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The Bartle-Dunford-Schwartz integral
IV. Applications to integration in locally compact Hausdorff spaces-Part I

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This part consists of Sections 17, 18 and 19. In the sequel T will denote a locally compact Hausdorff space, $\mathcal{B}(T)$ the σ-algebra of the Borel sets in T and $\delta(\mathcal{C})$ the δ-ring generated by the family $\mathcal{C}$ of compact sets in T. The classical Vitali-Carathédory integrability criterion theorem is generalized in Section 17 to $\mathcal{L}_1(\mathbf{m})$ (resp. to $\mathcal{L}_1(\sigma(\mathcal{P}), \mathbf{m})$), where $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\mathbf{m} : \delta(\mathcal{C}) \to X$) is σ-additive and Borel regular (resp. and $\delta(\mathcal{C})$-regular) and X is a quasicomplete or a sequentially complete lcHs. Section 18 is devoted to the study of the Baire version of the classical Dieudonné-Grothendieck theorem and its generalizations to Banach space-valued and sequentially complete lcHs-valued σ-additive regular Borel measures (see Theorems 18.6, 18.21 and 18.23). In Section 19 the concepts of weakly compact and prolongable Radon operators are introduced and several characterizations of these operators are given.

17. GENERALIZATIONS OF THE VITALI-CARATHÉODORY INTEGRABILITY CRITERION THEOREM

Enumerations of sections will be continued from Part III. We adopt the same notation and terminilogy in Parts I,II and III.

The results of the present section are needed in Section 23 of [P13] to describe the duals of $\mathcal{L}_1(\mathbf{m})$ and $\mathcal{L}_1(\mathbf{n})$, where $\mathbf{m} : \mathcal{B}(T) \to X$ (resp. $\mathbf{n} : \delta(\mathcal{C}) \to X$) is σ-additive and $\mathcal{B}(T)$-regular (resp. and $\delta(\mathcal{C})$-regular), T is a locally compact Hausdorff space and X is a Banach space.

In the sequel, T denotes a locally compact Hausdorff space and $\mathcal{U}$, $\mathcal{C}$, and $\mathcal{C}_0$ are as in Definition 16.4 of [P12]; i.e., $\mathcal{U}$ is the family of open sets in T, $\mathcal{C}$ that of compact sets in T and $\mathcal{C}_0$ that of compact G_δ sets in T. Then $\mathcal{B}(T) = \sigma(\mathcal{U})$, the σ-algebra of the Borel sets in T; $\mathcal{B}_c(T) = \sigma(\mathcal{C})$, the σ-ring of the σ-Borel sets in T and $\mathcal{B}_0(T) = \sigma(\mathcal{C}_0)$, the σ-ring of the Baire sets in T. $\delta(\mathcal{C})$ and $\delta(\mathcal{C}_0)$ are the δ-rings generated by $\mathcal{C}$ and $\mathcal{C}_0$.

As in Parts I and III, X denotes a Banach space or an lcHs over $\boldsymbol{K}(= \boldsymbol{R}$ or $\boldsymbol{C})$ with Γ, the family of all continuous seminorms on X, unless otherwise mentioned and it will be explicitly specified whether X is a Banach space or an lcHs. Let $\mathcal{R} = \mathcal{B}(T)$ or $\delta(\mathcal{C})$ and a σ-additive set function $\mathbf{m} : \mathcal{R} \to X$ is said to be $\mathcal{R}$-regular if it satisfies the conditions in Definition 16.7 of [P12].

Lemma 17.1.

Let X be a quasicomplete (resp. sequentially complete) lcHs,$\mathbf{m}: \mathcal{B}(T) \to X$ be σ-additive and Borel regular and $f: T \to [0,\infty)$ (resp. and be $\mathcal{B}(T)$-measurable). Then $f \in \mathcal{L}_1(\mathbf{m})$ (resp. $f \in \mathcal{L}_1(\mathcal{B}(T), \mathbf{m})$) if and only if, given $\epsilon > 0$ and $q \in \Gamma$, there exist functions $u^{(q)}$ and $v^{(q)}$ on T such that $u^{(q)} \le f \le v^{(q)}\, \mathbf{m}_q$-a.e. in T, $u^{(q)}$ is upper semicontinuous in T with compact support and with $u^{(q)}(T) \subset [0,\infty)$, $v^{(q)}$ is lower semicontinuous and $\mathbf{m}_q$-integrable in T, and $(\mathbf{m}_q)^\bullet_1(v^{(q)} - u^{(q)}, T) < \epsilon$.

Proof. In the light of Theorem 15.13(i) of [P12] which states that

$\mathcal{L}_1(\mathbf{m}) = \bigcap_{q\in\Gamma} \mathcal{L}_1(\mathbf{m}_q)$ (resp. $\mathcal{L}_1(\mathcal{B}(T), \mathbf{m}) = \bigcap_{q\in\Gamma} \mathcal{L}_1(\mathcal{B}(T), \mathbf{m}_q)$), it suffices to prove the result for Banach spaces. So we shall assume X to be a Banach space. Suppose the conditions are satisfied for each $\epsilon > 0$. If f is not $\mathcal{B}(T)$-measurable, we first show that it is $\mathbf{m}$-measurable. By hypothesis, for each n, there exist such functions u_n and v_n with $0 \le u_n \le f \le v_n$ $\mathbf{m}$-a.e. in T and with $\mathbf{m}^\bullet_1(v_n - u_n, T) < \frac{1}{n}$. Since v_n is $\mathbf{m}$-integrable in T, by the domination principle (see Theorem 3.5(vii) and Remark 4.3 of [P10]) u_n is also $\mathbf{m}$-integrable in T. Let $g_n = \max_{1\le i\le n} u_i$ and $h_n = \min_{1\le i\le n} v_i$. Then by Theorems 3.3 and 3.5, §3, Ch. III of [MB], g_n is upper semicontinuous and h_n is lower semicontinuous for each n. Moreover, $g_n \nearrow$ and $h_n \searrow$ and by hypothesis and by Theorem 3.5(vii) and Remark 4.3 of [P10], g_n and h_n are $\mathbf{m}$-integrable in T for each n. Let $g = \sup_n g_n$ and $h = \inf_n h_n$. Then $0 \le g \le f \le h$ $\mathbf{m}$-a.e. in T and g and h are $\mathcal{B}(T)$-measurable. Moreover, $0 \le h_n - g_n \le v_1 \in \mathcal{L}_1(\mathbf{m})$ and $h_n - g_n \to h - g$ pointwise in T. Hence by LDCT (see Theorem 3.7 and Remark 4.3 of [P10]), $h - g \in \mathcal{L}_1(\mathbf{m})$ and $\mathbf{m}^\bullet_1((h-g) - (h_n - g_n), T) \to 0$. Moreover, as $0 \le h_n - g_n \le v_n - u_n$, by Theorem 5.11(i) of [P11], $\mathbf{m}^\bullet_1(h_n - g_n, T) \le \mathbf{m}^\bullet_1(v_n - u_n, T) < \frac{1}{n}$ for $n \in \mathbb{N}$. Consequently, by Theorem 5.13(ii) of [P11], $\mathbf{m}^\bullet_1(h - g, T) \le \mathbf{m}^\bullet_1((h-g) - (h_n - g_n), T) + \mathbf{m}^\bullet_1(h_n - g_n, T) \to 0$ as $n \to \infty$. Hence by Theorem 5.18(ii) of [P11], $h = g$ $\mathbf{m}$-a.e. in T. Then $f = g = h$ $\mathbf{m}$-a.e. in T and hence f is $\mathbf{m}$-measurable. Moreover, as $\mathbf{m}^\bullet_1(v_n - f, T) \le \mathbf{m}^\bullet_1(v_n - u_n, T) < \frac{1}{n}$ and as $\mathcal{L}_1(\mathbf{m})$ is complete by Theorem 6.8 of [P11], we conclude that f is $\mathbf{m}$-integrable in T. If f is $\mathcal{B}(T)$-measurable, then by the above argument $\lim_n \mathbf{m}^\bullet_1(v_n - f, T) = 0$ and hence by the second part of Theorem 6.8 of [P11], $f \in \mathcal{L}_1(\mathcal{B}(T), \mathbf{m})$.

To prove the converse, let us assume that f is not identically zero, $f \ge 0$ and $f \in \mathcal{L}_1(\mathbf{m})$ (resp. $f \in \mathcal{L}_1(\mathcal{B}(T), \mathbf{m})$). By Definition 3.1 of [P10], there exists a $\mathcal{B}(T)$-measurable function $\hat{f}: T \to [0,\infty)$ such that $f = \hat{f}$ $\mathbf{m}$-a.e. in T or $\hat{f} = f$ if f is $\mathcal{B}(T)$-measurable. Arguing as in the first paragraph on p.51 of [Ru1], we have

$$\hat{f}(t) = \sum_{i=1}^{\infty} c_i \chi_{E_i}(t), \quad t \in T$$

where $c_i > 0$ and $E_i \in \mathcal{B}(T)$ for all i. Let

$$f_n = \sum_{i=1}^{n} c_i \chi_{E_i}, \quad n \in \mathbb{N}.$$

Then $0 \le f_n \nearrow \hat{f}$ and hence by LDCT (see Theorem 3.7 and Remark 4.3 of [P10]) $\lim_n \mathbf{m}^\bullet_1(\hat{f} -$

$f_n, T) = 0$. Thus, given $\epsilon > 0$, there exists n_0 such that $\mathbf{m}_1^\bullet(\hat{f} - f_{n_0}, T) < \frac{\epsilon}{2}$. That is,

$$\mathbf{m}_1^\bullet(\sum_{n_0+1}^{\infty} c_i\chi_{E_i}, T) < \frac{\epsilon}{2}. \qquad (17.1.1)$$

By the Borel regularity of $\mathbf{m}$ there exist compact sets $(K_i)_1^\infty$ and open sets $(V_i)_1^\infty$ such that $K_i \subset E_i \subset V_i$ and such that

$$c_i||\mathbf{m}||(V_i \backslash K_i) < \frac{\epsilon}{2^{i+1}} \qquad (17.1.2)$$

for $i \in I\!\!N$. Let $v = \sum_{i=1}^\infty c_i\chi_{V_i}$ and $u = \sum_{i=1}^{n_0} c_i\chi_{K_i}$. Then by Theorems 3.3, 3.4 and 3.5 and ex.2 of §3, Ch. III of [MB], v is lower semicontinuous and u is upper semicontinuous in T, $u(T) \subset [0, \infty)$, $\operatorname{supp} u$ is compact and $u \leq \hat{f} \leq v$ in T. If $v_n = \sum_{i=1}^n c_i\chi_{V_i}$, then $(v_n)_1^\infty \subset \mathcal{L}_1(\mathbf{m})$ as v_n are $\mathcal{B}(T)$-simple functions. For $A \subset T$, by Theorem 5.3 of [P11] we have

$$\mathbf{m}_1^\bullet(\chi_A, T) = \sup_{|x^*|\leq 1} \int_T \chi_A dv(x^*\mathbf{m}) = \sup_{|x^*|\leq 1} v(x^*\mathbf{m})(A) = ||\mathbf{m}||(A). \qquad (17.1.3)$$

Now, by (17.1.1), (17.1.2) and (17.1.3) and by Theorem 5.13(ii) of [P11] we have $\mathbf{m}_1^\bullet(v - v_n, T) \leq \mathbf{m}_1^\bullet(\sum_{n+1}^{\infty} c_i\chi_{E_i}, T) + \mathbf{m}_1^\bullet(\sum_{n+1}^{\infty} c_i\chi_{V_i\backslash E_i}, T) < \frac{\epsilon}{2} + \sum_{n+1}^{\infty} c_i||\mathbf{m}||(V_i\backslash E_i) < \epsilon$

for $n \geq n_0$. As $\mathcal{L}_1(\mathbf{m})$ is complete (resp. as v is $\mathcal{B}(T)$-measurable and as $\mathcal{L}_1(\mathcal{B}(T), \mathbf{m})$ is complete) by Theorem 6.8 of [P11], v is $\mathbf{m}$-integrable in T.

Finally, by Theorems 5.11 and 5.13(ii) of [P11] and by (17.1.3) we have

$$\begin{aligned} \mathbf{m}_1^\bullet(v - u, T) &\leq \mathbf{m}_1^\bullet(\sum_1^{n_0} c_i\chi_{V_i\backslash K_i}, T) + \mathbf{m}_1^\bullet(\sum_{n_0+1}^{\infty} c_i\chi_{V_i}, T) \\ &\leq \sum_1^{n_0} c_i||\mathbf{m}||(V_i\backslash K_i) + \mathbf{m}_1^\bullet(\sum_{n_0+1}^{\infty} c_i\chi_{E_i}, T) + \mathbf{m}_1^\bullet(\sum_{n_0+1}^{\infty} c_i\chi_{V_i\backslash K_i}, T) \\ &\leq \sum_1^{\infty} c_i||\mathbf{m}||(V_i\backslash K_i) + \mathbf{m}_1^\bullet(\sum_{n_0+1}^{\infty} c_i\chi_{E_i}, T) < \epsilon. \end{aligned}$$

Hence the lemma holds.

Theorem 17.2 (Generalization of the Vitali-Carathéodory integrability criterion theorem for Borel regular m). Let X be a quasicomplete (resp. sequentially complete) lcHs, $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive and Borel regular and $f : T \to \boldsymbol{R}$(resp. and be $\mathcal{B}(T)$-measurable). Then $f \in \mathcal{L}_1(\mathbf{m})$ (resp. $f \in \mathcal{L}_1(\mathcal{B}(T), \mathbf{m})$) if and only if, given $\epsilon > 0$ and $q \in \Gamma$, there exist functions $u^{(q)}$ and $v^{(q)}$ on T such that $u^{(q)} \leq f \leq v^{(q)}$ $\mathbf{m}_q$-a.e. in T, $u^{(q)}$ is upper semicontinuous, bounded above and $\mathbf{m}_q$-integrable in T, $v^{(q)}$ is lower semicontinuous, bounded below and $\mathbf{m}_q$-integrable in T and $(\mathbf{m}_q)_1^\bullet(v^{(q)} - u^{(q)}, T) < \epsilon$.

Proof. In the light of Theorem 15.13(i) of [P12], it suffices to prove the result for a Banach space X and hence let X be a Banach space. Suppose the conditions hold for each $\epsilon > 0$. If f is not $\mathcal{B}(T)$-measurable, we first show that f is $\mathbf{m}$-measurable. For each n, there exist such functions u_n and v_n with $u_n \leq f \leq v_n$ $\mathbf{m}$-a.e. in T and $\mathbf{m}_1^\bullet(v_n - u_n, T) < \frac{1}{n}$. Let $g_n = \max_{1\leq i\leq n} u_i$ and $h_n = \min_{1\leq i\leq n} v_i$, $g = \sup_n g_n$ and $h = \inf_n h_n$. Then by Theorems 3.3 and 3.5, §3, Ch. III of [MB], $(g_n)_1^\infty$ are upper semicontinuous and $(h_n)_1^\infty$ are lower semicontinuous. Moreover, $g_n \nearrow g$ and $h_n \searrow h$. Hence g and h are $\mathcal{B}(T)$-measurable and $g \leq f \leq h$ $\mathbf{m}$-a.e. in T. Now, $0 \leq h_n - g_n \leq v_n - u_n$ and by hypothesis, $h_n - g_n$ and $v_n - u_n$ are well defined on T. By hypothesis, v_n and u_n are $\mathbf{m}$-integrable in T and hence by Theorem 5.12(ii) of [P11], v_n and u_n are finite $\mathbf{m}$-a.e. in T. Hence $v_n - u_n$ is $\mathbf{m}$-integrable in T. Consequently, by Theorem 3.5(vii) and Remark 4.3 of [P10], $h_n - g_n$ is $\mathbf{m}$-integrable in T for $n \in \mathbb{N}$ and as $0 \leq h_n - g_n \leq v_1 - u_1$, $h_n - g_n \to h - g$ in T and $v_1 - u_1$ is $\mathbf{m}$-integrable in T, by LDCT (see Theorem 3.7 and Remark 4.3 of [P10]), $\lim_n \mathbf{m}_1^\bullet((h_n - g_n) - (h - g), T) = 0$. Moreover, $\mathbf{m}_1^\bullet(h_n - g_n, T) \leq \mathbf{m}_1^\bullet(v_n - u_n, T) < \frac{1}{n} \to 0$ as $n \to \infty$. Hence $\mathbf{m}_1^\bullet(h - g, T) = 0$ so that by Theorem 5.18(ii) of [P11], $h = g$ $\mathbf{m}$-a.e. in T. Consequently, $f = h = g$ $\mathbf{m}$-a.e. in T and hence f is $\mathbf{m}$-measurable.

As $\mathbf{m}_1^\bullet(v_n - f, T) \leq \mathbf{m}_1^\bullet(v_n - u_n, T) < \frac{1}{n} \to 0$ as $n \to \infty$, by Theorem 6.8 of [P11] the function $f \in \mathcal{L}_1(\mathbf{m})$ (resp. $f \in \mathcal{L}_1(\mathcal{B}(T), \mathbf{m})$).

Conversely, let $f \in \mathcal{L}_1(\mathbf{m})$ (resp. $f \in \mathcal{L}_1(\mathcal{B}(T), \mathbf{m})$). By Theorem 3.5(vii) and Remark 4.3 of [P10], $f^+, f^- \in \mathcal{L}_1(\mathbf{m})$ (resp. $f^+, f^- \in \mathcal{L}_1(\mathcal{B}(T), \mathbf{m})$). Using Lemma 17.1 above and Theorem 3.5(vii) and Remark 4.3 of [P10] and arguing as in the general case in the proof of Theorem 2.24 of [Ru1], one can prove the converse. Details are left to the reader.

