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"INEQUALITIES FOR MODULI OF CONTINUITY AND REARRANGEMENTS"

BY

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1. Introduction. Let $X(0,1)$ be a rearrangement invariant space of Lebesgue measurable functions on $(0,1)$, to this space we associate a modulus of continuity defined as follows: $\omega_X(f,t) = \sup_{0\leq h\leq t} ||\Delta_h f||_X$, where

$\Delta_h f(x) = [f(x+h) - f(x)]\chi_{(0,1-h)}(x)$.

In this paper we announce several elementary inequalities relating the behaviour of f^* (then non-increasing rearrangement of f) and $\omega_X(f,t)$. These results have interesting applications in the theory of embedding of r.i. spaces and Lipschitz spaces, Fourier analysis and other areas (cf. [1], [2], [4]). We outline some applications in §3.

2. The inequalities. Let f be a non-increasing function, $f \in X$, then $\forall n \geq 2$ we have $\omega_X(f,1/n). \frac{1}{\phi_X(1/2n)} \geq f(\frac{1}{2n}) - f(1/n)$ and therefore if we let $n = 2^k$, $k = 1 \ldots$, and sum, we get $f(1/2^{m+1}) - f(1/2) \leq \sum_{k=1}^{m} \frac{1}{\phi_X(\frac{1}{2^{k+1}})} \omega_X(f, \frac{1}{2^k})$.

This last inequality combined with the facts that $\omega_X(f,t)\nearrow$ and $\phi_X(t)t^{-1}\searrow$ yields

Theorem 1. Suppose that $f \in X$, and $f\searrow$, then

$$(1) \qquad f(s) - f(1/2) \leq 8 \int_s^1 \frac{\omega_X(f,u)}{\phi_X(u)} \frac{du}{u}, \quad 0 \leq s \leq 1/2.$$

For a detailed proof see [4]. (1) gives nearly the result we are looking for, the complete result follows from

Theorem 2. $\omega_X(f^*,t) \leq 7\omega_X(f,t)$, $f \in X$.

The proof of this result is given in [5] (cf. also [8]) and depends on the possibility of approximating f by suitable averages and the fact that

$||f^*-g^*||_X \leq ||f-g||_X$.

Combining (1) and Theorem 2 we get

Corollary 3. $f^*(s) - f^*(1/2) \leq 54 \int_s^1 \frac{\omega_X(f,u)}{\phi_X(u)} \frac{du}{u}$, $0 \leq s \leq 1/2$.

Garsia and Rodemich [1] have given a proof of Corollary 3, using a general (and complicated) combinatorial argument, for the case where $X = L^p$.

Let us remark that we can in fact obtain a stronger result, indeed use (1) with $f = f^{**}$, Theorem 2 and the following

Theorem 4. $\omega_X(f^{**},t) \leq \omega_X(f^*,t)$.

We get

$$(2) \qquad f^{**}(s) - f^{**}(1/2) \leq \text{const} \int_s^1 \frac{\omega_X(f,u)}{\phi_X(u)} \frac{du}{u}, \quad 0 \leq s \leq 1/2.$$

Next we look at the dual problem of estimating $\omega_X(f^*,t)$, we obtain the following

Theorem 5. Let X be a separable r.i. space, and let $f \in X$, then

$$(3) \qquad \omega_X(f^*,t) \leq ||H||_{X\to X} ||f^* \cdot \chi_{(0,t)}||_X$$

where $(Hf)(t) = \frac{1}{t} \int_0^t f(s)ds$.

A proof is given in [4] and again the ideas are elementary: we look at $(\Delta_h f^*)^{**}$, where $|f| = c\chi_E$ and show that $(\Delta_h f^*)^{**}(t) \leq H(f^*\chi_{(0,h)})(t)$, then we extend the result to arbitrary simple functions.

3. Applications. Corollary 3 and (2) can be used to obtain embedding theorems of various types. We give several examples.

Consider spaces $\Lambda(A,\phi_Z,\phi_Y) = \{f \in M(0,1) : ||f^*\phi_Z||_{L_A(d\mu)} < \infty\}$,

where $d\mu(t) = \phi_Y(t)\,\frac{dt}{t}$ (see [6]).

Theorem 6. Let A be a Young's functions with the Δ_2 property, and suppose the following conditions are satisfied

(i) $\int_t^\infty \phi_{Z'}(u)\,\frac{du}{u^2} \leq \theta_1\phi_{Z'}(t).t^{-1}$

(ii) $\int_t^\infty \phi_Z(u^{-1})\phi_Y(u^{-1})\,\frac{du}{u} \leq \theta_2\phi_Z(t^{-1})\phi_Y(t^{-1})$

(iii) $f \in X$, and $||\omega_X(f,t)\phi_Z(t)/\phi_X(t)||_{L_A(d\mu)} < \infty$.

Then, $f \in \Lambda(A,\phi_Z,\phi_Y)$.

The proof follows readily from Corollary 3 and the extention of Hardy's - inequality given in [6]. (See also [7].)

Let $Lip(A,\phi_X,\phi_Y) = \{f \in M(0,1) : ||\omega_Y(f,t)/\phi_X(t)||_{L_A(\frac{dt}{t})} < \infty\}$.

Using Corollary 3 we obtain the following (cf. Herz [3], Theorem 5).

Theorem 7. Let A be a Young's function with the Δ_2 property and suppose that ϕ_Z is bounded away by powers from 1 and t, then

$Lip(A,\phi_X,Y) \subseteq \Lambda(A,\phi_Z,1)$, whenever there exists $\theta > 0$ such that

$\phi_Z(t)\phi_X(t) \leq \theta\phi_Y(t)$, $\forall\, t > 0$.

We remark that similar results hold for generalized Young's functions such that $A(t).t^{-1} \searrow$ (cf. [6] and [7]). Partial converses follow from Theorem 5.

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References

[1] A.M. Garsia, Combinatorial inequalities and smoothness of functions, Bull. Amer. Math. Soc. 82(1976), 157-170.

[2] ___________, A remarkable inequality and the uniform convergence of Fourier series, Indiana Univ. Math. J. 25(1976), 85-102.

[3] C.S. Herz, Lipschitz spaces and Bernstein's theorem on absolutely conver gent Fourier transforms, J. Math. Mech. 18(1968), 283-324.

[4] M. Milman, Embedding of rearrangement invariant spaces in Lorentz spaces, Acta Sc. Math. (to apper).

[5] ___________, An inequality for generalized modulus of continuity, Analysis Mathematica (to apper).

[6] ___________, Some new function spaces and their tensor products, Australian National University 1977.

[7] A. Torchinsky, Interpolation of operation and Orlicz classes, Studia Math. (to appear).

[8] ___________, Mat. Zam. 18(1975), 63-66.

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