THE SEPARABILITY OF THE STRICT TOPOLOGIES ON $C_B(X)$

BY

JORGE VIELMA

ABSTRACT.

THE CONCEPT OF A V-SEPARABILY SUBMETRIZABLE SPACE IS INTRODUCED AND IT IS PROVED THAT FOR SUCH SPACES ALL STRICT TOPOLOGIES ON $C_B(X)$ ARE SEPARABLE.

AMS(MOS) SUBJECT CLASSIFICATIONS:
PRIMARY: 46E10, 28A10
SECONDARY: 54D20

KEY WORDS AND PHRASES: STRICT TOPOLOGY, SEPARABLE SPACE, τ-ADDITIVE MEASURE, σ-ADDITIVE MEASURE, GROTHENDIECK MEASURE, PERFECT MEASURE.

THE SEPARABILITY OF THE STRICT TOPOLOGIES ON $C_B(X)$

BY

JORGE VIELMA

ABSTRACT. The concept of a V-separably submetrizable space is introduced and it is proved that for such spaces all strict topologies on $C_b(X)$ are separable.

INTRODUCTION. All topological spaces consider here are assume to be completely regular Hausdorff. If X is such a space, then $C_b(X)$ denotes the space of all bounded real-valued contінous functions on X. Sentilles, [5], consider locally convex topologies β_o, β_τ and β_σ on $C_b(X)$ which yield the space $M_t(X)$, $M_\tau(X)$ and $M_\sigma(X)$ of tight, τ-additive and σ-additive Baire measures as duals. Koumoulllis, [3], introduces the topology β_p on $C_b(X)$ which yields the space $M_p(X)$ of Baire perfect measures as dual, and Wheeler, [7] , discusses the topology β_g on $C_b(X)$ which yields the space $M_g(X)$ of Grothendieck measures as dual. Summers, [6], proved that β_o is separable if and only if X is a separable submetrizable space.

Let us use β_z as a generic symbol for β_p, β_g, β_τ and β_σ.

If $f: X \to Y$ is continous, then the map $T_f: (C_b(Y), \beta_z) \to (C_b(X), \beta_z)$ defined as $T_f(g) = g \circ f$ is also continous and its adjoint $T_f^* : (M_z(X), w^*) \to (M_z(Y), w^*)$ is continous in their respective weak*-topologies. A space X is said to be a V-separably submetrizable space if there exist a separable metric space Y and a one-to-one continous function f from X onto Y such that the adjoint map $T_f^*: M_\sigma(X) \to M_\sigma(Y)$ is one-one.It is not difficult to prove that $\mathbb{R}_e$, the real numbers with the lower limit topology is a V-separably submetrizable space.

THEOREM 1. Let X be a V-separably submetrizable space, then $(C_b(X), \beta_\sigma)$ is separable. Conversely if $(C_b(X), \beta_\sigma)$ is separable then X is separably submetrisable.

PROOF. Let Y be a separable metric space and $f: X \to Y$ be a one-to-one continous onto map such that $T_f^* : M_\sigma(X) \to M_\sigma(Y)$ is one-to-one. Since Y is measure compact and since β_τ is separable for separable metric spaces, we have that $(C_b(Y), \sigma(C_b(Y), M_\sigma(Y)))$ is separable. Also the hypothesis implies that $T_f(C_b(Y))$ is a $\sigma(C_b(X), M_\sigma(X))$-dense subset of $C_b(X)$. Therefore $(C_b(X), \beta_\sigma)$ is separable.

For the converse, let us note that $\beta_o \leq \beta_\sigma$ then if $(C_b(X), \beta_\sigma)$ is separable. We have that $(C_b(X), \beta_o)$ is separable, which implies that X is separably submetrizable, [6].

COROLARY 2. If X is a V-separably submetrisable space, then $(C_b(X), \beta_z)$ is separable.

THEOREM 3. Let X be a metric space. Then $(C_b(X), \beta_p)$ is separable if and only if card $(X) \leq C$.

PROOF. If $(C_b(X),\beta_p)$ is separable, then X is separably submetrizable, then card$(X) \leq C$.

Suppose card$(X) \leq C$, then there is a countable subset $\{f_n\}$ of $C_b(X)$ which separates the points of X. Now define $F: X \to \mathbb{R}^N$ by $F(x) = (f_1(x), f_2(x),\ldots,f_n(x),\ldots)$ then F is a continous, one-to-one map from X onto a separable metric subspace of $\mathbb{R}^{\mathbb{N}}$. Then X is separably submetrizable which also implies that X is real compact [2]. Then $M_p(X) = M_t(X)$, [4]. Now since $(C_b(X),\beta_o)$ is separable we conclude that $(C_b(X),\beta_p)$ is also separable.

Universidad de los Andes
Facultad de Ciencias
Departamento de Matemática
Mérida - Venezuela