Theorem 17.3 (Generalization of the Vitali-Carathéodory integrability criterion theorem for $\delta(\mathcal{C})$-regular n). Let X be a quasicomplete (resp. sequentially complete) lcHs, $\mathbf{n} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular and $f : T \to \mathbb{R}$(resp. and be $\mathcal{B}_c(T)$-measurable). Then $f \in \mathcal{L}_1(\mathbf{n})$ (resp. $f \in \mathcal{L}_1(\mathcal{B}_c(T), \mathbf{n})$) if and only if, given $q \in \Gamma$ and $\epsilon > 0$, there exist functions $u^{(q)}$ and $v^{(q)}$ on T such that $u^{(q)} \leq f \leq v^{(q)}$ $\mathbf{n}_q$-a.e. in T, $v^{(q)}$ is lower semicontinuous, bounded below and $\mathbf{n}_q$-integrable in T, $u^{(q)}$ is upper semicontinuous, bounded above and $\mathbf{n}_q$-integrable in T and $(\mathbf{n}_q)_1^\bullet(v^{(q)} - u^{(q)}, T) < \epsilon$.

Proof. First we observe that a Borel measurable function h with

$N(h) = \{t \in T : h(t) \neq 0\}$ σ-bounded (i.e. contained in a countable union of compact sets) is necessarily $\mathcal{B}_c(T)$-measurable and hence $\mathbf{n}_q$-integrable upper semicontinuous and lower semicontinuous functions are $\mathcal{B}_c(T)$-measurable. Using this observation and arguing quite similar to the proof of the sufficiency part of Theorem 17.2 in which $\mathcal{B}(T)$ is replaced by $\mathcal{B}_c(T)$ and $\mathbf{m}$ by $\mathbf{n}$, one can show that the conditions are sufficient.

As seen in the proof of Lemma 17.1, we prove the result assuming X to be a Banach space and $f \in \mathcal{L}_1(\mathbf{n})$(resp. $f \in \mathcal{L}_1(\mathcal{B}_c(T), \mathbf{n})$), $f \geq 0$ and f not identically zero. Then there exists an $\mathbf{n}$-null set $N \in \mathcal{B}_c(T)$ such that $\hat{f} = f\chi_{T\setminus N}$ is $\mathcal{B}_c(T)$-measurable and $N = \emptyset$ when f is $\mathcal{B}_c(T)$-measurable.

Let $s_0 = 0$ and

$$s_n(t) = \begin{cases} \frac{i-1}{2^n} & \text{if } \frac{i-1}{2^n} \leq \hat{f}(t) < \frac{i}{2^n},\ i = 1, 2, ..., 2^n n \\ n & \text{if } \hat{f}(t) \geq n. \end{cases}$$

Then $s_n \nearrow \hat{f}$ in T and $\hat{f}(t) = \sum_{n=1}^{\infty}(s_n - s_{n-1})(t)$. As $N(\hat{f}) \in \mathcal{B}_c(T)$, there exists $(A_n)_1^{\infty} \subset \delta(\mathcal{C})$ with $A_n \nearrow N(\hat{f})$. Then
$\sum_{n=1}^{k}(s_n(t)\chi_{A_n}(t) - s_{n-1}(t)\chi_{A_{n-1}}(t)) = s_k(t)\chi_{A_k}(t)$ and hence
$\hat{f}(t) = \lim_k s_k(t) = \lim_k s_k(t)\chi_{A_k}(t)$. Moreover, it is easy to check that
$\sum_1^{\infty}(s_n\chi_{A_n} - s_{n-1}\chi_{A_{n-1}})$ is of the form $\sum_1^{\infty} c_k\chi_{E_k}$ with $c_k > 0$ for each k and $(E_k)_1^{\infty} \subset \delta(\mathcal{C})$ since $A \cap B \in \delta(\mathcal{C})$ for $A \in \delta(\mathcal{C})$ and $B \in \mathcal{B}_c(T)$. Thus

$$\hat{f}(t) = \sum_{k=1}^{\infty} c_k\chi_{E_k},\ c_k > 0 \text{ for each } k \text{ and } (E_k)_1^{\infty} \subset \delta(\mathcal{C}).$$

As $\mathbf{n}$ is $\delta(\mathcal{C})$-regular, given $\epsilon > 0$, there exist an open set $V_n \in \delta(\mathcal{C})$ and a compact K_n such that $K_n \subset E_n \subset V_n$ and such that $||\mathbf{n}||(V_n \backslash K_n) < \frac{\epsilon}{2^{n+1}}$ for $n \in \mathbb{N}$. Then arguing as in the proof of the converse part of Lemma 17.1, one can show the existence of u and v as in the said lemma with u and v $\mathcal{B}_c(T)$-measurable. Then arguing as in the proof of the necessity part of Theorem 17.2 the theorem is proved for real valued f.

Remark 17.4. Arguing as in the proof of Lemma 17.1 and Theorem 17.2, a result generalizing Corollary of Theorem 3, no.4, §4, Ch. IV of [B] can be obtained for $f \in \mathcal{L}_1(\mathbf{m})$ where $\mathbf{m}$ is as in Theorem 17.2. Similarly, an analogous result is true for $f \in \mathcal{L}_1(\mathbf{n})$ where $\mathbf{n} : \delta(\mathcal{C}) \to X$ is σ-additive and $\delta(\mathcal{C})$-regular.

18. THE BAIRE VERSION OF THE DIEUDONNÉ-GROTHENDIECK THEOREM AND ITS VECTOR-VALUED GENERALIZATIONS

We show that the boundedness hypothesis in Corollary 1 of [P4] is redundant and thereby we obtain the Baire version of the Dieudonné-Grothendieck theorem in Theorem 18.6 below. Then using the ideas in the proofs of Proposition 2.11 and Theorem 2.12 of [T], we generalize Theorem 18.6 to σ-additive Borel regular vector measures. (See Theorems 18.21 and 18.23.)

Notation 18.1. $C_c(T) = \{f : T \to \mathbb{K}, f \text{ continuous with compact support}\}$; $C_0(T) = \{f : T \to \mathbb{K}, f \text{ continuous and vanishes at infinity in } T\}$, both the spaces being provided with norm $||\cdot||_T$. $M(T)$ denotes the dual of $(C_0(T), ||\cdot||_T)$ and each member of $M(T)$ is considered as a σ-additive Borel regular scalar measure on $\mathcal{B}(T)$. We write $|\mu|(\cdot) = v(\mu, \mathcal{B}(T))(\cdot)$ for $\mu \in M(T)$. Then $||\mu|| = |\mu|(T)$, for $\mu \in M(T)$. $\mathcal{V}$ is the family of relatively compact open sets in T. $C_c(T)$ endowed with the inductive limit locally convex toplogy as in §1, Ch. III of [B] is denoted by $\mathcal{K}(T)$.

Lemma 18.2. $\delta(\mathcal{C}) = \{A \in \mathcal{B}(T) : \bar{A} \in \mathcal{C}\}$ and $\delta(\mathcal{C}_0) = \{A \in \mathcal{B}_0(T) : \bar{A} \in \mathcal{C}\}$ where $\bar{A}$ denotes the closure of A.

Proof. Let $\mathcal{F}$ be the family of all closed sets in T. Since $\delta(\mathcal{C}) \subset \mathcal{B}_c(T)$ and $\delta(\mathcal{C}_0) \subset \mathcal{B}_0(T)$, and since each member of $\delta(\mathcal{C}) \cup \delta(\mathcal{C}_0)$ is relatively compact, it suffices to show that $\{A \in \mathcal{B}(T) : \bar{A} \in \mathcal{C}\} \subset \delta(\mathcal{C})$ and $\{A \in \mathcal{B}_0(T) : \bar{A} \in \mathcal{C}\} \subset \delta(\mathcal{C}_0)$. Let $A \in \mathcal{B}(T)$ (resp. $A \in \mathcal{B}_0(T)$) with $\bar{A} \in \mathcal{C}$. Then by Theorem 50.D of [H] there exists $C_0 \in \mathcal{C}_0$ such that $\bar{A} \subset C_0$ and hence by Theorem 5.E of [H] we have $A = A \cap C_0 \in \sigma(\mathcal{F}) \cap C_0 = \sigma(\mathcal{F} \cap C_0) = \sigma(\mathcal{C} \cap C_0) = \delta(\mathcal{C} \cap C_0) \subset \delta(\mathcal{C})$ (resp. $A = A \cap C_0 \in \sigma(\mathcal{C}_0) \cap C_0 = \sigma(\mathcal{C}_0 \cap C_0) = \delta(\mathcal{C}_0 \cap C_0) \subset \delta(\mathcal{C}_0)$). Hence the lemma holds.

Lemma 18.3. A σ-compact open set in T is a Baire set. Conversely, every open Baire set in T is σ-compact.

Proof. Let U be open in T and let $U = \bigcup_1^\infty K_n$, $(K_n)_1^\infty \subset \mathcal{C}$. Then by Theorem 50.D of [H], for each K_n there exists $C_n \in \mathcal{C}_0$ such that $K_n \subset C_n \subset U$ and hence $U = \bigcup_1^\infty C_n \in \mathcal{B}_0(T)$. Conversely, if $U \in \mathcal{B}_0(T)$ is open in T, then U is σ-bounded so that there exists $(K_n)_1^\infty \subset \mathcal{C}$ such that $U \subset \bigcup_1^\infty K_n$. Then by Theorem 50.D of [H] and by the previous part there exist relatively compact open Baire sets $(V_n)_1^\infty$ such that $K_n \subset V_n$ for each n. Then $U = \bigcup_1^\infty (U \cap V_n)$ and by Lemma 18.2, each $U \cap V_n \in \delta(\mathcal{C}_0)$. Then U is σ-compact by Proposition 15, §14 of [Din].

Lemma 18.4. Let $(\mu_n)_1^\infty \subset M(T)$ (resp. $\mathbf{m}_n : \mathcal{B}(T) \to X$, $n \in \mathbb{N}$, be σ-additive and Borel regular, where X is an lcHs). Then:

(i) For each open set U in T, there exists an open Baire set V_U in T such that $V_U \subset U$ and $|\mu_n|(U \backslash V_U) = 0$ for all n and consequently,
$\mu_n(U) = \mu_n(V_U)$ for all n (resp. given $q \in \Gamma$, there exists an open Baire set $V_U^{(q)}$ in T such that $V_U^{(q)} \subset U$ and $||\mathbf{m}_n||_q(U \backslash V_U^{(q)}) = 0$ for all n and hence $|\mathbf{m}_n(U) - \mathbf{m}_n(V_U^{(q)})|_q = 0$ for all n).

(ii) If, for each open Baire set V in T, $\sup_n |\mu_n(V)| < \infty$, then

$$\sup_n |\mu_n(U)| < \infty$$

for each open set U in T and consequently, $\sup_n ||\mu_n|| < \infty$.

Proof.
Claim 1. Given an open set U in T, (resp. and $q \in \Gamma$), for each $n \in \mathbb{N}$, there exists an open Baire set V_n in T such that $V_n \subset U$ and $|\mu_n|(U \backslash V_n) = 0$ so that $\mu_n(U) = \mu_n(V_n)$ (resp. there exists an open Baire set $V_n^{(q)}$ in T such that $V_n^{(q)} \subset U$ and $||\mathbf{m}_n||_q(U \backslash V_n^{(q)}) = 0$ so that $|\mathbf{m}_n(U) - \mathbf{m}_n(V_n^{(q)})|_q = 0$).

In fact, let $\nu_n = |\mu_n|$ or $||\mathbf{m}_n||_q$ as the case may be. Then, given $\epsilon = \frac{1}{k}$, $k \in \mathbb{N}$, by the Borel regularity of μ_n and of $\mathbf{m}_n$, there exists $K_k^{(n)} \in \mathcal{C}$, $K_k^{(n)} \subset U$ such that $\nu_n(U \backslash K_k^{(n)}) < \frac{1}{k}$. Then by Theorem 50.D of [H] and by Lemma 18.3 there exists an open Baire set $V_k^{(n)}$ in T such that $K_k^n \subset V_k^{(n)} \subset U$. Then $\nu_n(U \backslash V_k^{(n)}) < \frac{1}{k}$. Let $V_n = \bigcup_{k=1}^\infty V_k^{(n)}$. Then V_n is an open Baire set in T, $V_n \subset U$, and $\nu_n(U \backslash V_n) = 0$.

(i) Let $V_U = \bigcup_1^\infty V_n$ (resp. $V_U^{(q)} = \bigcup_{n=1}^\infty V_n^{(q)}$) where V_n (resp. $V_n^{(q)}$) are chosen as in Claim 1 with respect to U (resp. with respect to U and q). Clearly, V_U (resp. $V_U^{(q)}$) is an open Baire set in T, $V_U \subset U$ and $|\mu_n|(U\backslash V_U) = 0$ (resp. $V_U^{(q)} \subset U$ and $||\mathbf{m}_n||_q(U\backslash V_U^{(q)}) = 0$) for all n. Hence (i) holds.

(ii) By hypothesis and by (i), $\sup_n |\mu_n(U)| = \sup_n |\mu_n(V_U)| < \infty$ for each open set U in T. Then by Theorem T_4 in Appendix I of [T], $\sup_n ||\mu_n|| < \infty$.

The following result is an improvement of the remark under Theorem T_4 in Appendix I of [T].

Corollary 18.5. Let $(\mu_\alpha)_{\alpha\in I} \subset M(T)$. Suppose every sequence from $(\mu_\alpha)_{\alpha\in I}$ is bounded in each open Baire set in T. Then $\sup_{\alpha\in I} ||\mu_\alpha|| < \infty$.

Proof. Otherwise, for each $n \in \mathbb{N}$, there would exist $\alpha_n \in I$ such that $||\mu_{\alpha_n}|| > n$. On the other hand, the hypothesis and Lemma 18.4(ii) would imply that $\sup_n ||\mu_{\alpha_n}|| < \infty$, a contradiction.

Theorem 18.6 (The Baire version of the Dieudonné-Grothendieck theorem). A sequence (μ_n) in $M(T)$ is weakly convergent if and only if, for each open Baire set U in T, $\lim_n \mu_n(U)$ exists in $\boldsymbol{K}$ or equivalently, there exists $\mu \in M(T)$ such that

$$\lim_n \int_T f d\mu_n = \int_T f d\mu \qquad (18.6.1)$$

for each bounded Borel measurable scalar function f on T if and only if $\lim_n \mu_n(U)$ exists in $\boldsymbol{K}$ for each open Baire set U in T. In that case, μ is unique.

Proof. If (μ_n) converges weakly to $\mu \in M(T)$, then (18.6.1) holds and particularly, $\lim_n \mu_n(U) = \mu(U) \in \boldsymbol{K}$ holds for each open Baire set U in T.

Conversely, if $\lim_n \mu_n(U) \in \boldsymbol{K}$ for each open Baire set U in T, then $\sup_n |\mu_n(U)| < \infty$ for each open Baire set U in T and hence by Lemma 18.4(ii), $\sup_n ||\mu_n|| < \infty$. Consequently, by Corollary 1 of [P4], (μ_n) converges weakly to some $\mu \in M(T)$ so that (18.6.1) holds. Since the weak toplogy of $M(T)$ is Hausdorff, the weak limit μ is unique.

Remark 18.7. In the light of Lemma 18.4(ii), the boundedness hypothesis in Corollary 1 of [P4] is redundant. This has already been noted in Remark 9.18 of [P7].

The following theorem generalizes Lemma 18.4(ii) to lcHs-valued σ-additive regular Borel measures on T.

Theorem 18.8. Let X be an lcHs and let $\mathbf{m}_n : \mathcal{B}(T) \to X$ be σ-additive and Borel regular for $n \in I\!N$. If $(\mathbf{m}_n(V))_1^\infty$ is bounded for each open Baire set V in T, then $\sup_n ||\mathbf{m}_n||_q(T) < \infty$ for each $q \in \Gamma$.

Proof. Let $q \in \Gamma$ and let V be an open Baire set in T. Then by hypothesis, $\sup_n q(\mathbf{m}_n(V)) < \infty$. If $U_q = \{x \in X : q(x) \leq 1\}$, then by hypothesis, by Proposition 10.14(i) of [P12] and by Proposition 2.2 of [P10], $\sup_n \sup_{x^* \in U_q^o} |x^* \circ \mathbf{m}_n|(V) < \infty$. Consequently, by Corollary 18.5, $\sup_n \sup_{x^* \in U_q^o} |x^* \circ \mathbf{m}_n|(T) < \infty$ and hence by Proposition 10.14(ii)(c) of [P12] we have

$$\sup_n ||\mathbf{m}_n||_q(T) = \sup_n \sup_{x^* \in U_q^o} |x^* \circ \mathbf{m}_n|(T) < \infty.$$

Hence the theorem holds.

The rest of the section is devoted to generalize Theorem 18.6 to Banach space and sequentially complete lcHs valued σ-additive regular Borel measures.

We start recalling the following definition from [T].

Definition 18.9. Let X be an lcHs with topology τ. A locally convex Hausdorff topology τ' on X is said to possess the Orlicz property when all the formal series $\sum x_n$ of elements in X which are subseries convergent in the topology τ' are unconditionally convergent in τ. A subset H of X^* is said to possess the Orlicz property when the topology $\sigma(X, H)$ possesses the Orlicz property.

Notation and Terminology 18.10. Let X be an lcHs with topology τ. X^{**} is the bidual of X when X^* is endowed with the strong topology $\beta(X^*, X)$ generated by the seminorms $\{q_B : B \text{ bounded in } X\}$, where $q_B(x^*) = \sup_{x \in B} |x^*(x)|$ for $x^* \in X^*$. The topology τ_e on X^{**} of uniform convergence in equicontinuous subsets of X^* is generated by the seminorms $\{q_E : E \in \mathcal{E}\}$ (see Notation 10.10 of [P12]) where $q_E(x^{**}) = \sup_{x^* \in E} |x^{**}(x^*)|$ for $x^{**} \in X^{**}$. If $u : C_0(T) \to X$ is a continuous linear map, then the adjoint $u^* : (X^*, \beta(X^*, X)) \to M(T)$ and biadjoint $u^{**} : (C_0(T))^{**} \to (X^{**}, \tau_e)$ are continuous and linear and $u^{**}|_{C_0(T)} = u$, where (X, τ) is identified as a subspace of (X^{**}, τ_e). For details see [Ho]. By Theorem 1 of [P5], for each continuous linear mapping $u : C_0(T) \to X$ there exists a unique X^{**}-valued vector measure (i.e. additive set function) $\mathbf{m}$ on $\mathcal{B}(T)$ such that $x^* \circ \mathbf{m} = u^* x^* \in M(T)$ for $x^* \in X^*$, the mapping $x^* \to x^* \circ \mathbf{m}$ of X^* into $M(T)$ is weak*-weak* continuous and $x^* u(\varphi) = \int_T \varphi d(x^* \circ \mathbf{m})$ for each $\varphi \in C_0(T)$ and $x^* \in X^*$. Then $\mathbf{m}(A) = u^{**}(\chi_A)$ for $A \in \mathcal{B}(T)$ and $\{\mathbf{m}(A) : A \in \mathcal{B}(T)\}$ is τ_e-bounded in X^{**}. Such $\mathbf{m}$ is called the representing measure of u (see Definition 4 of [P5]).

