NOTAS DE MATEMATICAS

Nº 110

ON THE OPTIMAL CONTROL IN BANACH SPACES

BY

HUGO LEIVA AND DIOMEDES BARCENAS

UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMATICAS
MERIDA-VENEZUELA

1991

ON TIME OPTIMAL CONTROL IN BANACH SPACES

BY

HUGO LEIVA AND DIOMEDES BARCENAS

ABSTRACT.

In this paper we prove some properties of attanaible sets of the abstract control system $\dot{x} = Ax + Bu$, where the controls u take their values almost everywhere in a convex weakly compact subset Ω of the control space. Furthermore we characterize the extremal controls and give a neccesary and sufficient condition for the normality of the system. After we prove an existence theorem for time optimal control, and establish a Maximum Principle for that control. Moreover, under certain conditions, the minimal control is the unique control that satisfies the tranversality condition; that is we can obtain a sufficient condition for optimality.

1. INTRODUCTION AND PRELIMINARIES.

In this paper let X be a reflexive Banach space, U a

KEY WORDS: *Time optimal control, normal system, extremal control infinite-dimensional sistems.*

ON THE OPTIMAL CONTROL IN BANACH SPACES

BY

HUGO LEIVA AND DIOMEDES BARCENAS

separable reflexive Banach space, A-with domain D(A) - the infinitesimal generator of a strongly continuous semigroup S(t) ($t \geq 0$) in the Banach space X, B a linear bounded operator whose domain is U and its range is contained in X ($B \in L(U,X)$); X* and U* the respective dual spaces of X and U, <.,.> the duality and Ω a nonempty subset of U.

We will study the property of the set of attanaible points and the time optimal control problem associated with the infinite-dimensional system

$$(1.1) \qquad \begin{cases} \dot{x}(t) = Ax(t) + B\,u(t) & t > 0 \\ x(0) = x_0, & \end{cases}$$

where the state $x(t) \in X$ and the admissible controls are defined ou $\mathbb{R}_{+0} = [0,\infty)$ and take their values is Ω almost everywhere. Each control is a measurable and essentially bounded function on finite intervals of R_{+0} (we will denote this functions by $L_\infty^{loc}(\mathbb{R}_{+0}, U)$).

For each admissible control U(.), the mild solution of (1.1) is given by

$$(1.2) \qquad x_u(t) = S(t)x_o + \int_0^t S(t-\alpha)B\,u(\alpha)d\alpha, \qquad t \geq 0$$

DEFINITION 1.1. Given $t_1 > 0$ the set of admissibles control on $[0,t_1]$ is defined by

$$C(t_1) = \{u \in L_\infty(0,t_1; U) : u(t) \in \Omega \quad a.e\}$$

and the corresponding set of *attanaible points* by

$$K(t_1) = \{ x_u(t_1) : x_u(.) \text{ is a mild solution of } (1.1), u \in C(t_1)\}$$

DEFINITION 1.2. A control $u \in C(t_1)$ is called an *extremal control* if the corresponding solution x_u satisfies $X_u(t_1) \in \partial K(t_1)$.

DEFINITION 1.3. For each $t \geq 0$, consider a target set $G(t) \subset X$. Suppose $t^* \geq 0$ and $U^* \in C(t^*)$ such that $x^*(t^*) \in G(t^*)$. Then u^* is called an optimal control if

$$t^* = \inf \{t \in [0,\infty) : K(t) \quad G(t) \neq \emptyset\}.$$

In references [1] and [5] the authors the time optimal control problem for the case in that $G(t) = \{x_1\}$, but they did not characterize the normality either analyze the transversality as a sufficient condition of optimality.

We will use the following notation:

$$WC(X) = \{K \in 2^X \setminus \{\emptyset\} : K \text{ is convex and closed}\}.$$

DEFINITION 1.4. On $2^X \setminus \{\emptyset\}$ the *Hausdorff pseudometric* ρ

is defined by

$$\rho(K_1, K_2) = \max \{\sup_{a\varepsilon A} d(a,B), \sup_{b\varepsilon B} d(A,b)\},$$

Since $\rho(A,B) = 0 \iff \bar{A} = \bar{B}$, the pseudometric ρ restricted to the closed subsets of X is a metric which can take the value ∞ and if we constrain it to WC(X), we get that WC(X) is a complete metric space (see [8]).

To given an idea about this work, we recall some results on attainable sets time optimal control problem, for finite-dimensional linear control systems:

If $X = \mathbb{R}^n$, $V = \mathbb{R}^m$ and Ω is a nonempty compact subset of U, then in [4] and [7] it is proved that K(t) is convex, compact and varying continuously with t, respect to the Hausdorff metric. In this case to characterize the extremal controls they use the fact that in finite-dimensional space foreach boundary point of a closed convex set there is a support hyperplane and the following selection theorem.

For every non-trivial solution $\eta(\cdot)$ of adjoint equation $\dot{\psi} = -\psi A$, there is $u \in C(t_1)$ such that

$$\eta(t)Bu(t) = \max_{v\varepsilon\Omega} \eta(t)Bv \quad \text{a.e on } [0,t_1].$$

Then by mean of this characterization they find a maximum principle for optimal control.

In infinite-dimensional case we don't have either support hyperplane for each boundary point of a closed convex set or adjoint equation. However each statement of the following proposition can be seen in [9].

We recall that X and U are reflexive Banach spaces.

