa, where $c \neq 0$ is in K

2) $x\alpha = x + f(y)$, $y\alpha = y$ or vice-versa, where $f(y)$ is in $K[y]$.

Using Theorems 1.1 and 2.2 we get the following,

<u>Theorem 2.3</u>. Let $\alpha \neq 1$ be in A. Then $R^\alpha \neq K$ if and only if g.c.d. $(x\alpha - x, y\alpha - y) \neq 1$.

<u>Proof</u>. Let g.c.d $(x\alpha - x, y\alpha - y) \neq 1$. Then, by 5.11 of [3], $R^\alpha \neq K$.

Conversely let $R^\alpha \neq K$. By Theorem 1.1, α is of the form $\beta^{-1}\gamma\beta$, where $x\gamma = x + f(y)$, $y\gamma = y$ or vice-versa and β is in $Aut_K(R)$. By Theorem 2.1, we can write $\beta = \beta_1 \beta_2 \ldots \beta_r$, where each β_i is a primitive K-automorphis of R. Thus $\alpha = \beta_r^{-1}\beta_{r-1}^{-1}\ldots\beta_1^{-1}\gamma\beta_1\ldots\beta_{r-1}\beta_r$. By induction on r, we shall show that g.c.d.$(x\alpha - x, y\alpha - y) \neq 1$.

First of all note that $f(y)$ can not be a non-zero constant. For, let $f(y) = a \neq 0$ in K. Then $x\gamma = x + a$, $y\gamma = y$. Let $x\beta_1 = x + f_1(y)$, $y\beta_1 = y$ thus $x(\beta_1^{-1}\gamma\beta_1) = x + a$. If $x\beta_1 = cx$, $y\beta_1 = y$ then $x(\beta_1^{-1}\gamma\beta_1) = x + ca^{-1}$. From these observations we see that in general $x\alpha = x(\beta^{-1}\gamma\beta) = x + ba$ with b in K, $b \neq 0$.

Since $x\alpha - x$ is in M^2, we see that $a = 0$. Then α is the identity map.

For $r = 0$, we have $\alpha = \gamma$ and thus g.c.d.$(x\alpha - x, y\alpha - y) =$ g.c.d.$(f(y),0) = f(y) \neq 1$. Now let $r \geq 1$ and assume that the result holds $0 \leq i \leq r - 1$. Let $\alpha' = \beta_{r-1}^{-1} \dots \beta_1^{-1} \gamma \beta_1 \dots \beta_{r-1}$ be given by $x\alpha' = x + C$, $y\alpha' = y + D$. By induction hypothesis, g.c.d.$(x\alpha' - x, y\alpha' - y) =$ g.c.d.$(C,D) \neq 1$. Now $\alpha = \beta_r^{-1} \alpha' \beta_r$. Suppose that $x\beta_r = x$, $y\beta_r = cy$ with $c \neq 0$ in K. Then

$x\alpha = x (\beta_r^{-1} \alpha' \beta_r) = x + C\beta_r$ and

$y\alpha = y(\beta_r^{-1} \alpha' \beta_r) = y + c^{-1}D\beta_r$. Thus

g.c.d.$(x\alpha - x, y\alpha - y) =$ g.c.d.$(C\beta_r, D\beta_r) \neq 1$. Consider the case $x\beta_r = x + g(y)$, $y\beta_r = y$. Then

$x\alpha = x(\beta_r^{-1} \alpha' \beta_r) = x + g(y) + C\beta_r - g(y + D\beta_r)$ and $y\alpha = y+D\beta_r$.

Thus g.c.d.$(x\alpha - x, y\alpha - y) =$ g.c.d.$(C\beta_r, D\beta_r) \neq 1$. The rest of the cases can similarly be considered.

As a consequence of this we get the following,

<u>Theorem 2.4</u>. Let α, β be in A given by

$$x\alpha = x + f(y),\ y\alpha = y$$

and $x\beta = x$, $y\beta = y + g(x)$. If $\gamma = \alpha\beta$ then $R^{\gamma} = K$.