Proposition 18.11. Let X be an LcHs and let $u : C_0(T) \to X$ be a continuous linear mapping with the representing measure $\mathbf{m}$. Then each $\varphi \in C_0(T)$ is $\mathbf{m}$-integrable in the sense of Definition 1 of [P3] and $u(\varphi) = \int_T \varphi d\mathbf{m}$ (considering X as a subspace of X^{**}).

Proof. By Theorem 1 of [P5], the range of $\mathbf{m}$ is bounded in (X^{**}, τ_e). Since each $\varphi \in C_0(T)$ is a bounded Borel measurable function, there exists a sequence (s_n) of $\mathcal{B}(T)$-simple functions converging to φ uniformly in T. Hence φ is $\mathbf{m}$-integrable in the sense of Definition 1 of [P3] with $\int_T \varphi d\mathbf{m} \in \widetilde{(X^{**}, \tau_e)}$, the completion of (X^{**}, τ_e). Then, for $x^* \in X^*$, by Theorem 1 of [P5] and by Lemma 6 of [P3] we have, $x^* u(\varphi) = \int_T \varphi d(x^* \circ \mathbf{m}) = x^*(\int_T \varphi d\mathbf{m})$. As $u(\varphi) \in X$ and $\int_T \varphi d\mathbf{m} \in \widetilde{(X^{**}, \tau_e)}$, it follows that $q_E(u(\varphi) - \int_T \varphi d\mathbf{m}) = 0$ for each $E \in \mathcal{E}$. Hence $u(\varphi) = \int_T \varphi d\mathbf{m}$ so that $\int_T \varphi d\mathbf{m} \in X$.

Remark 18.12. In the light of the above proposition, the hypothesis of quasicompleteness in (vi) of Proposition 5 in [P6] is redundant.

The following lemma is needed in the proof of Theorem 18.14 which is an improvement of Theorem 3(vii) of [P5] and is motivated by Theorem 2.7 of [T] whose proof is adapted here.

Lemma 18.13. Let X be a normed space and let H be a norm determining set in X^*. Then $H \subset \{x^* \in X^* : |x^*| \leq 1\}$.

Proof. Let $x^* \in H$. Then, for $|x| \leq 1$, we have

$$|x^*(x)| \leq \sup_{y^* \in H} |y^*(x)| = |x| \leq 1$$

and hence $|x^*| = \sup_{|x| \leq 1} |x^*(x)| \leq 1$.

Theorem 18.14. Let X be a Banach space and let $u : C_0(T) \to X$ be a continuous linear mapping with the representing measure $\mathbf{m}$ on $\mathcal{B}(T)$. Let H be a norm determining set in X^* with the Orlicz property. Then u is weakly compact if and only if for each open Baire set U in T there exists a vector $x_U \in X$ such that

$$(x^* \circ \mathbf{m})(U) = x^*(x_U) \qquad (18.14.1)$$

for $x^* \in H$.

Proof. If u is weakly compact, then by Theorem 2(ii) of [P5], $\mathbf{m}$ has range in X and hence the condition is necessary.

Conversely, let (18.14.1) hold. Let (U_n) be a disjoint sequence of open Baire sets in T. For a subsequence P of $I\!\!N$, by (18.14.1) we have

$$x^*(x_{\bigcup_{n \in P} U_n}) = (x^* \circ \mathbf{m})(\bigcup_{n \in P} U_n) = \sum_{n \in P} (x^* \circ \mathbf{m})(U_n) = \sum_{n \in P} x^*(x_{U_n}) \qquad (18.14.2)$$

for each $x^* \in H$ and hence for each $x^* \in < H >$, where $< H >$ is the linear span of H. Since H is a norm determining set, $\sigma(X, H)$ is Hausdorff. Then by Theorem V.3.9 of [DS], $(X, \sigma(X, H))^* =< H >$ and hence (18.14.2) implies that $\sum_{n=1}^{\infty} x_{U_n}$ is subseries convergent in $\sigma(X, H)$. Then, as H has the Orlicz property by hypothesis, $\sum_1^{\infty} x_{U_n}$ is unconditionally convergent in the norm topology of X. Therefore, $\lim_n |x_{U_n}| = 0$ so that $\lim_n \sup_{x^* \in H} |x^*(x_{U_n})| = 0$. Consequently, by (18.14.1), $\lim_n \sup_{x^* \in H} |(x^* \circ \mathbf{m})(U_n)| = 0$. Then by Lemma 18.13 and by

Theorem 1 of [P4], $\mathbf{m}_H = \{x^* \circ \mathbf{m} : x^* \in H\}$ is relatively weakly compact in $M(T)$.

Claim 1. That $\mathbf{m}_H$ is relatively weakly compact in $M(T)$ implies that u is weakly compact.

In fact, by the said theorem of [P4], given an open Baire set U in T (resp. for T) and $\epsilon > 0$, there exists $K \in \mathcal{C}_0$ such that $K \subset U$ and

$$\sup_{x^* \in H} |x^* \circ \mathbf{m}|(U \backslash K) < \epsilon \qquad (18.14.3)$$

(resp.

$$\sup_{x^* \in H} |x^* \circ \mathbf{m}|(T \backslash K) < \epsilon. \qquad (18.14.3'))$$

Since $u^* x^* = x^* \circ \mathbf{m}$ by 18.10, by (18.14.3) and (18.14.3') we have

$$\sup_{x^* \in H} |u^* x^*|(U \backslash K) < \epsilon \qquad (18.14.4)$$

where U is the given open Baire set or $U = T$.

For such U, $\chi_{U \backslash K}$ is lower semicontinuous and hence we have

$$\begin{aligned} \sup_{\varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U\backslash K}} |u(\varphi)| &= \sup_{\varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U\backslash K}} \sup_{x^* \in H} |x^* u(\varphi)| \\ &= \sup_{x^* \in H} \sup_{\varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U\backslash K}} |(u^* x^*)(\varphi)|. \qquad (18.14.5) \end{aligned}$$

On the other hand,

$$\begin{aligned} \sup_{\varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U\backslash K}} |(u^* x^*)(\varphi)| &= \sup_{|\psi| \le |\varphi|, \psi, \varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U\backslash K}} |u^* x^*(\psi)| \\ &= \sup_{\varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U\backslash K}} |u^* x^*|(|\varphi|) \\ &= |u^* x^*|^*(\chi_{U\backslash K}) \\ &= |u^* x^*|^*(U \backslash K) \qquad (18.14.6) \end{aligned}$$

by (12) on p.55 of [B] and by Definitions 1 and 2, §1, Ch. IV of [B]. By Corollary 3 of Theorem 2, §5, no.5 of Ch. IV of [B], the Borel sets in T are $|u^* x^*|$-measurable and by an abuse of notation let us denote $|u^* x^*|^*|_{\mathcal{B}(T)}$ also by $|u^* x^*|$. Then by (18.14.5) and (18.14.6) we have

$$\sup_{\varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U\backslash K}} |u(\varphi)| = \sup_{x^* \in H} |(u^* x^*)|(U \backslash K). \qquad (18.14.7)$$

As $u^* x^* = x^* \circ \mathbf{m} \in M(T)$, by Theorem 4.11 of [P1], by the last part of Theorem 3.3 of [P2] and by Notation 18.1 above, we have

$$\mu_{|u^* x^*|} = var(\mu_{u^* x^*}, \mathcal{B}(T)) = |\mu_{u^* x^*}| \qquad (18.14.8)$$

where $\mu_{u^*x^*}$ is the complex Radon measure induced by u^*x^* in the sense of Definition 4.3 of [P1]. Note that $\mu_{u^*x^*}$ is the same as $x^* \circ \mathbf{m}$ as $u^*x^* \in M(T)$ (see 18.10). Then by (18.14.3), (18.14.3'), (18.14.7) and (18.14.8) we have

$$\begin{aligned}
\sup_{\varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U \backslash K}} |u(\varphi)| &= \sup_{x^* \in H} |(u^*x^*)|(U \backslash K) \\
&= \sup_{x^* \in H} \mu_{|u^*x^*|}(U \backslash K) \\
&= \sup_{x^* \in H} |\mu_{u^*x^*}|(U \backslash K) \\
&= \sup_{x^* \in H} |x^* \circ \mathbf{m}|(U \backslash K) < \epsilon \qquad (18.14.9)
\end{aligned}$$

where U is the given open Baire set or $U = T$.

On the other hand, by (18.14.4), (18.14.6) and (18.14.9) and by (12) on p.55 of [B] we have

$$\begin{aligned}
\epsilon > \sup_{\varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U \backslash K}} |u(\varphi)| &= \sup_{\varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U \backslash K}} \sup_{|x^*| \le 1} |(x^*u)(\varphi)| \\
&= \sup_{|x^*| \le 1} \sup_{\varphi \in \mathcal{K}(T), |\varphi| \le \chi_{U \backslash K}} |(u^*x^*)(\varphi)| \\
&= \sup_{|x^*| \le 1} |u^*x^*|^*(U \backslash K) \\
&= \sup_{|x^*| \le 1} |u^*x^*|(U \backslash K) \\
&= \sup_{|x^*| \le 1} v(x^* \circ \mathbf{m})(U \backslash K) \qquad (18.14.10)
\end{aligned}$$

since $U \backslash K$ is $|u^*x^*|$-measurable by Corollary 3 of Theorem 2, §5, no.5 of Ch. IV of [B].

Since $\mathbf{m} : \mathcal{B}(T) \to X^{**}$ is additive and $|x^{**}| = \sup_{|x^*| \le 1} |x^{**}(x^*)|$ for $x^{**} \in X^{**}$, by an argument similar to the proof of Proposition 10.12(iii) of [P12] and by (18.14.10) we have

$$||\mathbf{m}||(U \backslash K) = \sup_{|x^*| \le 1} v(x^* \circ \mathbf{m})(U \backslash K) < \epsilon$$

where U is the given open Baire set or $U = T$. Therefore, $\mathbf{m}$ is Baire inner regular in each open Baire set U in T and in the set T in the norm topology of X^{**}, which is the same as τ_e for X^{**}. Hence by Theorem 8(xxix) of [P5], u is weakly compact.

Corollary 18.15. Let X be a Banach space, H be a norm determining subset of X^* and $u : C_0(T) \to X$ be a continuous linear mapping. Let $\mathcal{K}(T)_b^* = (C_c(T), ||\cdot||_T)^*$- the set of all bounded linear functionals on $\mathcal{K}(T)$ (see pp.65 and 69 of [P2]). If $\boldsymbol{\eta} : \mathcal{K}(T)_b^* \to M(T)$ is the isometric isomorphism given in Theorem 5.3 of [P2] ($\boldsymbol{\eta} = \Phi^{-1}_{\mathcal{B}(T)}$ in the notation of Theorem 5.3 of [P2]), then $\boldsymbol{\eta}(\theta) = \mu_\theta|\mathcal{B}(T)$ for $\theta \in \mathcal{K}(T)_b^*$, and hence $\boldsymbol{\eta}(x^*u) = x^* \circ \mathbf{m}$ for $x^* \in X^*$ where $\mathbf{m}$ is the representing measure of u. Moreover, if $\boldsymbol{\eta}\{x^*u : x^* \in H\}$ is relatively weakly compact in $M(T)$, then u is weakly compact.

Proof. Clearly, x^*u is a bounded linear functional on $C_0(T)$ and hence $x^*u \in \mathcal{K}(T)_b^*$. Moreover, $(x^*u)(\varphi) = (u^*x^*)(\varphi) = (x^* \circ \mathbf{m})(\varphi)$ for $\varphi \in C_0(T)$ (see 18.10). Then

$$(x^*u)(\varphi) = \int_T \varphi d(x^* \circ \mathbf{m}), \quad \varphi \in C_0(T).$$

Consequently, by Theorem 5.3 of [P2], $\boldsymbol{\eta}(x^*u) = x^* \circ \mathbf{m}$, $x^* \in X^*$. Then by hypothesis, $\{x^* \circ \mathbf{m} : x^* \in H\}$ is relatively weakly compact in $M(T)$ and hence by Claim 1 in the proof of Theorem 15.14, u is weakly compact.

The following theorem is motivated by Theorem 2.7 bis in [T] and its proof in [T] is adapted here.

Theorem 18.16. Let X be a quasicomplete lcHs with topology τ and let H be a subset of X^* having the Orlicz property such that the topology τ is identical with the topology of uniform convergence in equicontinuous subsets of H. Let $u : C_0(T) \to X$ be a continuous linear mapping with the representing measure $\mathbf{m}$. Then u is weakly compact if and only if for each open Baire set U in T there exists a vector $x_U \in X$ such that

$$(x^* \circ \mathbf{m})(U) = x^*(x_U) \qquad (18.16.1)$$

for $x^* \in H$.

Proof. If u is weakly compact, then by Theorem 2(ii) of [P5] the condition is necessary.

Conversely, let (18.16.1) hold. Let $H_{\mathcal{E}} = \{E \subset H : E \text{ is equicontinuous}\}$. If $x^*(x) = 0$ for each $x^* \in H$, then for $E \in H_{\mathcal{E}}$, $q_E(x) = \sup_{x^* \in E} |x^*(x)| = 0$. Since τ is the same as the locally convex topology generated by $\{q_E : E \in H_{\mathcal{E}}\}$, it follows that $x = 0$ and hence $\sigma(X, H)$ is Hausdorff.

Let (U_n) be a disjoint sequence of open Baire sets in T. Arguing as in the proof of Theorem 18.14, for a subsequence P of $\mathbb{N}$, by (18.16.1) we have $\sum_{n \in P} x^*(x_{U_n})$ is subseries convergent for $x^* \in < H >$, the linear span of H. Since $(X, \sigma(X,H))^* = < H >$ by Theorem V.3.9 of [DS], $\sum_1^\infty x_{U_n}$ is subseries convergent in $\sigma(X,H)$. By hypothesis, H has the Orlicz property and hence $\sum_1^\infty x_{U_n}$ is unconditionally convergent in τ. Let $E \in H_{\mathcal{E}}$. Then $\lim_n q_E(x_{U_n}) = 0$ and hence by (18.16.1) we have

$$\lim_n \sup_{x^* \in E} |(x^* \circ \mathbf{m})(U_n)| = 0. \qquad (18.16.2)$$

Since the range of $\mathbf{m}$ is bounded in τ_e by Theorem 1 of [P5], $\sup_{A \in \mathcal{B}(T)} q_E(\mathbf{m}(A)) < \infty$ and hence $\{x^* \circ \mathbf{m} : x^* \in E\}$ is bounded in $M(T)$. Then by (18.16.2) and by Theorem 1 of [P4]

$$(*)\, \{x^* \circ \mathbf{m} : x^* \in E\} \text{ is relatively weakly compact in } M(T).$$

Let $\widetilde{X}$ be the completion of X. Let $\Pi_{q_E} : \widetilde{X} \to \widetilde{X}_{q_E} \subset \widetilde{\widetilde{X}_{q_E}}$ for $E \in H_{\mathcal{E}}$, where $\widetilde{\widetilde{X}_{q_E}}$ is the completion of the normed space $\tilde{X}_{q_E}$. If Ψ_{x^*} is as in Proposition 10.12(i) of [P12], then by

Proposition 10.12(ii) $\{\Psi_{x^*} : x^* \in E\}$ is a norm determining set for $\widetilde{X}_{q_E}$, $E \in H_{\mathcal{E}}$. Then by Proposition 10.12(i) and by (18.10) we have $(\Psi_{x^*} \circ \Pi_{q_E} \circ u)(\varphi) = (x^*u)(\varphi) = \int_T \varphi d(x^* \circ \mathbf{m})$ for $\varphi \in C_0(T)$ and clearly, $\Psi_{x^*} \circ \Pi_{q_E} \circ u \in \mathcal{K}(T)^*_b$. Therefore, $\boldsymbol{\eta}(\Psi_{x^*} \circ \Pi_{q_E} \circ u) = x^* \circ \mathbf{m}$ where $\boldsymbol{\eta}$ is as in Corollary 18.15. Then by (*) and by the latter corollary, $\Pi_{q_E} \circ u$ is weakly compact for each $E \in H_{\mathcal{E}}$. Consequently, by Lemma 2.21 of [T], which holds for complex lcHs too, we conclude that u is weakly compact.

Theorem 18.17. Let X be a quasicomplete lcHs with topology τ, H be a subset of X^* having the Orlicz property such that τ is the same as the topology of uniform convergence in equicontinuous subsets of H and $u : C_0(T) \to X$ be a continuous linear mapping with the representing measure $\mathbf{m}$. Suppose u_H is the same as u on $C_0(T)$ with X provided with the topology $\sigma(X, H)$. Then u is weakly compact if and only if for each open Baire set U in T there exists a vector $x_U \in X$ such that

$$(x^*u_H^{**})(\chi_U) = x^*(x_U) \qquad (18.17.1)$$

for $x^* \in H$. Moreover, condition (18.17.1) is the same as $(x^* \circ \mathbf{m})(U)) = x^*(x_U)$ for open Baire sets U in T and for $x^* \in H$.