PROPOSITION 1.1. If Ω is convex and weakly compact, then

a) $C(t_1)$ is convex and weakly compact in

$$L_p(0,t_1;U) \quad (1 < p < \infty).$$

b) $K(t_1)$ is convex and weakly compact in X.

c) The family of functions

$$f_t: X^* \longrightarrow \mathbb{R} \qquad (t \in [0,t_1])$$

defined by

$$f_t(x^*) = \max_{v\in\Omega} \langle x^*, S(t)Bv\rangle$$

is equicontinuous.

d) For each $x^* \in X^*$, the mapping

$$t \in [0,t_1] \longrightarrow \max_{v\in\Omega} \langle x^*, S(t)Bv\rangle$$

is continuous.

e) If $0 \in \Omega$, for each $x^* \in X^*$ there exists $u \in C(t_1)$ such that

$$\langle x^*, S(t)\, Bu(t)\rangle = \max_{v\varepsilon\Omega} \langle x^*, S(t)\, Bv\rangle \text{ a.e}$$

in $[0,t_1]$.

REMARK. The statements (a)-(d) had been obtained in a more general context by Bárcenas-Leiva [2].

Throught this work we will suppose K(t) has nonempty interior (int $K(t) \neq \emptyset$).

Linear control systems with this property (int $K(t) \neq \Phi$) had been broadly characterized (see for example [3], [9]).

In all this work we will suppose $0 \varepsilon \Omega$ and Ω is a weakly compact set.

2. CHARACTERIZATION OF EXTREMAL CONTROLS.

PROPOSITION 2.1. K(t) varies continuously with $t \geq 0$ respect to the Hausdorff metric.

PROOF. Let $t_1 > 0$ be fixed and $\varepsilon > 0$, we must find $\delta > 0$ such that

$$|t_1 - t_2| < \delta \Rightarrow \rho(K(t_1), K(t_2) < \varepsilon.$$

Let t_2 be with $|t_1 - t_2| < t_1$. If $x \varepsilon K(t_1)$, there exists $u \varepsilon C(t_1)$ such that

$$x = S(t_1)x_0 + \int_0^{t_1} S(t_1-\alpha)\; Bu(\alpha)d\alpha.$$

If we define the control

$$\bar{u}(t) = \begin{cases} u(t) & \text{if } 0 \le t \le t_1 \\ u(t_1) & \text{if } t_1 \le t \le 2t_1, \end{cases}$$

then

$$y = S(t_2)x_o + \int_0^{t_2} S(t_2-\alpha)B\bar{u}(\alpha)\,d\alpha \in K(t_2)$$

and

$$\|x-y\| \le \|S(t_2)x_o - S(t_1)x_o\| + \left\|\int_{t_1}^{t_2} S(t_2-\alpha)B\bar{u}(\alpha)\,d\alpha\right\|$$

$$+ \int_0^{t_1} \|S(t_2-\alpha)Bu(\alpha) - S(t_1-\alpha)\,Bu(\alpha)\|\,d\alpha. \qquad (1)$$

It is easy to see that

$$\lim_{t_2 \to t_1} \|S(t_2)x_o - S(t_1)x_o\| = 0$$

and

$$\lim_{t_2 \to t_1} \left\|\int_{t_1}^{t_2} S(t_2-\alpha)B\bar{u}(\alpha)\,d\alpha\right\| = 0.$$

Since

$$\lim_{t_2 \to t_1} \|S(t_2-\alpha)Bu(\alpha) - S(t_1-\alpha)Bu(\alpha)\| = 0,$$

and, by Hille-Yosida's Theorem there exist $M, w > 0$ such that

$$\| S(t_2-\alpha)Bu(\alpha)-S(t_1-\alpha)Bu(\alpha)\| \leq Me^{w(t_1-\alpha)} ,$$

so by applying dominated convergence theorem to the first term on the right of (1), we can find $0 < \delta < 1$ such that

$$|t_2 - t_1| < \delta \Rightarrow \|x-y\| < \varepsilon.$$

This concludes the proof when $t_1 > 0$. The case $t_1 = 0$ is evident. #

COROLLARY 2.1. If $p \in \text{int } K(t_1)$, there is a neigborhood N of p and $\delta > 0$ such that $N \subset K(t_2)$ for $|t_2-t_1| < \delta$.

PROOF. Since $p \in \text{int } K(t_1)$ there is $r > 0$ such that $\overline{B(p,r)} \subset K(t_1)$. We put

$$\alpha = \inf\{\|x-y\| : x \in \partial B(p,r),\ y \in \partial K(t_1)\} > 0$$

without loss of generality can suppose $\alpha-2r > 0$. Hence, by proposition 2.1 there is $\delta > 0$ such that

$$|t_2 - t_1| < \delta \Rightarrow \rho(K(t_1), K(t_2) < \frac{\alpha-2r}{4} . \tag{2}$$

If $x_o \in B(p,r) \subset K(t_2)$, there is $x^* \in X^*$ which satisfies the condition

$$K(t_2) \subset \{x \in X: x^*(x) < x^*(x_o)\}.$$

On the other hands we have

$$\|x_o-y\| \geq \|y-z\| - \|x_o-z\|$$

$$\geq \alpha - \|x_o-z\| ; \qquad y \in \partial K(t_1), \; z \in \partial B,$$

hence

$$\|x_o-y\| \geq \alpha r \qquad y \in \partial K(t_1)$$

which implies that there exists $y_o \in \partial K(t_1)$ such that

$$\inf_{x \in \Pi x_o} \|x-y_o\| > \frac{\alpha-2r}{2}$$

where $\Pi x_o = \{x \in X : \langle x^*, x-x_o \rangle = 0\}$.