Proof. Note that $x\gamma = x\alpha\beta = x + f(y + g(x))$ and $y\gamma = y(\alpha\beta) = y + g(x)$ imply that g.c.d.$(x\gamma - x, y\gamma - y)$ = g.c.d.$(f(y), g(x)) = 1$. Thus, by Theorem 2.3, $R^{\gamma} = K$.

We have remarked in the introduction that if α is in A such that $R^{\alpha} = K[f] \neq K$, then $V(f)\bar{\alpha} \subseteq V(f)$. By a deeper result of Algebraic geometry [2, Theorem 8, p. 292], there exists an integer $n \geq 1$ such that $\bar{\alpha}^n$ fixes $V(f)$ pointwise. Using our previons results we show that $\bar{\alpha}$ itself fixes $V(f)$ pointwise.

Theorem 2.5. Let $R^{\alpha} = K[f] \neq K$ with α in A. Then $\bar{\alpha}$ fixes $V(f)$ pointwise.

Proof. As f is irreducible, by 5.10 of [3], we see that $V(f)$ is fixed pointwise by $\bar{\alpha}$ if and only if f divides the g.c.d.$(x\alpha - x, y\alpha - y)$. Now there exists an $n \geq 1$ such that $\bar{\alpha}^n$ fixes $V(f)$ pointwise and thus f divides g.c.d.$(x\alpha^n - x, y\alpha^n - y)$. We shall show below that g.c.d.$(x\alpha - x, y\alpha - y)$ = g.c.d.$(x\alpha^n - x, y\alpha^n - y)$ and whence the required result will follow.

Now $R^{\alpha} \neq K$ implies that there is β in $Aut_K(R)$ such that $\alpha = \beta^{-1} \gamma \beta$ with γ given by $x\gamma = x + f(y)$, $y\gamma = y$ or vice-versa. By Theorem 2.2, $\beta = \beta_1 \beta_2 \ldots \beta_r$ with each β_i either of type 1 or type 2. We proceed, as in Theorem 2.3, by induction on r. For $r = 0$, $\alpha = \gamma$ and then $x\alpha^n = x + n f(y)$, $y\alpha^n = y$ and the result is immediate. Suppose that $r \geq 1$ and the result is true for $0 \leq i \leq r-1$. Let $\alpha' = \beta_{r-1}^{-1} \ldots \beta_1^{-1} \gamma \beta_1 \ldots \beta_{r-1}$. Then

$x'^n = \beta_{r-1}^{-1} \ldots \beta_1^{-1} \gamma^n \beta_1 \ldots \beta_r$. By induction hypotheses,

$g.c.d.(x\alpha' - x, y\alpha' - y) = g.c.d.(x\alpha'^n - x, y\alpha'^n - y)$.

Now $\alpha = \beta_r^{-1} \gamma' \beta_r$ and $\alpha^n = \beta_r^{-1} \alpha'^n \beta_r$. Let $x\alpha' = x + C$, $y\alpha' = y + D$ and $x\alpha'^n = x + C_1$, $y\alpha'^n = y + D_1$. As seen in the last part of Theorem 2.3, we get

$g.c.d.(x\alpha^n - x, y\alpha^n - y) = g.c.d.(C_1 \beta_r, D_1 \beta_r) =$

$= g.c.d.(C\beta_r, D\beta_r) = g.c.d.(x\alpha - x, y\alpha - y)$ and hence the theorem.

Acknowledgement. We would like to thank John David for drawing our attention to Theorem 2.2.

R E F E R E N C E S

[1.] S. S. Abhyankar, T.T. Moh "Embeddings of the line in the Plane" J. Reine Angew. Math. Vol, 276 (1975) pp. 148-166.

[2.] T.L. Coolidge "A Treatise on Algebraic plane Curves" Dover, New York, 1959.

[3.] M. Fraser and A. Maser "The Structure of the Automorphism Group of Polynomial Rings" J. Alg, vol. 25 (1973), pp.25-39.

RM/JP/codea

R. MARKANDA J. PASCUAL

"FIXED RINGS OF AUTOMORPHISMS OF $K[x,y]$"