Proof. As observed in the proof of Theorem 18.16, the hypothesis on τ implies that the topology $\sigma(X, H)$ is Hausdorff and hence $(X, \sigma(X, H))$ is an lcHs. As $\sigma(X, H)$ is weaker than τ, $u : C_0(T) \to (X, \sigma(X, H))$ is continuous and hence u_H is a continuous linear map. Therefore, $x^*u_H^{**}(\chi_A) = (u_H^*x^*)(\chi_A)$ for $x^* \in (X, \sigma(X, H))^*$ and for $A \in \mathcal{B}(T)$. Since $(X, \sigma(X, H))^* =< H >$, the linear subspace spanned by H by Theorem V.3.9 of [DS], particularly we have

$$x^*u_H^{**}(\chi_A) = (u_H^*x^*)(\chi_A) \qquad (18.17.2)$$

for $x^* \in H$. Since

$$(u_H^*x^*)(\varphi) = x^*(u_H\varphi) = x^*(u\varphi) = u^*x^*(\varphi)$$

for $\varphi \in C_0(T)$ and for $x^* \in H$, we have

$$u_H^*x^* = u^*x^* \qquad (18.17.3)$$

for $x^* \in H$. Consequently, by (18.17.2) and (18.17.3) we have

$$x^*u_H^{**}(\chi_A) = (u_H^*x^*)(\chi_A) = (u^*x^*)(\chi_A) = x^*u^{**}(\chi_A) = (x^* \circ \mathbf{m})(A) \qquad (18.17.4)$$

for $x^* \in H$ and for $A \in \mathcal{B}(T)$, since $\mathbf{m}$ is the representing measure of u. Hence the hypothesis (18.17.1) is equivalent to saying that

$$(x^* \circ \mathbf{m})(U) = x^*u_H^{**}(\chi_U) = x^*(x_U)$$

for $x^* \in H$ and for open Baire sets U in T. Consequently, by Theorem 18.16, u is weakly compact.

Conversely, if u is weakly compact, then by Theorem 2(ii) of [P5] $\mathbf{m}$ has range in X and hence by (18.17.4), (18.17.1) holds.

Lemma 18.18. Let X be a sequentially complete lcHs and let $\mathbf{m}_n : \mathcal{B}(T) \to X$ be σ-additive and Borel regular for $n \in I\!N$. Then $\lim_n \mathbf{m}_n(U) \in X$ for each open Baire set U in T if and only if $\lim_n \mathbf{m}_n(U) \in X$ for each open set U in T.

Proof. Clearly the condition is sufficient. Conversely, let $\lim_n \mathbf{m}_n(V) \in X$ for each open Baire set V in T. Let U be an open set in T and $q \in \Gamma$. Then by Lemma 18.4(i) there exists an open Baire set $V_U^{(q)} \subset U$ such that $|\mathbf{m}_n(U) - \mathbf{m}_n(V_U^{(q)})|_q = 0$ for all n. By hypothesis, $\lim_n \mathbf{m}_n(V_U^{(q)}) = x_q$ (say) exists in X, for each $q \in \Gamma$. Then

$$|\mathbf{m}_n(U) - x_q|_q \leq |\mathbf{m}_n(U) - \mathbf{m}_n(V_U^{(q)})|_q + |\mathbf{m}_n(V_U^{(q)}) - x_q|_q \to 0$$

as $n \to \infty$ and hence $|\mathbf{m}_n(U) - \mathbf{m}_k(U)|_q \to 0$ as $n, k \to \infty$. Since q is arbitrary in Γ, this implies that $(\mathbf{m}_n(U))$ is Cauchy in X. As X is sequentially complete, there exists $x_U \in X$ such that $\lim_n \mathbf{m}_n(U) = x_U$. Hence the lemma holds.

Lemma 18.19. Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{B}(T) \to X$ be σ-additive. Then each $\varphi \in C_0(T)$ is $\mathbf{m}$-integrable in the sense of Definition 1 of [P3] as well as $\mathbf{m}$-integrable in the sense of Definition 12.1 of [P12] and both the integrals of φ coincide. If $u : C_0(T) \to X$ is given by $u(\varphi) = \int_T \varphi d\mathbf{m}$ for $\varphi \in C_0(T)$, then u is a weakly compact operator. If $\mathbf{m}$ is further Borel regular, then $\mathbf{m}$ is the representing measure of u (see Notation and Terminology 18.10).

Proof. Since $\mathbf{m}$ is σ-additive on the σ-algebra $\mathcal{B}(T)$, $||\mathbf{m}||_q(T) < \infty$ for each $q \in \Gamma$. Since $\varphi \in C_0(T)$ is bounded and Borel measurable, there exists a sequence (s_n) of $\mathcal{B}(T)$-simple functions such that $s_n \to \varphi$ uniformly in T with $|s_n| \nearrow |\varphi|$. Then, given $q \in \Gamma$,

$$q(\int_T s_n d\mathbf{m} - \int_T s_k d\mathbf{m}) \leq ||s_n - s_k||_T ||\mathbf{m}||_q(T) \to 0$$

as $n, k \to 0$ and hence $(\int_A s_n d\mathbf{m})_1^\infty$ is Cauchy in X for each $A \in \mathcal{B}(T)$. Since X is sequentially complete, φ is $\mathbf{m}$-integrable in the sense of Definition 1 of [P3] and $\int_A \varphi d\mathbf{m} = \lim_n \int_A s_n d\mathbf{m}$ for $A \in \mathcal{B}(T)$. On the other hand, by Definition 12.1' in Remark 12.11 and by Remark 12.13 of [P12], the $\mathcal{B}(T)$-measurable function φ is $\mathbf{m}$-integrable in the sense of Definition 12.1 of [P12] with (BDS)$\int_A \varphi d\mathbf{m} = \int_A \varphi d\mathbf{m}$ for $A \in \mathcal{B}(T)$.

Clearly, u is linear. Moreover, by Theorem 11.9(ii)(b) of [P12] we have

$$q(u\varphi) = q(\int_T \varphi d\mathbf{m}) \leq ||\varphi||_T \cdot ||\mathbf{m}||_q(T)$$

for each $q \in \Gamma$ and hence u is continuous. (See Remark 12.5 of [P12].)

Let $\Sigma(\mathcal{B}(T))$ be the Banach space of all bounded complex functions which are uniform limits of sequences of $\mathcal{B}(T)$-simple functions with norm the supremum norm $||\cdot||_T$. Then $C_0(T)$ is a subspace of $\Sigma(\mathcal{B}(T))$. If $\Phi : \Sigma(\mathcal{B}(T)) \to X$ is given by $\Phi(\varphi) = \int_T \varphi d\mathbf{m}$ with the integral defined in the sense of Definition 1 of [P3], then by Lemma 6 of [P3], Φ is a continuous linear map and

$\mathbf{m}$ is the representing measure of Φ in the sense of Definition 2 of [P3]. Since $\mathcal{B}(T)$ is a σ-algebra and $\mathbf{m}$ is σ-additive on $\mathcal{B}(T)$, $\mathbf{m}$ is strongly additive on $\mathcal{B}(T)$ and hence by Theorem 1 of [P3], Φ is weakly compact. Consequently, $u = \Phi|_{C_0(T)}$ is weakly compact.

Now suppose $\mathbf{m}$ is further Borel regular. Then by Theorem 2(ii) of [P5], the representing measure $\hat{\mathbf{m}}$ (in the sense of Definition 4 of [P5]) of the weakly compact operator u has range in X and by Theorem 1 of [P5], $x^* \circ \hat{\mathbf{m}} \in M(T)$ for $x^* \in X^*$ and $x^*u(\varphi) = \int_T \varphi d(x^* \circ \hat{\mathbf{m}})$ for $\varphi \in C_0(T)$. On the other hand, $u(\varphi) = \int_T \varphi d\mathbf{m}$ and hence by Lemma 6(iii) of [P3] we have $\int_T \varphi d(x^* \circ \mathbf{m}) = x^*u(\varphi)$ for $\varphi \in C_0(T)$. Thus we have

$$x^*u(\varphi) = \int_T \varphi d(x^* \circ \mathbf{m}) = \int_T \varphi d(x^* \circ \hat{\mathbf{m}}), \ \ \varphi \in C_0(T).$$

Since $x^* \circ \mathbf{m} \in M(T)$ by hypothesis, by the uniqueness part of the Riesz representation theorem, $x^* \circ \hat{\mathbf{m}} = x^* \circ \mathbf{m}$ for $x^* \in X^*$ and consequently, by the Hahn-Banach theorem we have $\mathbf{m} = \hat{\mathbf{m}}$ and hence $\mathbf{m}$ is the representing measure of u (in the sense of (18.10)).

The proof of (i) in the following lemma is motivated by the proof of Theorem 2.12 of [T].

Lemma 18.20. Let X be a sequentially complete lcHs and let $\mathbf{m}_n : \mathcal{B}(T) \to X$ be σ-additive and Borel regular for $n \in \mathbb{N}$. Suppose $\lim_n \mathbf{m}_n(U)$ exists in X for each open Baire set U in T. Let $u_n : C_0(T) \to X$ be given by $u_n(\varphi) = \int_T \varphi d\mathbf{m}_n$ for $\varphi \in C_0(T)$. Then:

(i) $\lim_n u_n(\varphi) = u(\varphi)$ (say) exists in X for each $\varphi \in C_0(T)$.

(ii) u is an X-valued continuous linear mapping on $C_0(T)$.

Proof. By hypothesis and by Lemma 18.18,

$$\lim_n \mathbf{m}_n(U) = \mathbf{m}(U) \,(\text{say}) \qquad (18.20.1)$$

exists in X for each open set U in T and moreover, by Theorem 18.8,

$$\sup_n ||\mathbf{m}_n||_q(T) = M_q \,(\text{say}) < \infty \qquad (18.20.2)$$

for each $q \in \Gamma$.

(i) Let $\varphi \in C_0(T)$, $\varphi \geq 0$. Then there exists a sequence (s_n) of $\mathcal{B}(T)$-simple functions such that $s_n \to \varphi$ uniformly in T and

$$s_n = \sum_{i=2}^{n\cdot 2^n} \frac{i-1}{2^n} \chi_{E_{i,n}}$$

where $n \geq ||\varphi||_T$ and $E_{i,n} = \varphi^{-1}([\frac{i-1}{2^n}, \frac{i}{2^n})) = \varphi^{-1}((-n, \frac{i}{2^n}))\backslash\varphi^{-1}((-n, \frac{i-1}{2^n}))$ for $i = 2, 3, ..., n\cdot 2^n$. Then $E_{i,n}$ is the difference of two open sets and hence s_n is a real linear combination of the characteristic functions of open sets. Consequently, each $\varphi \in C_0(T)$ is the uniform limit of a sequence (s'_n) of $\mathcal{B}(T)$-simple functions with $|s'_n| \nearrow |\varphi|$ and with each s'_n being a complex linear combination of the characteristic functions of open sets. Thus, given $\varphi \in C_0(T)$, $q \in \Gamma$ and $\epsilon > 0$, there exists s of the form $s = \sum_{i=1}^{k} \alpha_i \chi_{U_i}$, U_i open in T, $||s||_T \leq ||\varphi||_T$ and

$$||s - \varphi||_T < \frac{\epsilon}{4M_q}. \qquad (18.20.3)$$

Then by (18.20.1) we have

$$\lim_n \int_T s d\mathbf{m}_n = \lim_n \sum_1^k \alpha_i \mathbf{m}_n(U_i) = \sum_1^k \alpha_i \mathbf{m}(U_i) = x \,(\text{say}).$$

Then there exists n_0 such that

$$|\int_T s d\mathbf{m}_n - x|_q < \frac{\epsilon}{4} \qquad (18.20.4)$$

for $n \geq n_0$. Then by (18.20.3) and (18.20.4) and by Theorem 11.9(i)(b) and Remark 12.5 of [P12] we have

$$\begin{aligned} |u_n(\varphi) - u_r(\varphi)|_q &\leq |u_n(\varphi) - \int_T s d\mathbf{m}_n|_q + |\int_T s d\mathbf{m_n} - \int_T s d\mathbf{m}_r|_q \\ &+ \ |\int_T s d\mathbf{m}_r - u_r(\varphi)|_q \\ &\leq \ ||\varphi - s||_T ||\mathbf{m}_n||_q(T) + |\int_T s d\mathbf{m}_n - x|_q + |\int_T s d\mathbf{m}_r - x|_q + ||s - \varphi||_T ||\mathbf{m}_r||_q(T) \\ &< \ ||\varphi - s||_T \cdot (2M_q) + 2\frac{\epsilon}{4} < \epsilon \end{aligned}$$

for $n, r \geq n_0$. Since q is arbitrary in Γ, this implies that $(u_n(\varphi))$ is Cauchy in X and as X is sequentially complete, there exists a vector $u(\varphi)$ (say) in X such that $\lim_n u_n(\varphi) = u(\varphi)$ for $\varphi \in C_0(T)$. Hence (i) holds.

(ii) Clearly, $u : C_0(T) \to X$ is linear and u is continuous by(i) and by Theorem 2.8 of [Ru2].

The proof of the following theorem is a vector measure adaptation of the proof of Proposition 2.11 of [T].

Theorem 18.21 (Generalization of Theorem 18.6 to Banach space valued σ-additive regular Borel measures). Let X be a Banach space and let $\mathbf{m}_n : \mathcal{B}(T) \to X$ be σ-additive and Borel regular for $n \in \mathbb{N}$. Then $\lim_n \mathbf{m}_n(U) \in X$ for each open Baire set U in T if and only if there exists an X-valued σ-additive measure $\mathbf{m}$ on $\mathcal{B}(T)$ such that

$$\lim_n \int_T f d\mathbf{m}_n = \int_T f d\mathbf{m} (\in X) \qquad (18.21.1)$$

for each bounded $\mathcal{B}(T)$-measurable scalar function f on T. In that case, $\mathbf{m}$ is unique and is Borel regular.

Proof. Suppose $\lim_n \mathbf{m}_n(U) \in X$ for each open Baire set U in T. Let $c_X = \{(x_n)_1^\infty \subset X : \lim_n x_n \in X\}$ be provided with norm $||(x_n)_1^\infty|| = \sup_n |x_n|$. Let $u_n(\varphi) = \int_T \varphi d\mathbf{m}_n$, $\varphi \in C_0(T)$. Then by hypothesis and by Lemma 18.19, $u_n,\ n \in \mathbb{N}$, are X-valued weakly compact operators on $C_0(T)$ with the representing measure $\mathbf{m}_n$. Let $\Phi : C_0(T) \to c_X$ be defined by $\Phi(\varphi) = (u_n(\varphi))_1^\infty$ for $\varphi \in C_0(T)$. By Lemma 18.20(i), Φ is well defined and clearly, linear. By hypothesis and by Theorem 18.8,

$$\sup_n ||\mathbf{m}_n||(T) = M \text{ (say) } < \infty. \qquad (18.21.2)$$

Then by Theorem 11.9(i)(b) and Remark 12.5 of [P12],

$$||\Phi(\varphi)|| = \sup_n |u_n(\varphi)| = \sup_n |\int_T \varphi d\mathbf{m}_n| \leq ||\varphi||_T \cdot \sup_n ||\mathbf{m}_n||(T) = M||\varphi||_T$$

and hence Φ is continuous.

Claim 1. Φ is weakly compact.

In fact, let $H = \{I_{n,x^*} : x^* \in X^*, |x^*| \leq 1, n \in \mathbb{N}\}$, where $< I_{n,x^*}, (x_k)_1^\infty >= x^*(x_n)$. Clearly, $H \subset (c_X)^*$ is a norm determining set for c_X. The proof of Corollary II.5 in Appendix II of [T] holds for complex spaces too and hence by the complex version of the said corollary, H has the Orlicz property for $(c_X, ||\cdot||)$. Let $\Phi : C_0(T) \to (c_X, \sigma(c_X, H))$ be designated as Φ_H so that $\Phi_H(\varphi) = (u_n(\varphi))_1^\infty$, $\varphi \in C_0(T)$. Clearly, Φ_H is continuous as $\sigma(c_X, H)$ is weaker than the norm topology of c_X. Moreover,
$< \Phi^* I_{n,x^*}, \varphi >=< I_{n,x^*}, \Phi(\varphi) >=< I_{n,x^*}, (u_k(\varphi))_1^\infty >= x^* u_n(\varphi) =< u_n^* x^*, \varphi >$ for $\varphi \in C_0(T)$ and hence

$$\Phi^* I_{n,x^*} = u_n^* x^* \qquad (18.21.3)$$

for $I_{n,x^*} \in H$.

On the other hand, by Theorem V.3.9 of [DS], $(c_X, \sigma(c_X, H))^* =< H >\subset (c_X)^*$ where $< H >$ is the linear span of H, and hence we have $< \Phi^* I_{n,x^*}, \varphi >=< I_{n,x^*}, \Phi(\varphi) >=< I_{n,x^*}, \Phi_H(\varphi) >=< \Phi_H^* I_{n,x^*}, \varphi >$ for $\varphi \in C_0(T)$ and hence $\Phi^* I_{n,x^*} = \Phi_H^* I_{n,x^*}$ for each $I_{n,x^*} \in H$. Then by (18.21.3) we have

$$\Phi_H^* I_{n,x^*} = u_n^* x^* \qquad (18.21.4)$$

for $I_{n,x^*} \in H$. By hypothesis, given an open Baire set U in T there exists a vector $x_U = (\mathbf{m}_n(U))_1^\infty \in c_X$. Then, as u_n is a weakly compact operator with the representing measure $\mathbf{m}_n$ by Lemma 18.19, for the open Baire set U in T with x_U as above , we have $< \Phi_H^{**}(\chi_U), I_{n,x^*} >=< \chi_U, \Phi_H^* I_{n,x^*} >=< \chi_U, u_n^* x^* >=< u_n^{**}(\chi_U), x^* >=< \mathbf{m}_n(U), x^* >=< x_U, I_{n,x^*} >$. Thus,

$$I_{n,x^*} \Phi_H^{**}(\chi_U) = I_{n,x^*}(x_U) \qquad (18.21.5)$$

for $I_{n,x^*} \in H$. Then by (18.21.5) and by Theorem 18.17, Φ is weakly compact and hence the claim holds.