Thus

$$\inf_{x \in K(t_2)} \|x-y_o\| > \frac{\alpha-2r}{2} .$$

This contradicts the formula (2). #

THEOREM 2.1. A control $u \in C(t_1)$ is extremal if and only if there is a non-zero $x^* \in X^*$ such that

$$\langle x^*, S(t_1-t)Bu(t) \rangle = \max_{V \in \Omega} \langle x^*, S(t_1-t)Bv \rangle$$

a.e on $[0,t_1]$.

PROOF. If u is an extremal control, the corresponding solution $x(.)$ satisfies $x(t_1) \varepsilon \partial K(t_1)$. Since $K(t_1)$ is convex and weakly compact and it$K(t_1) \neq \emptyset$ $(t_1 > 0)$ there is $x^* \varepsilon X^*$; $x^* \neq 0$ such that

$$<x^*, x - x(t_1)> \leq 0 \qquad \forall x \varepsilon K(t_1).$$

Suppose that there exists $E \subset [0,t_1]$ with positive measure such that

$$<x^*, S(t_1-t) Bu(t) > = < \max_{v\varepsilon\Omega} <x^*, S(t,-t), Bv> \qquad (t \varepsilon E).$$

By proposition 1.1, there exists $\tilde{u} \varepsilon C(t_1)$ such that

$$<x^*, S(t_1-t)B\tilde{u}(t)> = \max_{v\varepsilon\Omega} <x^*, S(t_1-t)Bv>$$

a.e on $[0,t_1]$ and the corresponding solution $x(.)$ satisfies the inequality

$$<x^*, \overline{x}(t_1) - x(t_1)> =$$

$$= \int_0^{t_1} [<x^*, S(t_1-\alpha)B\tilde{u}(\alpha)> - <x^*, S(t_1-\alpha)Bu(\alpha)>] d\alpha$$

which contradicts the choices of x^*. Hence

$$<x^*, S(_1-t)Bu(t)> = \max_{v\varepsilon\Omega} <x^*, S(t_1-t)Bv>$$

a.e on $[0,t_1]$.

Conversely if $u \in C(t_1)$ and $x^* \in X^*$ different from zero for which

$$<x^*, S(t_1-t)\, Bu(t)> = \max_{v\in\Omega} <x^*, S(t_1-t)Bv> \text{ a.e}$$

on $[0,t_1]$ and $x(.)$ is the corresponding solution for u; for each $\tilde{x}(t_1) \in K(t_1)$ there is $\tilde{u} \in C(t_1)$ such that

$$\tilde{x}(t_1) = S(t_1)x_o + \int_0^{t_1} S(t_1-\alpha)\; B\tilde{u}(\alpha)\,d\alpha.$$

Hence

$$<x^*, \tilde{x}(t_1) - x(t_1)> = \int_0^{t_1} <x^*, S(t_1-\alpha)B(\tilde{u}(\alpha)-u(\alpha))d\alpha>$$

$$\leq 0,$$

which implies $x(t_1) \in \partial K(t_1)$. #

COROLLARY 2.2. Let $u \in C(t_1)$ be an extremal control with solution $x(.)$. Then for every $\tau \in (0,t_1]$, the restriction of u to the interval $[0,\tau]$ is an extremal control. Furthermore, if $x^* \in X^*$ separates $x(t_1)$ and $K(t_1)$, then x^* separates $x(\tau)$ and $X(\tau)$.

PROOF. It is not hard.

3. NORMAL CONTROL SYSTEMS.

DEFINITION 3.1. The control system (1.1) is called normal if the following implication holds: If $u_1, u_2 \in C(t_1)$ transfer x_o to the same $p \in \partial K(t_1)$, then $u_1 = u_2$ a.e on $[0,t_1]$.

Theorem 3.1. If the control system (1.1) is normal, then $K(t_1)$ is strictly convex.

PROOF. Suppose there exists a support hyperplane Π_{t_1} for $K(t_1)$ for which $\Pi_{t_1} \cap K(t_1)$ contains a line's segment L. Let $p_a \neq p_b \in L$ with $u_a, u_b \in C(t_1)$ their corresponding controls respectively.

We now considerer the Banach space $Y = X \times X$ with the norm

$$\|y\|_Y = \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_Y = \|x_1\|_X + \|x_2\|_X$$

and the function

$$f(t) = \begin{bmatrix} S(t_1-t)\ Bu_a(t) \\ S(t_1-t)\ Bu_b(t) \end{bmatrix}, \quad t \in J = [0,t_1],$$

with values in Y. Clearly $f \in L(0,t_1,Y)$. By Liapunov's theorem (see [5]) the set

$$F = \{w(D) = \int_D f(t)dt : D \subset J \text{ is measurable}\}$$

has convex closure. Thus

$$\frac{1}{2} w(J) = \frac{1}{2} w(J) + \frac{1}{2} w(\emptyset) \in \overline{F} ;$$

therefore there exists a sequence $\{w(D_n)\}$ contained in F such that

$$\lim_{n\to\infty} w(D_n) = \frac{1}{2} w(J); \quad \lim_{n\to\infty} w(J \setminus D_n) = \frac{1}{2} w(J).$$