Let $\hat{\mathbf{m}}$ be the representing measure of Φ. Then by Theorem 2(ii) of [P5], $\hat{\mathbf{m}}$ has range in c_X so that $\hat{\mathbf{m}}(A) = \Phi^{**}(\chi_A) \in c_X$ for $A \in \mathcal{B}(T)$ and let $\hat{\mathbf{m}}(A) = (x_n)_1^\infty \in c_X$. Then by (18.21.3) we have $x^*(x_n) = I_{n,x^*}\hat{\mathbf{m}}(A) = I_{n,x^*}\Phi^{**}(\chi_A) =< \Phi^* I_{n,x^*}, \chi_A >=< u_n^* x^*, \chi_A >=< x^*, u_n^{**}(\chi_A) >=< x^*, \mathbf{m}_n(A) >$ for $I_{n,x^*} \in H$ and hence $x^*(x_n) = (x^* \circ \mathbf{m}_n)(A))$ for $x^* \in X^*$ and for $A \in \mathcal{B}(T)$. Then by the Hahn-Banach theorem, $x_n = \mathbf{m}_n(A)$ for all n and hence $(\mathbf{m}_n(A))_1^\infty = \hat{\mathbf{m}}(A) \in c_X$. This implies that $\lim_n \mathbf{m}_n(A) = \mathbf{m}(A)\,(\text{say})$ exists in X for each $A \in \mathcal{B}(T)$. Then by VHSN (see Proposition 2.4 of [P10]), $\mathbf{m} : \mathcal{B}(T) \to X$ is σ-additive and hence $||\mathbf{m}||(T) < \infty$.

Let $M_0 = \max(M, ||\mathbf{m}||(T))$ where M is as in (18.21.2). Let f be a bounded $\mathcal{B}(T)$-measurable scalar function. Then there exists a sequence (s_n) of Borel simple functions such that $|s_n| \nearrow |f|$ and $||s_n - f||_T \to 0$ as $n \to \infty$. Thus, given $\epsilon > 0$, there exists n_0 such that

$$||s_{n_0} - f||_T < \frac{\epsilon}{3M_0}. \qquad (18.21.6)$$

Let $s = s_{n_0} = \sum_1^r \alpha_i \chi_{A_i}$, $(A_i)_1^r \subset \mathcal{B}(T)$. Then by (18.21.6) and by Theorem 11.9(i)(b) and Remark 12.5 of [P12] we have

$$|\int_T f d\mathbf{m}_n - \int_T s d\mathbf{m}_n| \leq ||f - s||_T \cdot ||\mathbf{m}_n||(T) < \frac{\epsilon}{3} \qquad (18.21.7)$$

for all n and

$$|\int_T f d\mathbf{m} - \int_T s d\mathbf{m}| \leq ||f - s||_T \cdot ||\mathbf{m}||(T) < \frac{\epsilon}{3}. \qquad (18.21.8)$$

As $\lim_n \mathbf{m}_n(A_i) = \mathbf{m}(A_i)$ for $i = 1, 2, ..., r$, there exists n_1 such that

$$|\alpha_i||\mathbf{m}_n(A_i) - \mathbf{m}(A_i)| < \frac{\epsilon}{3r} \qquad (18.21.9)$$

for $n \geq n_1$ and for $i = 1, 2, ..., r$. Then by (18.21.7), (18.21.8) and (18.21.9) we have

$$|\int_T f d\mathbf{m}_n - \int_T f d\mathbf{m}| < \epsilon$$

for $n \geq n_1$. Hence $\lim_n \int_T f d\mathbf{m}_n = \int_T f d\mathbf{m}$.

The converse is evident. The uniqueness of $\mathbf{m}$ is immediate from (18.21.1) if we take $f = \chi_A$ with $A \in \mathcal{B}(T)$.

Claim 2. $\mathbf{m}$ is Borel regular.

In fact, by Tehorem 6 of [P5], $\hat{\mathbf{m}}$ is Borel regular and hence, given $A \in \mathcal{B}(T)$ and $\epsilon > 0$, there exist an open set U and a compact K in T such that $K \subset A \subset U$ and $||\hat{\mathbf{m}}||(U \backslash K) < \epsilon$.

Since H is norm determining, arguing as in the proof of Proposition 10.12(iii) of [P12] we have

$$\begin{aligned}\epsilon > ||\hat{\mathbf{m}}||(U\backslash K) &= \sup_{n\in\mathbb{N}, x^*\in X^*, |x^*|\leq 1} v(I_{n,x^*}\hat{\mathbf{m}})(U\backslash K)\\ &= \sup_{n\in\mathbb{N}, x^*\in X^*, |x^*|\leq 1} v(I_{n,x^*}(\mathbf{m}_n(U\backslash K))_1^\infty)\\ &= \sup_{n\in\mathbb{N}, x^*\in X^*, |x^*|\leq 1} v(x^*\mathbf{m}_n)(U\backslash K)\\ &= \sup_{n\in\mathbb{N}} ||\mathbf{m}_n||(U\backslash K)\end{aligned}$$

and hence A is uniformly $\mathbf{m}_n$-regular (Borel regular) for $n \in \mathbb{N}$. Moreover,

$$\mathbf{m}(U\backslash K) = \lim_n \mathbf{m}_n(U\backslash K)$$

and

$$\begin{aligned}||\mathbf{m}||(U\backslash K) &= \sup_{|x^*|\leq 1} v(x^*\circ \mathbf{m})(U\backslash K)\\ &= \sup_{|x^*|\leq 1} v(x^*\circ \lim_n \mathbf{m}_n)(U\backslash K)\\ &= \sup_{|x^*|\leq 1} v(\lim_n (x^*\circ \mathbf{m}_n)(U\backslash K)\\ &\leq \sup_n ||\mathbf{m}_n||(U\backslash K)\end{aligned}$$

and hence the claim holds.

This completes the proof of the theorem.

Remark 18.22. Unlike Theorem 18.6 the above result has nothing to do with the weak convergence of $(\mathbf{m}_n)_1^\infty$ since $\mathbf{m}_n \notin M(T)$ for $n \in N$ We also give in [P14] an improved version of Theorem 2.12 of [T].

Theorem 18.23 (Generalization of Theorem 18.6 to sequentially complete lcHs-valued σ-additive regular Borel measures). Let X be a sequentially complete lcHs and let $\mathbf{m}_n : \mathcal{B}(T) \to X$, $n \in \mathbb{N}$, be σ-additive and Borel regular. Then $\lim_n \mathbf{m}_n(U) \in X$ for each open Baire set U in T if and only if there exists an X-valued σ-additive measure $\mathbf{m}$ on $\mathcal{B}(T)$ such that

$$\lim_n \int_T f d\mathbf{m}_n = \int_T f d\mathbf{m} \in X$$

for each bounded $\mathcal{B}(T)$-measurable scalar function f on T. In that case, $\mathbf{m}$ is Borel regular and unique.

Proof. For each $q \in \Gamma$, let $(\mathbf{m}_n)_q = \Pi_q \circ \mathbf{m}_n$. Then $(\mathbf{m}_n)_q : \mathcal{B}(T) \to X_q \subset \widetilde{X_q}$ is σ-additive and Borel regular for each $n \in \mathbb{N}$. Suppose there exists $x_U \in X$ such that $\lim_n \mathbf{m}_n(U) = x_U$

for each open Baire set U in T. Then $\lim_n(\mathbf{m}_n)_q(U) = \Pi_q(x_U) \in X_q \subset \widetilde{X_q}$ for each $q \in \Gamma$. Then by Theorem 18.21 applied to $(\mathbf{m}_n)_q$, $n \in \mathbb{N}$, there exists a Borel regular σ-additive measure $\boldsymbol{\gamma}_q : \mathcal{B}(T) \to \widetilde{X_q}$ such that

$$\lim_n \int_T f d(\mathbf{m}_n)_q = \int_T f d\boldsymbol{\gamma}_q (\in \widetilde{X_q}) \qquad (18.23.1)$$

for each bounded $\mathcal{B}(T)$-measurable scalar function f on T. Then $|(\int_T f d\mathbf{m}_n - \int_T f d\mathbf{m}_k)|_q \to 0$ for each $q \in \Gamma$ and hence $(\int_T f d\mathbf{m}_n)_1^\infty$ is Cauchy in X. Consequently, as X is sequentially complete, $\lim_n \mathbf{m}_n(A) = \mathbf{m}(A)$ (say) exists in X, for each $A \in \mathcal{B}(T)$. Clearly, $\mathbf{m} : \mathcal{B}(T) \to X$ is additive. Moreover, $\lim_n(\mathbf{m}_n)_q(A) = (\Pi_q \circ \mathbf{m})(A)$ for $A \in \mathcal{B}(T)$ and for $q \in \Gamma$. But $\lim_n(\mathbf{m}_n)_q(A) = \boldsymbol{\gamma}_q(A)$ by (18.23.1) for $q \in \Gamma$. Hence

$$(\Pi_q \circ \mathbf{m})(A) = \boldsymbol{\gamma}_q(A) \qquad (18.23.2)$$

for $A \in \mathcal{B}(T)$.

Claim 1. $\mathbf{m} : \mathcal{B}(T) \to X$ is σ-additive.

In fact, let $(A_i)_1^\infty \subset \mathcal{B}(T)$ be a disjoint sequence. Given $q \in \Gamma$ and $\epsilon > 0$, there exists $n_0(q)$ such that $|\boldsymbol{\gamma}_q(\bigcup_1^\infty A_i) - \sum_1^n \boldsymbol{\gamma}_q(A_i)|_q < \epsilon$ for $n \geq n_0(q)$, since $\boldsymbol{\gamma}_q$ is σ-additive on $\mathcal{B}(T)$. Then by (18.23.2) we have, $|\mathbf{m}(\bigcup_1^\infty A_i) - \sum_1^n \mathbf{m}(A_i)|_q = |(\Pi_q \circ \mathbf{m})(\bigcup_1^\infty A_i) - \sum_1^n (\Pi_q \circ \mathbf{m})(A_i)|_q = |\boldsymbol{\gamma}_q(\bigcup_1^\infty A_i) - \sum_1^n \boldsymbol{\gamma}_q(A_i)|_q < \epsilon$ for $n \geq n_0(q)$. Then, as $q \in \Gamma$ is arbitrary, it follows that $\mathbf{m}(\bigcup_1^\infty (A_i) = \sum_1^\infty \mathbf{m}(A_i)$ and hence the claim holds.

Then by Theorem 11.9' in Remark 12.11 of [P12], each bounded Borel function f is $\mathbf{m}$-integrable in T. Now, for $q \in \Gamma$, by (18.23.1) and (18.23.2), by Claim 1 and by Remark 12.5' (see Remark 12.11 of [P12]) and Theorem 11.8(v) of [P12], we have $|\int_T f d\mathbf{m}_n - \int_T f d\mathbf{m}|_q = |\Pi_q(\int_T f d\mathbf{m}_n - \int_T f d\mathbf{m})|_q = |\int_T f d(\Pi_q \circ \mathbf{m}_n) - \int_T f d(\Pi_q \circ \mathbf{m})|_q = |\int_T f d(\mathbf{m}_n)_q - \int_T f d(\boldsymbol{\gamma}_q)|_q \to 0$ as $n \to \infty$. Hence $\lim_n \int_T f d\mathbf{m}_n = \int_T f d\mathbf{m}$ and therefore, the condition is necessary.

Evidently, the condition is also sufficient. The uniqueness of $\mathbf{m}$ is immediate from the equality

$$\lim_n \int_T f d\mathbf{m}_n = \int_T f d\mathbf{m}$$

by taking $f = \chi_A$ with $A \in \mathcal{B}(T)$. $\mathbf{m}$ is Borel regular as $\Pi_q \circ \mathbf{m} = \boldsymbol{\gamma}_q$ for $q \in \Gamma$ by (18.23.2) and as $\boldsymbol{\gamma}_q$ is Borel regular for each $q \in \Gamma$. Hence the theorem holds.

Remark 18.24. Only the Banach space version of Theorem 18.17 which is deduced from Theorem 18.16 is used in the proof of Theorem 18.21. However, Theorem 18.16 in its generality is needed in the proof of Theorem 22.8 of [P13] which improves Theorem 12.2 of [P12] when $\mathcal{P} = \delta(\mathcal{C})$, $\mathbf{m}$ is $\delta(\mathcal{C})$-regular and σ-additive and X is a complete lcHs. Theorem 18.14 is used in the proof of Theorem 22.4 of [P13] which strengthens Theorem 4.2 of [P10] when $\mathcal{P}$ and $\mathbf{m}$ are as above and $\mathbf{m}$ is Banach space-valued.

19. WEAKLY COMPACT AND PROLONGABLE RADON OPERATORS

Notation 19.1. $C_c(T)$ always denoted the normed space $(C_c(T), ||\cdot||_T)$. For $C \in \mathcal{C}$, let $C_c(T, C) = \{f \in C_c(T) : \operatorname{supp} f \subset C\}$ and let $I_C : C_c(T, C) \hookrightarrow C_c(T)$ be the canonical injection. Let ξ be the inductive limit locally convex topology on $C_c(T)$ induced by the family $\{C_c(T, C), I_C\}$, where $C_c(T, C)$ are provided with the topology τ_u of uniform convergence. Then we denote $(C_c(T), \xi)$ by $\mathcal{K}(T)$. It is well known that $\mathcal{K}(T)$ is an lcHs and $\mathcal{K}(T)^*$ denotes the topological dual of $\mathcal{K}(T)$. See §1, Ch. III of [B].

For the convenience of the reader, let us recall the following notation given in the end of Notation 18.1.

Notation 19.2. $\mathcal{V}$ **denotes the family of all relatively compact open sets in** T**.**

In this section, following Thomas [T], we introduce the notions of weakly compact and prolongable Radon operators on $\mathcal{K}(T)$ with values in a quasicomplete lcHs and using the results of [P5] and those of Section 18 above, we give several characterizations of weakly compact Radon operators which are not included in [P5].

Definition 19.3. Let X be an lcHs and let $u : \mathcal{K}(T) \to X$ be a continuous linear mapping. This means, for each $C \in \mathcal{C}$ and $q \in \Gamma$, there exists a finite constant $M_{C,q}$ such that $|u(\varphi)|_q \leq M_{C,q}||\varphi||_T$ for all $\varphi \in C_c(T, C)$. Such a mapping u is called an X-valued Radon operator on $\mathcal{K}(T)$. (Thomas calls it an X-valued Radon measure in [T].)

Theorem 19.4 (Integral representation of Radon operators). Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a Radon operator. Then there exists a vector measure $\mathbf{m} : \delta(\mathcal{C}) \to X^{**}$ such that (i) $x^* \circ \mathbf{m} : \delta(\mathcal{C}) \to \boldsymbol{K}$is σ-additive and $\delta(\mathcal{C})$-regular for each $x^* \in X^*$; (ii) $\{\mathbf{m}(A) : A \in \mathcal{B}(V)\}$ is τ_e-bounded in X^{**} (see Notation and Terminology 18.10) for each $V \in \mathcal{V}$ and (iii) for each $\varphi \in C_c(T)$, $u(\varphi) = \int_T \varphi d\mathbf{m}$ (in the sense of Definition 1 of [P3]), where X is identified as a subspace of (X^{**}, τ_e). Finally, (i)-(iii) determine $\mathbf{m}$ uniquely.

Proof. Let $V \in \mathcal{V}$ and let $u_V = u|_{C_c(V)}$. Let $q \in \Gamma$. For $\varphi \in C_c(V)$, $\operatorname{supp}\varphi \subset \bar{V} \in \mathcal{C}$ and hence $|u_V(\varphi)|_q = |u(\varphi)|_q \leq M_{\bar{V},q}||\varphi||_T$ so that u_V is continuous. As X is sequentially complete, u_V has a unique continuous linear extension $\widetilde{u_V}$ to the whole of $C_0(V)$ with values in X. Then by Theorem 1 of [P5], $\widetilde{u_V}$ has the representing measure $\mathbf{m}_V$ (as an additive set function) on $\mathcal{B}(V)$ with values in X^{**} and $\mathbf{m}_V(A) = \widetilde{u_V}^{**}(\chi_A) = u_V^{**}(\chi_A)$ for $A \in \mathcal{B}(V)$; $x^* \circ \mathbf{m}_V : \mathcal{B}(V) \to \boldsymbol{K}$ is σ-additive and $\mathcal{B}(V)$-regular for $x^* \in X^*$, the mapping $x^* \to x^* \circ \mathbf{m}_V$ of X^* into $M(V)$ is weak*-weak* continuous,

$$x^* u_V(\varphi) = \int_T \varphi d(x^* \circ \mathbf{m}_V) \qquad (19.4.1)$$

for $\varphi \in C_0(V)$ and for $x^* \in X^*$, and $\{\mathbf{m}_V(A) : A \in \mathcal{B}(V)\}$ is τ_e-bounded in X^{**}.

Let $A \in \delta(\mathcal{C})$. Then there exists $V \in \mathcal{V}$ such that $A \subset V$. Let $\mathcal{U}$ be the family of open sets in T and $\mathcal{U}_V$ be that of all open sets in V. Then by Lemma 18.2, $A \in \mathcal{B}(T)$ and by Theorem 5.E of [H], $A \in \sigma(\mathcal{U}) \cap V = \sigma(\mathcal{U} \cap V) = \sigma(\mathcal{U}_V) = \mathcal{B}(V)$ and hence $A \in \mathcal{B}(V)$. Let $\mathbf{m}(A) = \mathbf{m}_V(A)$.

Claim 1. $\mathbf{m} : \delta(\mathcal{C}) \to X^{**}$ is a well defined, vector measure (i.e., an additive set function).