Consider the controls

$$u_n^{(2)}(t) = \begin{cases} u_a(t) & , \quad t \in D_n \\ u_b(t) & , \quad t \in J \setminus D_n \end{cases}$$

$$u_n^{(2)}(t) = \begin{cases} u_a(t) & t \in J \setminus D_n \\ u_b(t) & t \in D_n . \end{cases}$$

with corresponding solutions $x_n^{(1)}(.)$ and $x_n^{(2)}$. It is easy to see that

$$\lim_{n\to\infty} x_n^{(1)}(t_1) = \lim_{n\to\infty} x_n^{(2)}(t_1) = \frac{1}{2} p_a + \frac{1}{2} p_b .$$

Since $C(t_1)$ is weakly compact in $L_p\ (0,t_1;\ U)$, $(1 < p < \infty)$; we can suppose that the sequences $\{u_n^{(1)}\}$ and $\{u_n^{(2)}\}$ converge weakly to the controls $u_1,\ u_2 \in C(t_1)$ respectively. Therefore for each $x^* \in X^*$ we have

$$\lim_{n\to\infty} \langle x^*, x_n^{(1)}(t_1)\rangle = \langle x^*,\ S(t_1)x_o + \int_0^t S(t_1-\alpha)Bu_1(\alpha)d\alpha\rangle$$

$$= \langle x^*, \frac{1}{2} p_a + \frac{1}{2} p_b \rangle = \langle x^*, S(t)x_o + \int_0^t S(t_1-\alpha)Bu_2(\alpha)d\alpha \rangle$$

$$= \lim_{n\to\infty} \langle x^*, x_n^{(2)}(t_1)\rangle .$$

This implies

$$S(t_1)x_o + \int_0^{t_1} S(t_1-\alpha)Bu_1(\alpha)d\alpha$$

$$= S(t_1)x_o + \int_0^{t_1} s(t_1-\alpha)Bu_2(\alpha)d\alpha \ ;$$

because X is a reflexive Banach space and so $S^*(t)$ $(t \geq 0)$ is a strongly continuous semigroup on X^*(see [1]).

Thus, by normality of the system (1.1) we obtain

(3.1) $$u_1(t) = u_2(t) \text{ a.e on } J = [0,t_1].$$

Since $C(t_1) \subset L_p\ (0,t_1;U)$ $(1 < p < \infty)$, from equality (3.1) we get

$$\lim_{n\to\infty} \langle u^*, u_n^{(1)} - u_n^{(2)} \rangle = 0 \text{ for each } u^* \in L_p\ (0,t_1;U)^* =$$

$$= L_q\ (0,t_1;\ U^*),$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Therefore, by $u_n^{(1)}$ and $u_n^{(2)}$, definitions we get

$$\langle u^*, u_a - u_b \rangle = 0 \quad \text{for each} \quad u^* \in L_q(0,t_1; U^*).$$

Thus $u_a(t) = u_b(t)$ a.e on $J = [0,t_1]$; and consequently $p_a = p_b$ which is a contradiction. This concludes the proof. #

THEOREM 3.2. The control system (1.1) is normal if and only if for each $x^* \in X^*$ non-zero and for each pair of controls $u_1, u_2 \in C(t_1)$, such that

$$(3.2) \quad \langle x^*, S(t_1-t)Bu_1(t) \rangle = \langle x^*, S(t_1-t)Bu_2(t) \rangle$$

$$= \max_{v\in\Omega} \langle x^*, S(t_1-t)Bv \rangle \text{ a.e on } [0,t_1]$$

we have $u_1 = u_2$ a.e on $[0,t_1]$.

PROOF. We suppose the control system (1.1) normal and considerer $x^* \in X^*$ a continuous linear functional non-zero. Let $u_1(.)$, $u_2(.)$ be controls in $C(t_1)$ with the corresponding solutions $x_1(.)$, $x_2(.)$ such that

$$\langle x^*, S(t_1-t)Bu_1(t) \rangle = \langle x^*, S(t_1-t)Bu_2(t) \rangle$$

$$= \max_{v\in\Omega} \langle x^*, S(t_1-t)Bv \rangle \text{ a.e ou } [0,t_1].$$

Let Π be the hyperplane defined by

$$\Pi = \{x \in X : \langle x^*, x - x_1(t_1) \rangle = 0\} .$$

That hyperplane is supporting $K(t_1)$ in the points $X(t_1)$ and $X_2(t_2)$ and by previous theorem $X_1(t_1) = X_2(t_1)$. Since (1.1) is supposed normal, then $u_1(t) = u_2(t)$ a.e on J.

Conversely, if we suppose (3.2) and $p \varepsilon \partial K(t_1)$ then there exists $x^* \varepsilon X^*$; $x^* \neq 0$ such that

$$\Pi = \{x \varepsilon X ; \quad \langle x^*, x - p\rangle = 0\}$$

is supporting $K(t_1)$ at p because int $K(t_1) \neq \emptyset$.

Let u_1, u_2 be controls belonging $C(t_1)$ which transfer x_o to p. Then, by Theorem 2,1,

$$\langle x^*, S(t_1-t)Bu_1(t)\rangle = \langle x^*, S(t_1-t)Bu_2(t)\rangle$$

$$= \max_{v\varepsilon\Omega} \langle x^*, S(t_1-t)Bv \quad \text{a.e ou} \quad [0,t_1].$$

Therefore, by (3.2) we obtain $u_1(t) = u_2(t)$ a.e on $[0,t_1]$. Hence the system (1.1) is normal. #

COROLLARY 3.1. If the system (1.1) is normal on $[0,t_1]$ then it is normal on $[0,\tau]$ for $0 < \tau < t_1$.

PROOF. It is easy to get. #

In following we will give an example in which the system (1.1) is normal and, of course, $K(t_1)$ is strictly convex.

EXAMPLE 3.1. Let Ω be a strictly convex and weakly compact set and suppose Ker $B^*S^*(t) = \{0\}$, for each $t \in [0,t_1]$.

Under these conditions the system (1.1) is normal. In fact, by theorem 3.2 it is sufficient to prove:

$$u_1, u_2 \in C(t_1) \text{ and } x \in X^*, \quad x^* \neq 0;$$

$$<x^*, S(t)\, Bu_1(t)> = <x^*, S(t)Bu_2(t)>$$

$$= \max_{v\in\Omega} <x^*, S(t)Bv> \quad \text{a.e on } [0,t_1]$$

because $x_t^* = B^*\ S^*(t)x^* \neq 0$, x_t^* attains a maximum on Ω at a unique point. So, the system (1.1) is normal and consequently $K(t_1)$ is strictly convex. #

4. TIME OPTIMAL CONTROL.

In this section we will prove an existence theorem for time optimal control according to definition 1.3.

THEOREM 4.1. Suppose the target set $G(t) \subset X$ is convex, weakly compact and varies continuously with t on $0 \leq t \leq \tau_1$. If there exists a control $u \in C(t_1)$ such that $x_u(t_1) \in G(t_1)$, then there exists a time optimal control $u^* \in C(t^*)$.

PROOF. We put

$$H = \{t \in [0,\tau_1] : K(t) \cap G(t) \neq \Phi\}.$$

To prove that H is a compact subset of $\mathbb{R}$ it is sufficient to prove that H is closed. If $\{t_n\}$ is a sequence in H with $\lim_{n\to\infty} t_n = t'$ and $K(t') \cap G(t') = \emptyset$, then

$$d = \inf\{\|x-y\| : x \in K(t'),\ y \in G(t')\} > 0.$$

On the other hands, there exists $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow \rho(K(t_n), K(t')) < \frac{d}{4}, \quad \rho(G(t_n), G(t')) < \frac{d}{4}.$$

Then, by the definition of the Hausdorff metric there exists $\bar{x} \in K(t')$, $\bar{y} \in G(t')$ such that

$$\| x_N - \bar{x} \| < \frac{d}{4}, \qquad \| x_N - \bar{y} \| < \frac{d}{4}.$$

With $x_N \in G(t_N) \cap K(t_N)$.

This implies,

$$\| \bar{x} - \bar{y} \| < \frac{d}{2}$$

which contradicts the choice of d.

Hence, if $t^* = \min H$, then $K(t^*) \cap G(t^*) \neq \emptyset$

If $u^* \in C(t^*)$ such that the corresponding solution $x_*(.)$ satisfies $x_*(t^*) \in G(t^*)$, then $u^*(.)$ is an optimal control required. #

REMARK 4.1. Theorem 4.1 remains valid if we change convex and weakly compact target sets by compactness.

THEOREM 4.2. Under the hypotheses of previous theorem, if $u^* \varepsilon C(t^*)$, $0 < t^* < \tau_1$ is an iptimal control, with corresponding solution $x_*(.)$ then $u^*(.)$ is extremal. That is:

$$(4.1) \qquad m(t) = \max_{v\varepsilon\Omega} \langle x^*, S(t_1-t)Bv = \langle x^*, S(t_1-t)Bu(t)\rangle$$

a.e. on $[0,t^*]$, for some $0 \neq x^* \varepsilon X^*$. Moreover, if $G(t) \equiv G$ is constant, then

$$(4.2) \qquad x_*(t^*) \ \varepsilon \ [\partial K(t^*) \setminus \bigcup_{0<t<t^*} K(t)].$$

PROOF. If $x_*(t^*) \ \varepsilon \ \mathrm{int}K(t^*)$, then by corollary 2.1, there are an open subset N of X such that $x_*(t^*) \ \varepsilon \ N$ and $\delta > 0$ such that

$$(4.3) \qquad t^* - \delta < t < t^* \Rightarrow N \subset K(t).$$

If

$$G(t) \cap N = \emptyset \qquad t \ \varepsilon \ (t^*-\delta, t^*),$$

then $G(t) \subset N^c$, where N^c denotes the complement of N.

We assert that $G(t^*) \subset N^c$. In fact, if $x_o \ \varepsilon \ G(t^*)$ but $x_o \ \varepsilon \ N^c$, then

$$0 < d = \inf_{x\varepsilon N^c} \| x_o - x \| \leq \inf_{x\varepsilon G(t)} \| x_o - x \| \qquad (t \ \varepsilon \ (t^*-\delta, t())).$$

Hence we have

$\rho(G(t^*, G(t) \geq 0 \quad (t \in (t^*-\delta, t^*))$, which contradicts the continuity of $G(t)$ respect to the Hausdorff metric. Therefore $G(t^*) \subset N^c$. In particular $G(t^*) \cap N = \emptyset$. Which contradicts the definition of N. Thus there is $t_1 \in (t^*-\delta, t^*)$ such that $G(t_1) \cap N \neq \emptyset$. Hence from (4.3) we obtain

$$K(t_1) \cap G(t_1) \neq \emptyset ,$$

and this contradicts that $u^*(.)$ is a time optimal control. Thus $x_*(t^*) \in \partial K(t^*)$ which means that $u^*(.)$ is an extremal control. Hence by theorem 2.1 there exists $x^* \in X^*$ non-zero, such that

$$m(t) = \max_{v \in \Omega} \langle x^*, S(t^*-S)Bv \rangle = \langle x^*, S(t^*-S)Bu(s) \rangle$$

a.e on $[0,t^*]$. Moreover the hyperplane

$$\Pi(t^*) = \{x \in X^*: \langle x^*, x-x_*(t^*) \rangle = 0\}$$

is supporting $K(t^*)$ at $x_*(t^*)$.