In fact, let $A \in \delta(\mathcal{C})$ and let $V_1, V_2 \in \mathcal{V}$ such that $A \subset V_1 \cap V_2$. Then $A \in \mathcal{B}(V_i)$ and the continuous linear mapping $\widetilde{u_{V_i}}$ has the representing measure $\mathbf{m}_{V_i}$ for $i = 1, 2$. Clearly, $A \in \mathcal{B}(V_1 \cap V_2)$ and for $\varphi \in C_c(V_1 \cap V_2)$, $x^* u_{V_1}(\varphi) = x^* u_{V_2}(\varphi) = x^* u(\varphi) = x^* u_{V_1 \cap V_2}(\varphi)$ for $x^* \in X^*$ and hence by (19.4.1) we have

$$\int_T \varphi d(x^* \circ \mathbf{m}_{V_1}) = \int_T \varphi d(x^* \circ \mathbf{m}_{V_2}) = \int_T \varphi d(x^* \circ \mathbf{m}_{V_1 \cap V_2}) \qquad (19.4.2)$$

for $x^* \in X^*$. As $x^* \circ \mathbf{m}_{V_1 \cap V_2}$, $(x^* \circ \mathbf{m}_{V_1})|_{\mathcal{B}(V_1 \cap V_2)}$ and $(x^* \circ \mathbf{m}_{V_2})|_{\mathcal{B}(V_1 \cap V_2)}$ belong to $M(V_1 \cap V_2)$, by (19.4.2) and by the uniqueness part of the Riesz representation theorem we have $(x^* \circ \mathbf{m}_{V_1})(A) = (x^* \circ \mathbf{m}_{V_2})(A) = (x^* \circ \mathbf{m}_{V_1 \cap V_2})(A)$ for $x^* \in X^*$. As $\mathbf{m}_{V_1}(A)$, $\mathbf{m}_{V_2}(A)$ and $\mathbf{m}_{V_1 \cap V_2}(A)$ belong to X^{**}, we conclude that $\mathbf{m}_{V_1}(A) = \mathbf{m}_{V_2}(A) = \mathbf{m}_{V_1 \cap V_2}(A)$. Hence $\mathbf{m}$ is well defined. Moreover, let $A_1, A_2 \in \delta(\mathcal{C})$ with $A_1 \cap A_2 = \emptyset$. Let $V \in \mathcal{V}$ such that $A_1 \cup A_2 \subset V$. Then, as $\mathbf{m}_V$ is additive on $\mathcal{B}(V)$, we have $\mathbf{m}(A_1 \cup A_2) = \mathbf{m}_V(A_1 \cup A_2) = \mathbf{m}_V(A_1) + \mathbf{m}_V(A_2) = \mathbf{m}(A_1) + \mathbf{m}(A_2)$ and hence $\mathbf{m}$ is additive. Therefore, Claim 1 holds.

Claim 2. $x^* \circ \mathbf{m}$ is σ-additive on $\delta(\mathcal{C})$ for each $x^* \in X^*$.

In fact, let $(A_i)_1^\infty \subset \delta(\mathcal{C})$, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A = \bigcup_1^\infty A_i \in \delta(\mathcal{C})$. Then there exists $V \in \mathcal{V}$ such that $A \subset V$ so that $A, (A_i)_1^\infty \subset \mathcal{B}(V)$. Then $x^* \circ \mathbf{m}(A) = x^* \circ \mathbf{m}_V(A) = \sum_1^\infty x^* \circ \mathbf{m}_V(A_i) = \sum_1^\infty x^* \circ \mathbf{m}(A_i)$ for $x^* \in X^*$. Hence Claim 2 holds.

Claim 3. $x^* \circ \mathbf{m}$ is $\delta(\mathcal{C})$-regular for $x^* \in X^*$.

In fact, let $A \in \delta(\mathcal{C})$ and $\epsilon > 0$. Choose $V \in \mathcal{V}$ such that $A \subset V$. Then by the $\mathcal{B}(V)$-regularity of $x^* \circ \mathbf{m}_V$ there exist $K \in \mathcal{C}$ and a set U open in V such that $K \subset A \subset U$ and such that $v(x^* \circ \mathbf{m}_V, \mathcal{B}(V))(U \backslash K) < \epsilon$. Then U is also open in T. As $\mathbf{m}|_{\mathcal{B}(V)} = \mathbf{m}_V$, and as $v(x^* \circ \mathbf{m}, \delta(\mathcal{C}))(U \backslash K) = v(x^* \circ \mathbf{m}, \mathcal{B}(V))(U \backslash K) = v(x^* \circ \mathbf{m}_V, \mathcal{B}(V))(U \backslash K) < \epsilon$, Claim 3 holds.

By the above claims, $\mathbf{m}$ verifies (i) of the theorem. Since $\{\mathbf{m}(A) : A \in \mathcal{B}(V)\} = \{\mathbf{m}_V(A) : A \in \mathcal{B}(V)\}$, (ii) of the theorem also holds.

Let $\varphi \in C_c(T)$ and let $\operatorname{supp} \varphi = K$. Then choose $V \in \mathcal{V}$ such that $K \subset V$. As φ is a bounded $\mathcal{B}(V)$-measurable function and as $\mathbf{m}_V$ is an X^{**}-valued τ_e-bounded vector measure on $\mathcal{B}(V)$, by the proof of Proposition 18.11 above, φ is not only $\mathbf{m}_V$-integrable in the sense of Definition 1 of [P3], but also $u(\varphi) = u_V(\varphi) = \int_T \varphi d\mathbf{m}_V$, considering X as a subspace of (X^{**}, τ_e). Moreover, as $\mathbf{m}_V = \mathbf{m}|_{\mathcal{B}(V)}$ and as $\varphi \in C_c(V)$, we conclude that $u(\varphi) = \int_T \varphi d\mathbf{m}$, for $\varphi \in C_c(T)$. Thus $\mathbf{m}$

verifies (iii) of the theorem.

To prove the uniqueness of $\mathbf{m}$, if possible let $\mathbf{n}$ be another X^{**}-valued vector measure on $\delta(\mathcal{C})$ such that (i)-(iii) hold for $\mathbf{n}$. Then as $x^* \in X^*$ is continuous on (X^{**}, τ_e), by Lemma 6 of [P3] and by (iii) we have

$$x^*u(\varphi) = \int_T \varphi d(x^* \circ \mathbf{m}) = \int_T \varphi d(x^* \circ \mathbf{n}) \qquad (19.4.3)$$

for $\varphi \in \mathcal{C}_c(T)$ and for $x^* \in X^*$. Let $V \in \mathcal{V}$. As (19.4.3) holds for all $\varphi \in C_c(V)$, by the uniqueness part of the Riesz representation theorem we have $(x^* \circ \mathbf{n})|_{\mathcal{B}(V)} = (x^* \circ \mathbf{m})|_{\mathcal{B}(V)}$ for $x^* \in X^*$. Hence $\mathbf{n}(A) = \mathbf{m}(A)$ for $A \in \mathcal{B}(V)$. Since $\delta(\mathcal{C}) = \bigcup_{V \in \mathcal{V}} \mathcal{B}(V)$, it follows that $\mathbf{n} = \mathbf{m}$. Hence $\mathbf{m}$ is unique.

The following definition is suggested by Theorem 19.4.

Definition 19.5. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a Radon operator. The unique X^{**}-valued vector measure $\mathbf{m}$ on $\delta(\mathcal{C})$ satisfying (i)-(iii) of Theorem 19.4 is called the representing measure of u.

Following Thomas [T] we give the following definition.

Definition 19.6. Let X be a quasicomplete lcHs. A linear mapping $u : \mathcal{K}(T) \to X$ is called a weakly compact Radon operator if u is continuous on $C_c(T)$ for the topology of uniform convergence (i.e. for the topology induced by $||\cdot||_T$) and if its continuous extension to $(C_0(T), ||\cdot||_T)$ is weakly compact.

In the light of the above definition, weakly compact Radon operators on $\mathcal{K}(T)$ can be considered as weakly compact operators on $(C_0(T), ||\cdot||_T)$, and [P5] gives 35 characterizations of these operators. An alternative proof based on the Borel extension theorem is given in [P9] to obtain the said characterizations. The reader may also refer to [P8] for a simple proof of many of these characterizations where 3 new characterizations are also given. The following theorem gives some more characterizations of these operators when the lcHs X satisfies some additional hypothesis and these are suggested by [T]. See also Theorems 19.14 and 19.15 for further characterizations of these operators.

Theorem 19.7. Let X be a quasicomplete lcHs with topology τ and let $u : C_0(T) \to X$ be a continuous linear mapping. Then:

(i) u is weakly compact if and only if , for each uniformly bounded sequence $(\varphi_n)_1^\infty \subset C_0(T)$ with $\varphi_n(t) \to 0$ for $t \in T$, $u(\varphi_n) \to 0$ in X.

(ii) Let $H \subset X^*$ have the Orlicz property and let τ be identical with the topology of uniform convergence in equicontinuous subsets of H. Let $\mathbf{m}$ be the representing measure of u in the sense of Definition 4 of [P5]. Then the following statements are equivalent.

(a) u is weakly compact.

(b) For each open set U in T there exists a vector $x_U \in X$ such that $(x^* \circ \mathbf{m})(U) = x^*(x_U)$ for each $x^* \in H$.

(c) Similar to (b) with U σ-Borel open sets in T.

(d) Similar to (b) with U open Baire sets in T.

(e) Similar to (b) with U σ-compact open sets in T.

(f) Similar to (b) with U open and F_σ in T.

(g) For each closed set F in T there exists a vector $x_F \in X$ such that $(x^* \circ \mathbf{m})(F) = x^*(x_F)$ for each $x^* \in H$.

(h) Similar to (g) with F closed G_δs in T.

Proof. (i) Let $\mathbf{m}$ be the representing measure of u. Then by Theorem 1 of [P5], $u^*x^* = x^* \circ \mathbf{m}$ for $x^* \in X^*$ and $x^*u(\varphi) = \int_T \varphi d(x^* \circ \mathbf{m})$ for $\varphi \in C_0(T)$. Let $(\varphi_n)_1^\infty \subset C_0(T)$ be uniformly bounded and let $\varphi_n(t) \to 0$ for $t \in T$. Then $u(\varphi_n) \to 0$ if and only if $q_E(u(\varphi_n)) = \sup_{x^* \in E} |x^*u(\varphi_n)| = \sup_{x^* \in E} |\int_T \varphi_n d(x^* \circ \mathbf{m})| = \sup_{\mu \in u^*E} |\int_T \varphi_n d\mu| \to 0$ as $n \to \infty$ for each equicontinuous set E in X^* since the topology τ of X is the same as that of uniform convergence in the equicontinuous subsets of X^*. As u^*E is bounded by Lemma 2 of [P5], by Theorem 2 of [G] the above condition holds if and only if u^*E is relatively weakly compact in $M(T)$ and hence by Proposition 4 of [P5] or by Corollary 9.3.7 of [E], if and only if u is weakly compact. Hence (i) holds.

(ii) By Theorem 2 of [P5], (a)⇒(b)⇒(c)⇒(d) and by Theorem 18.16, (d) implies (a). By Lemma 18.3, (d)⇔(e). Clearly, (f)⇒(e) and (b)⇒(f). Hence (a)-(f) are equivalent.

(b)⇒(g) In fact, let F be a closed set in T. Let $U = T\backslash F$. Then by (b) there exist vectors $x_U, x_T \in X$ such that $x^*(x_U) = (x^* \circ \mathbf{m})(U)$ and $x^*(x_T) = (x^* \circ \mathbf{m})(T)$ for $x^* \in H$. Then $(x^* \circ \mathbf{m})(F) = x^*(x_T - x_U)$ for $x^* \in H$ and hence (g) holds. Similarly, (g) implies (b) as T is closed and as $F = T\backslash U$ is closed for an open set U in T.

By taking complements, we see that (h) and (f) are equivalent.

Hence the theorem holds.

Following Thomas [T] we give the following definition and its equivalence with Definition 3.1 of [T] will be proved in [P14].

Definition 19.8. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a Radon operator. Then u is said to be prolongable if, for each $V \in \mathcal{V}$, the continuous linear extension $\widetilde{u_V}$ to $C_0(V)$ of the continuous linear map $u_V = u|_{C_c(V)}$ is weakly compact.

The weakly compact Radon operators in Definition 19.6 and prolongable Radon operators in Definition 19.8 are called respectively weakly compact bounded Radon measures and prolongable

Radon measures in [T].

We can strengthen Theorem 19.4 as below when the Radon operator is prolongable.

Theorem 19.9 (Integral representation of prolongable Radon operators). Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a prolongable Radon operator. Then the representing measure $\mathbf{m}$ of u as in Definition 19.5 is X-valued, σ-additive and $\delta(\mathcal{C})$-regular (considering X as a subspace of (X^{**}, τ_e)) and

$$u(\varphi) = \int_T \varphi d\mathbf{m}, \ \varphi \in \mathcal{K}(T) \qquad (19.9.1)$$

where the integral is a (BDS)-integral.

Conversely, if $\mathbf{m}$ is an X-valued σ-additive $\delta(\mathcal{C})$-regular measure on $\delta(\mathcal{C})$, then the mapping $u : \mathcal{K}(T) \to X$ given by $u(\varphi) = \int_T \varphi d\mathbf{m}$, $\varphi \in \mathcal{K}(T)$ (the integral being a (BDS)-integral), is a prolongable Radon operator. Moreover, the representing measure of u is $\mathbf{m}$.

Proof. Let u be prolongable. Then by Theorem 19.4 there exists a unique X^{**}-valued vector measure $\mathbf{m}$ on $\delta(\mathcal{C})$ such that $x^* \circ \mathbf{m}$ is σ-additive and $\delta(\mathcal{C})$-regular for each $x^* \in X^*$; $u(\varphi) = \int_T \varphi d\mathbf{m}$ for $\varphi \in \mathcal{K}(T)$ in the sense of Definition 1 of [P3] and $\{\mathbf{m}(A) : A \in \mathcal{B}(V)\}$ is τ_e-bounded for each $V \in \mathcal{V}$. Let $V \in \mathcal{V}$ and let $\mathbf{m}_V = \mathbf{m}|_{\mathcal{B}(V)}$. Then, from the proof of Theorem 19.4 we note that $\mathbf{m}_V$ is the representing measure of the continuous linear map $\widetilde{u_V}$ on $(C_0(V), ||\cdot||_T)$ (in the sense of Definition 4 of [P5]) and by hypothesis, $\widetilde{u_V} : C_0(V) \to X$ is weakly compact. Then by Theorem 2 of [P5], $\mathbf{m}_V$ is σ-additive on $\mathcal{B}(V)$ and has range in X and by Theorem 6 of [P5], $\mathbf{m}_V$ is $\mathcal{B}(V)$-regular. Since V is arbitrary in $\mathcal{V}$ and since $\delta(\mathcal{C}) = \bigcup_{V \in \mathcal{V}} \mathcal{B}(V)$, we conclude that $\mathbf{m}$ is σ-additive on $\delta(\mathcal{C})$, is $\delta(\mathcal{C})$-regular and has range in X. Let $\varphi \in \mathcal{K}(T)$ with $\operatorname{supp} \varphi \subset K \in \mathcal{C}$. Let $V \in \mathcal{V}$ such that $K \subset V$. Then $\varphi \in C_c(V) \subset C_0(V)$ and $\mathbf{m}_V = \mathbf{m}|_{\mathcal{B}(V)}$ is σ-additive and X-valued. Then by Lemma 18.19, $\int_T \varphi d\mathbf{m} = \int_V \varphi d\mathbf{m}_V$ is a (BDS)-integral.

Conversely, let $\mathbf{m} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular. Let τ be the topology of X. If $V \in \mathcal{V}$, then $\mathbf{m}_V = \mathbf{m}|_{\mathcal{B}(V)}$ is σ-additive, $\mathcal{B}(V)$-regular and X-valued. Let $u : \mathcal{K}(T) \to X$ be given by $u(\varphi) = \int_T \varphi d\mathbf{m}$ for $\varphi \in \mathcal{K}(T)$ where the integral is a (BDS)-integral. Then by Theorem 11.9(i)(b) and Remark 12.5 of [P12], u is a continuous linear map. Let $U \in \mathcal{V}$. Then by Theorem 50.D of [H], there exists $W \in \mathcal{V}$ such that $\bar{U} \subset W$ and hence $C_0(U) \subset C_c(W)$. Then $\mathbf{m}_W = \mathbf{m}|_{\mathcal{B}(W)}$ is X-valued and σ-additive in τ and hence by Lemma 18.19, $\widetilde{u_W} : C_0(W) \to X$ given by $\widetilde{u_W}(\varphi) = \int_W \varphi d\mathbf{m}_W$ is weakly compact and hence $\widetilde{u_U} = \widetilde{u_W}|_{C_0(U)}$ is weakly compact. Hence u is prolongable. Clearly, $\mathbf{m}$ satisfies (i) of Theorem 19.4. Since $\mathbf{m}|_{\mathcal{B}(V)}$ is σ-additive, (ii) of Theorem 19.4 also holds since $\tau = \tau_e|_X$. By Lemma 18.19, the (BDS)-integral $\int_T \varphi d\mathbf{m}$ for $\varphi \in \mathcal{K}(T)$ is the same as the integral in the sense of Definition 1 of [P3], noting that $\tau = \tau_e|_X$ when X is considered as a subspace of (X^{**}, τ_e). Hence $\mathbf{m}$ is the representing measure of u.

Corollary 19.10. A linear functional θ belongs to $\mathcal{K}(T)^*$ (resp. $\mathcal{K}(T)^*_b$ (see [P2])) if and only if $\theta : \mathcal{K}(T) \to \boldsymbol{K}$ is a prolongable (resp. weakly compact) Radon operator. In that case, its representing measure $\mathbf{m}_\theta$ is the same as the complex Radon measure μ_θ induced by θ in the sense of Definition 4.3 of [P1].

Thus prolongable (resp. weakly compact) Radon operators on $\mathcal{K}(T)$ with values in a quasicomplete lcHs generalize complex measures (resp. bounded complex measures) in the sense of [B].

Theorem 19.11. Let X and Y be quasicomplete lcHs over $I\!K$. Let $u : \mathcal{K}(T) \to X$ be a prolongable (resp. weakly compact) Radon operator and let $v : X \to Y$ be a continuous linear mapping. Then:

(i) $v \circ u : \mathcal{K}(T) \to Y$ is a prolongable (resp. weakly compact) Radon operator.

(ii) $\mathbf{m}_{v\circ u}(A) = v(\mathbf{m}_u(A))$ for $A \in \delta(\mathcal{C})$ (resp. for $A \in \mathcal{B}(T)$), where $\mathbf{m}_u$ and $\mathbf{m}_{v\circ u}$ are the representing measures of u and $v \circ u$, respectively.

(iii) If $f \in \mathcal{L}_1(\mathbf{m}_u)$, then $f \in \mathcal{L}_1(\mathbf{m}_{v\circ u})$ and $\int_A f d\mathbf{m}_{v\circ u} = v(\int_A f d\mathbf{m}_u)$ for $A \in \mathcal{B}_c(T)$ (resp. for $A \in \mathcal{B}(T)$).

Proof. (i) If w is a weakly compact operator with range in X, then it is well known that $v \circ w$ is weakly compact. This result is used to prove (i).

(ii) Let u be prolongable and let $V \in \mathcal{V}$. Then $\widetilde{u_V} : C_0(V) \to X$ is weakly compact and its representing measure $(\mathbf{m}_u)_V$ on $\mathcal{B}(V)$ has range in X and is given by $(\mathbf{m}_u)_V = \widetilde{u_V}^{**}|_{\mathcal{B}(V)} = u_V^{**}|_{\mathcal{B}(V)}$. Let $A \in \delta(\mathcal{C})$ and choose $V \in \mathcal{V}$ such that $A \subset V$. Then $(\mathbf{m}_u)_V(A) = u_V^{**}(\chi_A)$ and $(v \circ u_V)^{**}(\chi_A) = (\mathbf{m}_{v\circ u})_V(A)$. Hence $\mathbf{m}_{v\circ u}(A) = (\mathbf{m}_{v\circ u})_V(A) = (v \circ u_V)^{**}(\chi_A) = v^{**}u_V^{**}(\chi_A) = v^{**}(\mathbf{m}_u)_V(A) = v^{**}\mathbf{m}_u(A) = v \circ \mathbf{m}_u(A)$ as $\mathbf{m}_u(A) \in X$ and $v^{**}|_X = v$. Similar argument holds when u is a weakly compact Radon operator.

(iii) Let $f \in \mathcal{L}_1(\mathbf{m}_u)$. Then by Theorem 11.8(v) and Remark 12.5 of [P12], $f \in \mathcal{L}_1(v \circ \mathbf{m}_u)$ and $\int_A f d(v \circ \mathbf{m}_u) = v(\int_A f d\mathbf{m}_u)$ for $A \in \mathcal{B}_c(T)$ (resp. for $A \in \mathcal{B}(T)$) if u is a prolongable (resp. weakly compact) Radon operator. As $\mathbf{m}_{v\circ u} = v \circ \mathbf{m}_u$ by (ii), (iii) holds.

The following theorem gives 24 characterizations for a Radon operator to be prolongable and [P5] plays a key role in the proof of the theorem. For the different concepts of regularity used in the following theorem see Definition 5 of [P5].

Theorem 19.12. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a Radon operator. Let $\mathbf{m} : \delta(\mathcal{C}) \to X^{**}$ be the representing measure of u and let $\mathbf{m}_0 = \mathbf{m}|_{\delta(\mathcal{C}_0)}$. Let us consider X as a subspace of (X^{**}, τ_e). Then the following statements are equivalent:

(1) u is prolongable.

(2) $\mathbf{m}$ has range in X.

(3) $\mathbf{m}$ is σ-additive in the topology τ_e of X^{**}.

(4) $\mathbf{m}(V) \in X$ for each $V \in \mathcal{V}$.

(5) $\mathbf{m}(V) \in X$ for each $V \in \mathcal{V} \cap \mathcal{B}_0(T)$.

(6) $\mathbf{m}(K) \in X$ for each $K \in \mathcal{C}$.

(7) $\mathbf{m}(K) \in X$ for each $K \in \mathcal{C}_0$.

(8) For each $U \in \mathcal{V}$ and for each increasing sequence $(f_n)_1^\infty \subset C_0(U)$ with $0 \leq f_n \leq 1$ in U for all n, (uf_n) converges weakly in X.

(9) Similar to (8) with $U \in \mathcal{V} \cap \mathcal{B}_0(T)$.

(10) $\mathbf{m}_0$ is σ-additive in τ_e.

(11) $\mathbf{m}_0$ has range in X.

(12) $\mathbf{m}$ is $\delta(\mathcal{C})$-regular (in τ_e).

(13) $\mathbf{m}$ is $\delta(\mathcal{C})$-inner regular (in τ_e).

(14) $\mathbf{m}$ is $\delta(\mathcal{C})$-inner regular (in τ_e) in $\mathcal{V}$.

(15) $\mathbf{m}$ is $\delta(\mathcal{C})$-outer regular (in τ_e) in each $K \in \mathcal{C}$.

(16) $\mathbf{m}_0$ is $\delta(\mathcal{C}_0)$-regular (in τ_e).

(17) $\mathbf{m}_0$ is $\delta(\mathcal{C}_0)$-inner regular (in τ_e).

(18) $\mathbf{m}_0$ is $\delta(\mathcal{C}_0)$-inner regular (in τ_e) in each open set $U \in \delta(\mathcal{C}_0)$.

(19) $\mathbf{m}_0$ is $\delta(\mathcal{C}_0)$-outer regular (in τ_e) in each $K \in \mathcal{C}_0$.

(20) All bounded Borel measurable functions f on T with compact support (equivalently, all bounded σ-Borel measurable functions f on T with compact support) are $\mathbf{m}$-integrable in T (in the sense of Definition 3 of [P5]) and $\int_T f d\mathbf{m} \in X$.

(21) All bounded Baire measurable functions f on T with compact support are $\mathbf{m}_0$-integrable in T (in the sense of Definition 3 of [P5]) and $\int_T f d\mathbf{m}_0 \in X$.

(22) All bounded functions f on T belonging to the first Baire class with compact support are $\mathbf{m}$-integrable in T (in the sense of Definition 3 of [P5]) and $\int_T f d\mathbf{m} \in X$.

(23) $u_V^{**} f \in X$ for all bounded functions f on T belonging to the first Baire class with compact support, the support being contained in $V \in \mathcal{V}$.

(24) For every uniformly bounded sequence (φ_n) of continuous functions vanishing in $T \backslash K$ for some $K \in \mathcal{C}$ (equivalently, by Urysohn's lemma for every sequence (φ_n) of continuous functions dominated by a member of $\mathcal{K}(T)$) with $\lim_n \varphi_n(t) = 0$ for each $t \in T$, $\lim_n u(\varphi_n) = 0$.

Proof. For $V \in \mathcal{V}$, let $u_V = u|_{(C_c(V),||\cdot||_T)}$. As u_V is continuous, it has a unique continuous linear extension $\widetilde{u_V}$ to $C_0(V)$. If $\mathbf{m}_V$ is the representing measure of $\widetilde{u_V}$ on $\mathcal{B}(V)$ as in Definition 4 of [P5], then from the proof of Theorem 19.4 it is clear that $\mathbf{m}_V = \mathbf{m}|_{\mathcal{B}(V)}$.

(1)⇔(2)(resp. (1)⇔(3)) By Theorem 2 of [P5], the range of $\mathbf{m}_V$ is contained in X (resp. $\mathbf{m}_V$ is σ-additive on $\mathcal{B}(V)$ in τ_e) if and only if $\widetilde{u_V}$ is weakly compact. Since $\delta(\mathcal{C}) = \bigcup_{V\in\mathcal{V}} \mathcal{B}(V)$, it follows that $\mathbf{m}$ has range in X (resp. $\mathbf{m}$ is σ-additive in τ_e) if and only if $\mathbf{m}_V(\mathcal{B}(V)) \subset X$ (resp. $\mathbf{m}_V$ is σ-additive on $\mathcal{B}(V)$ in τ_e) for each $V \in \mathcal{V}$ and hence if and only if $\widetilde{u_V}$ is weakly compact for each $V \in \mathcal{V}$. Hence the result holds.

Clearly, (2)⇒(4)⇒(5).

(5)⇒(1)

Claim 1. $\mathcal{B}_0(T)$ is the σ-ring generated by all relatively compact open Baire sets in T.

In fact, given $C \in \mathcal{C}_0$, by Theorem 50.D of [H] and by Lemma 18.3, there exists a relatively compact open Baire set U in T such that $C \subset U$. Then $C = U\backslash(U\backslash C)$ and hence the claim holds.

Let $V \in \mathcal{V}$. Then by Lemma 18.3 and Claim 1 above, $\mathcal{B}_0(V)$ is the σ-ring generated by $\mathcal{U}_V = \{U : U \text{open in} V, U \in \mathcal{B}_0(V)\} = \{U : U \text{ open in} V,\, U = \bigcup_1^\infty F_n,\, F_n \text{ compact in} V\}$. Since V is open in T, $U \in \mathcal{U}_V$ if and only if $U \subset V$ and U is open and σ-compact in T and hence by Lemma 18.3, $U \in \mathcal{U}_V$ if and only if $U \subset V$ and U is an open Baire set in T. Hence $\mathcal{U}_V = \{U \in \mathcal{V} \cap \mathcal{B}_0(T) : U \subset V\}$. Then by hypothesis (5), $\mathbf{m}_V(U) \in X$ for all open sets $U \in \mathcal{B}_0(V)$ and hence by Theorem 3(vii) of [P5], $\widetilde{u_V}$ is weakly compact. Therefore, (1) holds.

(4)⇒(6) Given $K \in \mathcal{C}$, by Theorem 50.D of [H] there exists an open set $V \in \mathcal{V}$ such that $K \subset V$. Then, as $K = V\backslash(V\backslash K)$, by hypothesis we have $\mathbf{m}(K) = \mathbf{m}(V) - \mathbf{m}(V\backslash K) \in X$.

(6)⇒(7) Obvious.

(7)⇒(5) Let $V \in \mathcal{V} \cap \mathcal{B}_0(T)$. Then as V is a relatively compact open Baire set in T, by Theorem 50.D of [H] there exists $K \in \mathcal{C}_0$ such that $V \subset K$. Then again by Theorem 50.D of [H] and by Lemma 18.3, there exists a relatively compact open Baire set U in T such that $K \subset U$. Then $V = K\backslash(K\backslash V)$ and $K\backslash V \in \mathcal{C}_0$ by Theorem 51.D of [H]. Then by hypothesis, $\mathbf{m}(V) \in X$.

(1)⇒(8) Let $U \in \mathcal{V}$. Then by (1), $\widetilde{u_U} : C_0(U) \to X$ is weakly compact, and by hypothesis, $f_n \nearrow$, $0 \le f_n \le 1$ in U and $(f_n)_1^\infty \subset C_0(U)$. Then by Theorem 3(xi) of [P5], $(uf_n) = (\widetilde{u_U} f_n)$ converges weakly in X and hence (8) holds.

(8)⇒(9) Obvious.

(9)⇒(1) Let $V \in \mathcal{V} \cap \mathcal{B}_0(T)$. Let (f_n) be an increasing sequence in $C_0(V)$ such that $0 \leq f_n \leq 1$ in V for $n \in I\!N$. Then by (9), $(\widetilde{u_V} f_n) = u(f_n)$ converges weakly in X. Then by Theorem 3(xi) of [P5], $\widetilde{u_V}$ is weakly compact. Now let $U \in \mathcal{V}$. Then by Theorem 50.D of [H] and by Lemma 18.3 there exists $V \in \mathcal{V} \cap \mathcal{B}_0(T)$ such that $\bar{U} \subset V$ and hence $\widetilde{u_U} = \widetilde{u_V}|_{C_0(U)}$ is weakly compact. Hence (1) holds.

(3)⇒(10) Obvious.

(10)⇒(1) Let $\mathcal{U}_0$ be the family of all open Baire sets in T. Let $V \in \mathcal{B}_0(T) \cap \mathcal{V}$. As seen in the proof of '(5)⇒(1)', $\mathcal{B}_0(V)$ is the σ-ring generated by $\{U \in \mathcal{U}_0 : U \subset V\}$ and hence $\mathcal{B}_0(V) = \sigma(\mathcal{U}_0 \cap V) = \sigma(\mathcal{U}_0) \cap V = \mathcal{B}_0(T) \cap V \subset \delta(\mathcal{C}_0)$ by Theorem 5.E of [H] and by Lemma 18.2 above. Then by hypothesis, $\mathbf{m}_0|_{\mathcal{B}_0(V)}$ is σ-additive on $\mathcal{B}_0(V)$ in τ_e and hence by Theorem 4(xiii) of [P5], $\widetilde{u_V}$ is weakly compact. Now if $W \in \mathcal{V}$, then by Theorem 50.D of [H] and by Lemma 18.3 above, there exists $V \in \mathcal{B}_0(T) \cap \mathcal{V}$ such that $\bar{W} \subset V$ and consequently, $C_c(W) \subset C_0(V)$. Then $\widetilde{u_W} = \widetilde{u_V}|_{C_0(W)}$ is weakly compact and hence (1) holds.

(2)⇒(11) Obvious.

(11)⇒(1) Arguing as in the proof of '(10)→(1)'and using Theorem 4(xv) of [P5] instead of Theorem 4(xiii) of [P5], we observe that $\widetilde{u_V}$ is weakly compact for each $V \in \mathcal{V} \cap \mathcal{B}_0(T)$. Then arguing as in the last part of the proof of '(10)⇒(1)', we can show that $\widetilde{u_W}$ is weakly compact for each $W \in \mathcal{V}$ and hence (1) holds.

(1)⇒(12) Let $A \in \delta(\mathcal{C})$. Then there exists $V \in \mathcal{V}$ such that $A \subset V$. By (1), $\widetilde{u_V} : C_0(V) \to X$ is weakly compact and hence by Theorem 6(xix) of [P5], $\mathbf{m}_V$ is $\mathcal{B}(V)$-regular. As $A \in \mathcal{B}(V)$, given $\epsilon > 0$ and $q \in \Gamma$, by the regularity of $\mathbf{m}_V$ and by Proposition 2.2 of [P10] there exist $K \in \mathcal{C}$ and a set U open in V such that $K \subset A \subset V$ and such that $||\mathbf{m}_V||_q(V \backslash K) < \epsilon$. Hence $||\mathbf{m}||_q(V \backslash K) < \epsilon$ since $\mathbf{m}_V = \mathbf{m}|_{\mathcal{B}(V)}$. As V is open in T and as $V \in \delta(\mathcal{C})$ by Lemma 18.2 , (12) holds.

(12)⇒(13)⇒(14) Obvious.

(14)⇒(1) Let $V \in \mathcal{V}$. Then by (14), $\mathbf{m}_V$ is $\mathcal{B}(V)$-inner regular in each open set in V and hence by Theorem 6(xxi) of [P5], $\widetilde{u_V}$ is weakly compact. Hence (1) holds.

(1)⇒(15) Let $K \in \mathcal{C}$. Then by Theorem 50.D of [H] there exists $V \in \mathcal{V}$ such that $K \subset V$ and (1) implies that $\widetilde{u_V}$ is weakly compact. Consequently, given $q \in \Gamma$ and $\epsilon > 0$, by Theorem 6(xxii) of [P5] and by Proposition 2.2 of [P10] there exists a set U open in V such that $K \subset U$ and such that $||\mathbf{m}_V||_q(U \backslash K) < \epsilon$. Hence $||\mathbf{m}||_q(U \backslash K) < \epsilon$ with U open in T and $U \in \delta(\mathcal{C})$ by Lemma 18.2. Hence (15) holds.

(15)⇒(19) Let $K \in \mathcal{C}_0$. Given $q \in \Gamma$ and $\epsilon > 0$, by (15) there exists U open in T with

$U \in \delta(\mathcal{C})$ such that $K \subset U$ and $||\mathbf{m}||_q(U\backslash K) < \epsilon$. Then by Theorem 50.D of [H] and by Lemma 18.3 above, there exists an open Baire set V in T such that $K \subset V \subset U$. Clearly, by Lemma 18.2 above, $V \in \delta(\mathcal{C}_0)$ and $||\mathbf{m}||_q(V\backslash K) < \epsilon$. Hence (19) holds.

(19)⇒(7) Let $K \in \mathcal{C}_0$. Then by Theorem 50.D of [H] and by Lemma 18.3 above there exists $V \in \mathcal{B}_0(T)\cap\mathcal{V}$ such that $K \subset V$. Let $q \in \Gamma$. Then by (19), for each $n \in I\!\!N$, there exists $V_n^{(q)} \in \delta(\mathcal{C}_0)$ such that $K \subset V_n^{(q)} \subset V$, $V_n^{(q)}$ open in V and $||\mathbf{m}_V||_q(V_n^{(q)}\backslash K) < \frac{1}{n}$. By Urysohn's lemma there exists $\varphi_n^{(q)} \in C_c(V_n^{(q)})$ such that $\varphi_n^{(q)}|_K = 1$ and $0 \le \varphi_n^{(q)} \le 1$ in $V_n^{(q)}$, $n \in I\!\!N$. Since $\varphi_n^{(q)} \in C_c(V)$, by Theorem 19.4 we have $u(\varphi_n^{(q)}) = u_V(\varphi_n^{(q)}) = \int_T \varphi_n^{(q)} d\mathbf{m}_V$. Then by Lemma 6(ii) of [P3] we have $||u(\varphi_n^{(q)}) - \mathbf{m}_V(K)||_q = ||\int_T(\varphi_n^{(q)} - \chi_K)d\mathbf{m}_V||_q = ||\int_{V_n^{(q)}\backslash K} \varphi_n^{(q)} d\mathbf{m}_V||_q \le ||\mathbf{m}_V||_q(V_n^{(q)}\backslash K) < \frac{1}{n}$ for each n. Thus, $\Pi_q(\mathbf{m}(K)) = \Pi_q(\mathbf{m}_V(K)) = \lim_n \Pi_q(u(\varphi_n^{(q)})) \in \widetilde{X_q}$. i.e., $\int_T \chi_K d\mathbf{m}_q \in \widetilde{X_q}$ for each $q \in \Gamma$. Then by Definition 12.1 of [P12], χ_K is $\mathbf{m}$-integrable in T and

$$\mathbf{m}(K) = (\text{BDS}) \int_T \chi_K d\mathbf{m} = \lim_{\longleftarrow} \int_T \chi_K d\mathbf{m}_q.$$

Then, as X is quasicomplete, by Theorem 12.3 of [P12], $\mathbf{m}(K) \in X$ and hence (7) holds.