Now, we suppose $G(t) = G$ is constant. Then, since u^* is an optimal control, $x_*(t^*) \in K(t^*) \cap G$; therefore $x_*(t^*) \notin K(t) \; \forall t \in [0,t^*)$. This implies

$$x_*(t^*) \in \left[\partial K(t^*) \setminus \bigcup_{0<t<t^*} K(t)\right]. \quad \#$$

REMARK 4.2. If $S(t)$, $(t \in \mathbb{R})$ is a strongly continuous group, then given $s \in \mathbb{R}$, for each $x_1^* \in X^*$ there exists $x_2^* \in X^*$ such that $S^*(-s)\, x_2^* = x_1^*$. In this case theorem 2.1 takes the following form:

A control $u \in C(t_1)$ is extremal if and only if, there is $0 \neq x^* \in X^*$ such that

$$\max_{v \in \Omega} \langle \eta(s), Bv \rangle = \langle \eta(s), Bu(s) \rangle \quad \text{a.e on } [0,t_1]$$

where

$$\eta(s) = S^*(-s)x^*, \quad 0 \leq s \leq t_1.$$

This is a generalization of a result for the finite-dimensional case (see [7]).

THEOREM 4.3. Suppose $x_o \in D(A)$, $S(t)$ $(t \in \mathbb{R})$ is a strongly continuous group on X, $G(t) = G$, is constant and $u^* \in C(t^*)$ is an optimal control which satisfies

$$(4.4) \quad \begin{aligned} &S(-s)\, Bu^*(s) \in D(A) \quad \text{a.e on } [0,t^*],\ t^* > 0 \\ &S(t-.)\, Bu^*(.) \in L_1(0,t;X) \quad \text{for all } t \in [0,t^*). \end{aligned}$$

Then

$$M(t) = \max_{v \in \Omega} \langle \eta(t), Ax_*(t) + Bv \rangle = \langle \eta(t), Ax_*^*(t) + Bu^*(t) \rangle$$

is defined a.e on $[0,t^*]$, where $x_*(.) = x_{u^*}(.)$ and $\eta(t) = S(-t)x^*$

with $x^* \neq 0$ according to Remark 4.2. Moreover we can choose $\eta(t^*)$ such that

(4.5) $$M(t^*) \geq 0.$$

Furthemore, if G is convex we can choose $\eta(t^*)$ satisfaying the <u>transversality condition</u>. Namely, the hyperplane

$$\Pi(t^*) = \{x: <\eta\ t^*,\ x-x_*(t^*)> = 0\}$$

separates $K(t^*)$ and G at the point $x_*(t^*)$.

PROOF. If hypothesis (4.4) is satisfied, then lemma 2.22 of [3] implies

$$\dot{x}_*(t) = Ax_*(t) + Bu^*(t) \quad \text{a.e on} \quad [0,t^*] ,$$

which implies $M(t)$ is well defined. We will use a limit process to prove $M(t^*) \geq 0$ because $x_*(.)$ may not be differentiable.

From (4.2) we obtain $x_*(t_*) \notin K(t)$, if $0 \leq t_1 < t^*$. Thus, by Theorem 9.1 of [3] there is $\eta(t_1) \in X^*$, with $\|\eta(t_1)\| = 1$ and

(4.6) $$0 < \inf_{x \in K(t_1)} \| x_*(t^*)-x \| = \inf_{x \in K(t_1)} <\eta(t_1),\ x_*(t^*)-x> .$$

Since $K(t_1)$ is weakly compact, there is $x(t_1) \varepsilon K(t_1)$ such that

$$(4.7) \quad 0 \leq \inf_{x \varepsilon K(t_1)} \| x_*(t^*) - x \| = \langle \eta(t_1), x_*(t^*)-x(t_1) \rangle .$$

From (4.6) and (4.7) we get

$$\langle \eta(t_1), x_*(t)-x \rangle \leq 0, \quad \text{for every} \quad x \varepsilon K(t_1).$$

This means that

$$(4.8) \quad \langle \eta(t_1), x_*(t^*) - x(t_1) \rangle > 0 \text{ and } \langle \eta(t_1), x-x(t_1) \rangle \leq 0$$
$$x \varepsilon K(t_1).$$

Thus $\eta(t_1)$ separates $x_*(t^*)$ of $K(t_1)$ at $x(t_1) \varepsilon K(t_1)$.

We will now prove that there is $\hat{t}_1 \varepsilon (t_1,t^*)$ such that $\langle \eta(t_1), \dot{x}(\hat{t}_1) \rangle > 0$. Otherwise, if for each $t \varepsilon (t_1,t^*)$, where $\dot{x}_*(t)$ exists we have

$$\langle \eta(t_1), \dot{x}_*(t) \rangle \leq 0 \text{ a.e on } [t_1,t^*],$$

then, since $x_*(.)$ is absolutely continuous,

$$\int_{t_1}^{t^*} \langle \eta(t_1), \dot{x}_*(t) \rangle \, dt \leq 0 \iff \langle \eta(t_1), x_*(t^*)-x_*(t_1) \rangle \leq 0$$

which implies

$$<\eta(t_1), x_*(t^*) - x(t_1)> = <\eta(t_1), x_*(t^*) - x_*(t_1)>$$

$$+ <\eta(t_1), x_*(t_1) - x(t_1)> .$$

This contradicts (4.8).

In this way we can choose a sequence

$$0 < t_1 < \hat{t}_1 < t_2 < \hat{t}_2 < \ldots < t_n < \hat{t}_n < \ldots < t^*$$

satisfying

$$(4.9) \qquad <\eta(t_n), \dot{x}_*(\hat{t}_n)> > 0 \iff <\eta(t_n), Ax_*(t_n) + Bu^*(\hat{t}_n)> > 0.$$

By weak compactness and uniform boundeness of the sets $K(t)$ $0 < t \leq t^*$, we can suppose:

$$(4.10) \qquad \begin{array}{l} \lim_{n\to\infty} u^*(\hat{t}_n) \overset{W}{=} u \in \Omega, \quad \lim_{n\to\infty} \eta(t_n) \overset{W}{=} \eta(t^*) \\ \lim_{n\to\infty} <\eta(t_n), x(t_n)> = \alpha \in \mathbb{R} \end{array}$$

where denotes the weak convergence.

Consider the hyperplane $\Pi(t^*)$ given by

$$\Pi(t^*) = \{x \in X: <\eta(t^*), x - x_*(t^*)> = 0\} .$$

We will show that $\Pi(t^*)$ separates $x_*(t^*)$ and $K(t^*)$. In fact, if there is $x_o \in K(t^*)$ such that

(4.11) $$\langle \eta(t^*),\ x_o - x_*(t^*) \rangle > 0,$$

since from (4.8) we get

(4.12) $$\langle \eta(t_n),\ x - x(t_n) \rangle \geq 0 \qquad (x \in K(t_n))$$

then, by the inequality

$$\langle \eta(t_n),\ x_*(t^*) - x(t_n) \rangle \leq \| x_*(t^*) - x_*(t_n) \|$$

and (4.10) we get

$$\lim_{n\to\infty} \langle \eta(t_n), x(t_n) \rangle = \langle \eta(t^*), x_*(t^*) \rangle$$

Since $\lim_{n\to\infty} \rho(K(t_n), K(t^*)) = 0$, we can get a sequence $\overline{x}_n \in K(t_n)$ $n=1,2,\ldots$ with $\lim_{n\to\infty} \| \overline{x}_n - x_o \| = 0$ and by (4.12) we have

(4.13) $$\langle \eta(t_n),\ \overline{x}_n - x(t_n) \rangle \leq 0, \qquad n=1,2,\ldots$$

Hence

$$0 \leq \lim_{n\to\infty} \langle \eta(t_n),\ \overline{x}_n - x(t_n) \rangle = \lim_{n\to\infty} [\langle \eta(t_1), \overline{x}_n - x_o \rangle$$

$$+ \langle \eta(t_n), x_o \rangle - \langle \eta(t_n), x(t_n) \rangle$$

$$= \langle \eta(t^*), x_o - x^*(t^*) \rangle > 0,$$

is a contradiction which (4.13). So $\Pi(t^*)$ separates $x_*(t^*)$ and $K(t^*)$.

Since A is a closed operator, from (4.4) we get

$$A x_*(\hat{t}_n) = AS(\hat{t}_n)x_o + AS(\hat{t}_n) \int_0^{\hat{t}_n} S(-s)Bu^*(s)ds$$

$$= S(\hat{t}_n)Ax_o + S(\hat{t}_n) \int_0^{t_n} S(-s) ABu^*(s)ds$$

which implies that

$$\lim_{n\to\infty} Ax_*(\hat{t}_n) = S(t^*)Ax_o + S(t^*) \int_0^{t^*} S(-s)ABu^*(s)ds$$

$$= Ax_*(t^*).$$

Hence, by taking limit in (4.9) we obtain $M(t^*) \geq 0$.

If G is convex, by an analogous process we can find a hyperplane $\Pi^*(t^*)$ which separates $K(t^*)$ and G through $x_o(t^*)$. #

COROLLARY 4.1. If A is a bounded operator and $u \in C(t_1)$ is an extremal control, then

$$(4.14) \qquad M(t) = \max_{v\in\Omega} \langle \eta(t), Ax(t) + Bv \rangle$$

$$= \langle \eta(t), Ax(t) + Bu(t) \rangle \quad \text{a.e on} \quad [0,t_1]$$

is well defined and constant.

PROOF. Since A is a bounded operator, $D(A) = X$ and $S(t) = \exp(tA)$ is a strongly continuous group. In this way the hypoteses of Theorem 4.3 are satisfied. Therefore $M(.)$ is well defined and proceeding as in Proposition 1.1, we get that $M(.)$ is absolutely continuous and hence differentiable almost everywhere on $[0,t_1]$.