(1)⇒(16) Let $A \in \delta(\mathcal{C}_0)$, $q \in \Gamma$ and $\epsilon > 0$. Then as in the proof of '(19)⇒(7)'there exists $V \in \mathcal{V} \cap \mathcal{B}_0(T)$ such that $A \subset V$. As $\widetilde{u_V}$ is weakly compact, by Theorem 8(xxvii) of [P5] and by Proposition 2.2 of [P10] there exist $K \in \mathcal{C}_0$ and an open set U in T belonging to $\mathcal{B}_0(V)$ such that $K \subset A \subset U$ and $||\mathbf{m}_V||_q(U\backslash K) < \epsilon$. From the proof of '(5)⇒(1)', we note that $U \in \mathcal{B}_0(T) \cap \mathcal{V} \subset \delta(\mathcal{C}_0)$ (by Lemma 18.2) and hence $\mathbf{m}_0$ is $\delta(\mathcal{C}_0)$-regular.

(16)⇒(17)⇒(18) Obvious.

(18)⇒(1) Let V be a relatively compact open Baire set in T. Then $V \in \mathcal{B}_0(V)$ since it is shown in the proof of '(5)⇒(1)' that the open Baire sets in $\mathcal{B}_0(V)$ are precisely the open Baire sets in T which are contained in V. Then the hypothesis implies that $\mathbf{m}_0$ is $\mathcal{B}_0(V)$-inner regular in each open Baire set in V and hence particularly in V. Therefore, by Theorem 8(xxix) of [P5], $\widetilde{u_V}$ is weakly compact. Then, given $U \in \mathcal{V}$, arguing as in the last part of the proof of '(10)⇒(1)', we can show that $\widetilde{u_U}$ is weakly compact. Hence (1) holds.

(1)⇒(20) Let f be a bounded Borel measurable function on T with support $K \in \mathcal{C}$. Then by Lemma 18.2, $N(f) \in \delta(\mathcal{C})$ and $N(f) \cap f^{-1}(U) \in \delta(\mathcal{C}) \subset \mathcal{B}_c(T)$ for open sets U in $I\!\!K$. Hence f is $\mathcal{B}_c(T)$-measurable. Clearly, $\mathcal{B}_c(T)$-measurable functions are Borel measurable. Let $V \in \mathcal{V}$ such that $K \subset V$. Let $\mathcal{U}$ be the family of open sets in T. $\mathcal{U}\cap V$ is the family of open sets in V and hence by Theorem 5.E of [H], $\mathcal{B}(V) = \sigma(\mathcal{U} \cap V) = \sigma(\mathcal{U}) \cap V = \mathcal{B}(T) \cap V$. Since f is $\mathcal{B}_c(T)$-measurable, $f^{-1}(U) \cap N(f) \in \mathcal{B}_c(T) \subset \mathcal{B}(T)$ for U open in $I\!\!K$ and clearly, $f^{-1}(U) \cap N(f) \subset V$. Hence $f^{-1}(U)\cap N(f)\cap V = f^{-1}(U)\cap N(f)$ and hence f is $\mathcal{B}(V)$-measurable. By (1), $\widetilde{u_V}$ is weakly compact. Then by Theorem 9(xxxi) of [P5], f is $\mathbf{m}_V$-integrable in T and $\int_T f d\mathbf{m} = \int_T f d\mathbf{m}_V \in X$. (See Definition 3 of [P5] and note that $\{\mathbf{m}_V(A) : A \in \mathcal{B}(V)\} = \{\mathbf{m}(A) : A \in \mathcal{B}(V)\}$ is τ_e-bounded in X^{**}.) Hence (20) holds.

(20)⇒(21)⇒(22) Obvious.

(22)⇒(5) Let $U \in \mathcal{V} \cap \mathcal{B}_0(T)$. Then by Lemma 18.3 there exists $(K_n)_1^\infty \subset \mathcal{C}$ such that $K_n \nearrow U$. Then by an argument based on Urysohn's lemma there exists $(f_n)_1^\infty \subset C_c(U)$ such that $f_n \nearrow \chi_U$ in T. Hence χ_U belongs to the first Baire class, and clearly has compact support. Then by hypothesis, $\mathbf{m}(U) \in X$ and hence (5) holds.

(22)⇒(23) Each bounded function f belonging to the first Baire class is Baire measurable. Moerover, let f have compact support K and choose $V \in \mathcal{V}$ such that $K \subset V$. Then by Definition 19.3, $u_V : C_c(V) \to X$ is continuous and hence has continuous extension $\widetilde{u_V}$ on $C_0(V)$. Then $\mathbf{m}_0(A) = \mathbf{m}_V(A) = u_V^{**}(\chi_A)$ for $A \in \mathcal{B}_0(V)$. Then $x^* \circ u_V \in \mathcal{K}(V)_b^*$ and $< \int_T f d(\mathbf{m}_0)_V, x^* >= \int_T f d(x^* \circ (\mathbf{m}_0)_V) = \int_T f d(u_V^* x^*) =< f, u_V^* x^* >=< u_V^{**} f, x^* >$ for $x^* \in X^*$. Hence $u_V^{**} f = \int_T f d(\mathbf{m}_0)_V = \int_T f d\mathbf{m} \in X$. Thus (23) holds.

(23)⇒(5) As seen in the proof of '(22)⇒(5)', χ_V belongs to the first Baire class for each $V \in \mathcal{V} \cap \mathcal{B}_0(T)$ and has compact support. Then by (23), $\mathbf{m}(V) = \mathbf{m}_V(V) = u_V^{**}(\chi_V) \in X$ and hence (5) holds.

(1)⇒(24) Let $(\varphi_n)_1^\infty$ satisfy the hypothesis so that $\sup_n ||\varphi_n||_T = M$ (say) $< \infty$, there exists $K \in \mathcal{C}$ such that $\varphi_n(t) = 0$ for $t \in T \backslash K$ and for all n and $\lim_n \varphi_n(t) = 0$ for $t \in T$. Let $V \in \mathcal{V}$ with $K \subset V$. Then $(\varphi_n)_1^\infty \subset C_0(V)$ and by (1), $\widetilde{u_V}$ is weakly compact. Then by Theorem 19.7(i), $u(\varphi_n) = \widetilde{u_V}(\varphi_n) \to 0$ in X.

(24)⇒(1) Let $V \in \mathcal{V}$. Let $(\varphi_n)_1^\infty \subset C_0(V)$ with $\lim_n \varphi_n(t) = 0$ for each $t \in V$ and $\sup_n ||\varphi_n||_T = M < \infty$. By Theorem 50.D of [H] there exists $U \in \mathcal{V}$ such that $\bar{U} \subset U$. As $(\varphi_n)_1^\infty \subset C_0(V)$, $\varphi_n, n \in \mathbb{N}$, vanish on $T \backslash \bar{V}$ and hence by (24), $\lim_n u(\varphi_n) = 0$. Then by Theorem 19.7(i), $\widetilde{u_V}$ is weakly compact and hence (1) holds.

This completes the proof of the theorem.

Theorem 19.13. Let X be a quasicomplete lcHs with topology τ and let $u : \mathcal{K}(T) \to X$ be a Radon operator with the representing measure $\mathbf{m}$. Let H be a set in X^* having the Orlicz property such that τ is identical with the topology of uniform convergence in equicontinuous subsets of H. Let μ_{x^*u} be the complex Radon measure induced by x^*u as in Definition 4.3 of [P1]. Then the following statements are equivalent:

(i) u is prolongable.

(ii) For each $V \in \mathcal{V}$, there exists $x_V \in X$ such that $x^*(x_V) = (x^* \circ \mathbf{m})(V)$ for each $x^* \in H$.

(iii) Similar to (ii) with $V \in \mathcal{B}_0(T) \cap \mathcal{V}$.

(iv) For each $K \in \mathcal{C}$, there exists $x_K \in X$ such that $x^*(x_K) = (x^* \circ \mathbf{m})(K)$ for each $x^* \in H$.

(v) Similar to (iv) with $K \in \mathcal{C}_0$.

(vi) For each $V \in \mathcal{V}$, there exists $x_V \in X$ such that $x^*(x_V) = \int_T \chi_V d(\mu_{x^*u})$ for $x^* \in H$.

(vii) Similar to (vi) with $V \in \mathcal{B}_0(T) \cap \mathcal{V}$.

(viii) Similar to (vi) with V replaced by $K \in \mathcal{C}$.

(ix) Similar to (viii) with $K \in \mathcal{C}_0$.

Proof. (i)⇒(ii)→(iii) by (2) of Theorem 19.12.

(iii)⇒(i) Let $V \in \mathcal{B}_0(T) \cap \mathcal{V}$. Then as observed in the proof of '(5)⇒(1)' in the proof of Theorem 19.12, $\mathcal{B}_0(V)$ is the σ-ring generated by the family $\mathcal{U}_V$ of all open Baire sets in T which are contained in V. If $U \in \mathcal{B}_0(V)$ and if U is open in T, then $U \in \mathcal{U}_V \subset \mathcal{V} \cap \mathcal{B}_0(T)$ by the proof of '(5)⇒(1)' in the proof of Theorem 19.12.

Hence, for each open set $U \in \mathcal{B}_0(V)$, by hypothesis there exists $x_U \in X$ such that $x^*(x_U) = (x^* \circ \mathbf{m})(U)$ for $x^* \in H$ and hence by Theorem 18.16, $\widetilde{u_V}$ is weakly compact. If $W \in \mathcal{V}$, then by Theorem 50.D of [H] and by Lemma 18.3, there exists $V \in \mathcal{B}_0(T) \cap \mathcal{V}$ such that $W \subset \bar{W} \subset V$. Then by the above argument $\widetilde{u_V}$ is weakly compact and hence $\widetilde{u_W} = \widetilde{u_V}|_{C_0(W)}$ is weakly compact and hence (i) holds.

By (2) of Theorem 19.12, (i)⇒(iv)⇒(v).

(v)⇒(iii) Let $V \in \mathcal{B}_0(T) \cap \mathcal{V}$. Then $\bar{V} \in \mathcal{C}$ and by Theorem 50.D of [H] there exists $K \in \mathcal{C}_0$ such that $\bar{V} \subset K$. Then by hypothesis, there exists $x_K \in X$ such that $x^*(x_K) = (x^* \circ \mathbf{m})(K)$ for $x^* \in H$. As $K \backslash V \in \mathcal{C}_0$ by Theorem 51.D of [H], by hypothesis there exists $x_{K\backslash V} \in X$ such that $x^*(x_{K\backslash V}) = (x^* \circ \mathbf{m})(K \backslash V)$ for $x^* \in H$. Then, as $V = K \backslash (K \backslash V)$, $(x^* \circ \mathbf{m})(V) = x^*(x_K - x_{K\backslash V})$ for $x^* \in H$ and hence (iii) holds.

First we prove the following result.

Claim 1. $(x^*u) \in \mathcal{K}(T)^*$ and $\mu_{x^*u} = x^* \circ \mathbf{m}$ on $\delta(\mathcal{C})$ for $x^* \in X^*$.

In fact, as u is continuous on $\mathcal{K}(T)$, $x^*u \in \mathcal{K}(T)^*$. By Theorem 19.4, $u(\varphi) = \int_T \varphi d\mathbf{m}$ for $\varphi \in \mathcal{K}(T)$ and consequently, by Lemma 6(ii) of [P3],

$$\int_T \varphi d(x^* \circ \mathbf{m}) = x^*u(\varphi) = \int_T \varphi d(\mu_{x^*u}) \tag{19.13.1}$$

for $\varphi \in \mathcal{K}(T)$. Choose $V \in \mathcal{V}$ such that $\operatorname{supp} \varphi \subset V$. Since $x^* \circ \mathbf{m}$ and μ_{x^*u} are regular on $\delta(\mathcal{C})$ by Theorem 19.4 above and by Theorem 4.4(i) of [P2], respectively, both of them are $\mathcal{B}(V)$-regular on $\mathcal{B}(V)$. Moreover, both of them are σ-additive on $\mathcal{B}(V)$. Since (19.13.1) holds for all $\psi \in C_c(V)$, by the uniqueness part of the Riesz representation theorem we have $(x^* \circ \mathbf{m})|_{\mathcal{B}(V)} = (\mu_{x^*u})|_{\mathcal{B}(V)}$. Since $\delta(\mathcal{C}) = \bigcup_{V \in \mathcal{V}} \mathcal{B}(V)$, we have $(x^* \circ \mathbf{m})(A) = \mu_{x^*u}(A)$ for $A \in \delta(\mathcal{C})$. Hence the claim holds.

Let u be prolongable. Let $V \in \mathcal{V}$, $U \in \mathcal{V} \cap \mathcal{B}_0(T)$, $K \in \mathcal{C}$ and $K_0 \in \mathcal{C}_0$. Then by (ii) (resp. (iii), (iv), (v)) there exists $x_V \in X$ (resp. $x_U \in X, x_K \in X, x_{K_0} \in X$) such that $(x^* \circ \mathbf{m})(V) = x^*(x_V)$ (resp. $(x^* \circ \mathbf{m})(U) = x^*(x_U), (x^* \circ \mathbf{m})(K) = x^*(x_K), (x^* \circ \mathbf{m})(K_0) = x^*(x_{K_0})$) for $x^* \in H$. Consequently, by Claim 1, (vi) (resp. (vii), (viii), (ix)) holds.

Finally, by Claim 1, (vi) (resp. (vii), (viii), (ix)) implies (ii) (resp. (iii), (iv), (v)) and hence each one implies that u is prolongable.

Theorem 19.14. Let X, u, H, μ_{x^*u} for $x^* \in H$ and $\mathbf{m}$ be as in Theorem 19.13. Then:
(a) The following statements are equivalent:

(i) u is a weakly compact Radon operator.

(ii) For each open set U in T there exists $x_U \in X$ such that $x^*(x_U) = \int_T \chi_U d(\mu_{x^*u})$ for $x^* \in H$.

(iii) Similar to (ii) with U σ-Borel open sets in T.

(iv) Similar to (ii) with U open Baire sets in T.

(v) Similar to (ii) with U F_σ- open sets in T.

(vi) Similar to (ii) with U σ-compact open sets in T.

(vii) Similar to (ii) with U replaced by closed sets F in T.

(viii) Similar to (vii) with F closed G_δ-sets in T.

(b) If u is a weakly compact Radon operator, then u is prolongable and the function χ_T is $\mathbf{m}_u$-integrable in T.

Proof. (a) is proved using Claim 1 in the proof of Theorem 19.13 and Theorem 19.7(ii). The first part of (b) is immediate from Definition 19.8. As $\mathbf{m}_u$ is X-valued and σ-additive on $\mathcal{B}(T)$ by Theorem 2 of [P5],χ_T is $\mathbf{m}_u$-integrable in T by Theorem 11.9(i) and Remark 12.5 of [P12].

Theorem 19.15. Let X be a quasicomplete lcHs and let $u : (C_c(T), ||\cdot||_T) \to X$ be a continuous linear mapping. Let $\tilde{u}$ be the continuous linear extension of u to $C_0(T)$ and let $\mathbf{m}$ be the representing measure of $\tilde{u}$ in the sense of Definition 4 of [P5]. Let $\mathbf{m}_c = \mathbf{m}|_{\mathcal{B}_c(T)}$ and $\mathbf{m}_0 = \mathbf{m}|_{\mathcal{B}_0(T)}$ .Then the following statements are equivalent:

(i) u is a weakly compact Radon operator.

(ii) u is prolongable and given $\epsilon > 0$ and $q \in \Gamma$, there exists $K \in \mathcal{C}$ such that $||\mathbf{m}||_q(T \backslash K) < \epsilon$.

(iii) u is prolongable and given $\epsilon > 0$ and $q \in \Gamma$, there exists $K \in \mathcal{C}$ such that $||\mathbf{m}_c||_q(T \backslash K) < \epsilon$ where $||\mathbf{m}_c||_q(T \backslash K) = \sup_{A \in \mathcal{B}_c(T), A \subset T \backslash K} ||\mathbf{m}_c||_q(A)$.

(iv) u is prolongable and given $\epsilon > 0$ and $q \in \Gamma$, there exists $K_0 \in \mathcal{C}_0$ such that $||\mathbf{m}_0||_q(T \backslash K_0) < \epsilon$ where $||\mathbf{m}_0||_q(T \backslash K_0) = \sup_{A \in \mathcal{B}_0(T), A \subset T \backslash K_0} ||\mathbf{m}_0||_q(A)$.

Proof. (i)⇒(ii) Let $V \in \mathcal{V}$. Then $\widetilde{u_V} = \tilde{u}|_{C_0(V)}$ is weakly compact and hence u is prolongable. By Theorem 6(xxi) of [P5] and by Proposition 2.2 of [P10] the other part of (ii) holds.

Clearly, (ii)⇒(iii)⇒(iv) since for $K \in \mathcal{C}$, by Theorem 50.D of [H] there exists $K_0 \in \mathcal{C}_0$ such that $K \subset K_0$.

(iv)⇒(i) Let $K_0 \in \mathcal{C}_0$. Choose $V \in \mathcal{V}$ such that $K_0 \subset V$. By hypothesis, $\widetilde{u_V}$ is weakly compact and hence by Theorem 8(xxx) of [P5] and by Proposition 2.2 of [P10], given $\epsilon > 0$ and $q \in \Gamma$, there exists $U \in \mathcal{B}_0(V)$, U open in V such that $K_0 \subset U$ and $||\mathbf{m}_V||_q(U \backslash K_0) < \epsilon$. As V is open in T, U is open in T and by Lemma 18.3, $U \in \mathcal{B}_0(T)$. This proves that $\mathbf{m}_0$ is Baire outer regular in each $K_0 \in \mathcal{C}_0$. The other hypothesis in (iv) implies that $\mathbf{m}_0$ is Baire inner regular in T and hence by Theorem 8(xxx) of [P5], $\tilde{u}$ is weakly compact. Hence (i) holds.

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