We will estimate the derivative of $M(t)$ at $t = \tau_1$ where it exists. Suppose $\tau_2 > \tau_1$. Then

$$\frac{M(\tau_2)-M(\tau_1)}{\tau_2 - \tau_1} \geq \frac{\langle \eta(\tau_2); Ax(\tau_2) + Bu(\tau_1) \rangle - \langle \eta(\tau_1), A(\tau_1)+Bu(\tau_1) \rangle}{\tau_2 - \tau_1}$$

$$= \langle \eta(\tau_2), A\,\frac{X(\tau_2)-x(\tau_1)}{\tau_2-\tau_1} \rangle + \langle \frac{\eta(\tau_2)-\eta(\tau_1)}{\tau_2-\tau_1}, Ax(\tau_1) \rangle$$

$$+ \langle \frac{\eta(\tau_2) - \eta(\tau_1)}{\tau_2 - \tau_1}, Bu(\tau_1) \rangle.$$

Whithout loss generality we can suppose that $\dot{x}(\tau_1)$ exists. If $x^* \in X^*$ satisfies the equation $\eta(t)=\exp(-A_t^*)x^*$, with $x^* \neq 0$ then

$$\eta\ (t) = - A^* \exp(-A^*t)x^* = -A^*\ \eta(t).$$

Thus, by taking the limit as $\tau_2 \to \tau_1$, we have

$$\frac{dM}{dt} \geq < \eta(\tau_1), A \dot{x}(\tau_1) > + < \dot{\eta}(\tau_1), Ax(\tau_1) > + < \dot{\eta}(\tau_1), Bu(\tau_1) >$$

$$= < \eta(\tau_1), A(Ax(\tau_1) + Bu(\tau_1) > - < A^*\eta(\tau_1), Ax(\tau_1) >$$

$$- < A^* \eta(\tau_1), Bu(\tau_1) > = 0.$$

Similar calculation shows that $\frac{dM}{dt}(\tau_1) \leq 0$. Consequently M is constant on $[0,t_1]$. #

The following theorem proves that, under normality conditions, the Maximum Principle (4.1) is sufficient for optimality, provided that the optimal control exists and is the unique extremal control which satisfies the transversality condition.

THEOREM 4.4. Let A be a bounded linear operator such that the following conditions are satisfied:

a) (1.1) is normal for $t > 0$

b) G is a convex and weakly compact subset of X

c) If $\bar{t} > 0$, $u \in C(\bar{t})$ and $x_u(\bar{t}) \in G$ then there exists a control $\bar{u} \in L_\infty^{loc}(\mathbb{R}_{+0}, U)$ such that $x_{\bar{u}}(t) \in G$ $(t \geq \bar{t})$ and $\bar{u}$ is not extremal at any $t > \bar{t}$.

Let $u_1 \in C(t_1)$, $u_2 \in C(t_2)$ satisfy the transversality conditions. Then $t_1 = t_2 = t^*$ and $u_2(t) = u_1(t)$ a.e on $[0,t^*]$. In particular $u_1 = u^*$ is the unique extremal control.

PROOF.

I) If $t_1 = t_2$, then $K(t_1)$ intersects G only at boundary points. Since $x_1(t_1)$ and $x_2(t_2)$ belong to ∂G and the problem (1.1) is normal, $K(t_1)$ is strictly convex. By the transversality condition there is a hyperplane which separates $K(t_1)$ and G; but the line segment joining $x_1(t_1)$ and $x_2(t_2)$ is contained in $K(t_1) \cap G$, then this segment is contained in $\partial K(t_1)$; which implies $x_1(t) = x_2(t)$. The normality implies now that $u_1(t) = u_2(t)$ a.e on $[0,t_1]$.

II) If $t_1 < t_2$, by the transversality condition we have that there is a hyperplane that separates $K(t_2)$ and G. By hypothesis (c) $K(t_2) \cap G \neq \emptyset$ which is a contradiction. Thus $t_1 = t_2$. #

ACKNOLEDGEMENT: Research supported by C.D.C.H.T. of Universidad de los Andes, under projec Nº C-391-89.

[1] N.V. Ahmed-K.L.Teo. Optimal control of distributed Parameter systems, North Holland, Amsterdam (1981).

[2] D. Bárcenas-H.Leiva. Controlabilidad con restricciones en espacios de Banach, Acta Científica Venezolana, 40, 3, 181-185 (1989).

[3] R.F. Curtain-A.J.Pritchard. "Infinite dimensional linear systems"Lecture Notes in control and information science, Springer-Verlag, Berlin-Heidelberg (1978).

[4] H. Hermes-J.P.Lassalle."Functional Analysis and time optimal control", Academic press, New York (1969).

[5] R.H.W. Hoppe."On the approximate solution of time-optimal control problems" Appl. Math. Optim. 9 263-290 (1983).

[6] V.I.Korobov-N.K. Shon. "Controllability of linear systems in Banach spaces with restrictions on the controls" translated from Differential'nye Uravneniya 16, 5, 806-817 (1980).

[7] E.B.Lee-L. Markus. "Foundations of optimal control Theory", John Wiley, New York (1967).

[8] N.S. Papageorgiu. "Properties of the solution and Attainable sets of differential inclusions in Banach spaces", Radovi Matematicki, 2, 247-261 (1986).

[9] G.Peichl - W. Schappecher. "Constrained controllability in Banach Spaces". SIAM J. Control and Optimization, 24, 6, 1261-1275 (1986).