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SOME REMARKS ON NONLINEAR INTERPOLATION

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Dedicated to Professor Władysław Orlicz
on the occasion of his 85th birthday

ABSTRACT. A simple proof of a more general version of the Orlicz interpolation theorem for Lipschitz operators is given along with some applications.

1. INTRODUCTION

Let (Ω,Σ,μ) be a σ-finite measure space and let $\phi: [0,\infty) \to [0,\infty)$ be an Orlicz function, i.e., a nondecreasing continuous convex function such that $\phi(0)=0$.

If $S(\mu)$ is the space of all real-valued measurable functions (two μ-almost everywhere equal functions are the same), then the functional $I_\phi: S(\mu) \to [0,\infty]$ defined by

$$I_\phi(x) = \int_\Omega \phi(|x(t)|)d\mu$$

is a modular on $S(\mu)$. The Orlicz space $L_\phi = L_\phi(\mu) = L_\phi(\Omega)$ is the space of all $x \in S(\mu)$ for which $I_\phi(rx) < \infty$ for some $r>0$, dependent on x.

The functional

$$||x||_\phi = \inf\{r>0 : I_\phi(x/r) \leq 1\}$$

defined on the whole of $S(\mu)$ is a norm on L_ϕ. The Orlicz class $L_\phi^0 = L_\phi^0(\mu)$ is the set of all $x \in S(\mu)$ for which $I_\phi(x) < \infty$. This set is, in general, only convex. The space $L_\phi^a = L_\phi^a(\mu)$ is the space of all $x \in S(\mu)$ for which $I_\phi(rx) < \infty$ for any $r>0$. This is a closed subspace in L_ϕ and the norm $||\cdot||_\phi$ on L_ϕ^a is continuous, i.e., if $x \in L_\phi^a$ and $|x| \geq x_n \downarrow 0$ then $||x_n||_\phi \to 0$. In the case when $\phi(u) = u^p$ we write, for brevity, the letter p instead of ϕ.

The first interpolation theorem considering Orlicz space (not only L_p space) as intermediate is due to Orlicz. He proved in [13] that any separable Orlicz space $L_\phi(a,b)$ is an interpolation space between $L_1(a,b)$ and $L_\infty(a,b)$ for linear operators. Next, in [14] he proved that any Orlicz space $L_\phi(a,b)$ is an interpolation space between $L_1(a,b)$ and $L_\infty(a,b)$, even for Lipschitz operators. In the present paper we give a generalization of the theorem of Orlicz with a simple proof and with some applications to inequalities related to rearrangement functions.

The nonincreasing left-continuous rearrangement of $x \in S(\mu)$ is the function $x^* = x^*_\mu : (0,\infty) \to [0,\infty]$ defined by

$$x^*_\mu(t) = \inf\{\lambda>0 : d_x(\lambda) < t\} ,$$

where $d_x(\lambda) = \mu(\{t \in \Omega : |x(t)| > \lambda\})$ and $\inf \phi = \infty$. The collection of all $x \in S(\mu)$ for which $d_x(\lambda) \not\equiv \infty$ will be denoted by $S_0(\mu)$. For every $x \in S_0(\mu)$ we have $d_x(\lambda) \to 0$ as $\lambda \to \infty$ and so $x^*(t)$ is finite for any $t > 0$.

The rearrangement has the following properties (cf. [7]):

(i) $0 \le x_n \uparrow x$ implies $x_n^*(t) \uparrow x^*(t)$ for all $t > 0$.

(ii) $0 \le x \le y$ implies $x^*(t) \le y^*(t)$ for all $t > 0$.

(iii) If m denotes the Lebesgue measure, then

$$\mu(\{t \in \Omega : |x(t)| > \lambda\}) = m(\{s>0 : x^*(s) > \lambda\}) \quad \text{for all } \lambda>0,$$

and we say that x and x^* are equimeasurable even though they are defined on different measure spaces. Moreover,

$$\int_\Omega |x|\,d\mu = \int_0^\infty x^*\,dm.$$

(iv) For functions x and y in $S(\mu)$ we have

$$\int_0^t (x+y)^*\,dm \le \int_0^t x^*\,dm + \int_0^t y^*\,dm \quad \text{for all } t > 0.$$

We introduce the <u>α-truncation</u> $(\alpha>0)$ of function x defined on Ω by

$$x^{(\alpha)}(t) = \min(|x(t)|,\alpha)\ \mathrm{sgn}\ x(t).$$

Let us note that for functions x and y defined on Ω we have

(1) $|x^{(\alpha)}(t) - y^{(\alpha)}(t)| \leq |x(t) - y(t)|$ for all $t\in\Omega$.

The paper is divided into four sections. In §2 we give two proofs of the interpolation theorem for almost Lipschitz mappings. The main theorems are proved in §3, including the proof of Orlicz theorem. Applications of these results to averaging operator, Jensen inequality, Hardy-Littlewood Pólya theorem, Lorentz-Shimogaki inequality and Brudnyĭ result about modulus of continuity of x and x^* are considered in section 4.

2. INTERPOLATION OF SEMI-LIPSCHITZ OPERATORS

In the proof of the first theorem we will need the following lemma about representation of Orlicz functions.

<u>Lemma 1.</u> Every Orlicz function ϕ has a representation

(2) $$\phi(u) = au + \int_0^\infty (u-s)_+\, dp(s),$$

where p is a nondecreasing nonnegative right-continuous function on $[0,\infty)$ and $a=p(0^+)$

<u>Proof</u>. It is well known that every Orlicz function ϕ can be represented in the form $\phi(u) = \int_0^u p(s)ds$, where p is the right-derivative of ϕ. By integration by parts we get

$$\int_0^u p(s)ds = up(u) - \int_0^u sdp(s) = up(0^+) + \int_0^u u\, dp(s) - \int_0^u sdp(s)$$

$$= up(0^+) + \int_0^u (u-s)dp(s) = ua + \int_0^\infty (u-s)_+dp(s).$$

Let us now explain with some examples how to interprete (2) with dp(s) as the measure:

1° If $p = \phi'$ is absolutely continuous then we have according to Lemma 1 that $\phi(u) = au + \int_0^\infty (u-s)_+ \phi''(s)ds$. Notice that in [6] it is assumed the existence of ϕ'' and that it is locally integrable.

2° If $\phi(u) = 0$ for $0 \leq u \leq 1$ and $u-1$ for $u>1$, then $p(s) = 0$ for $0 \leq s < 1$ and 1 for $s \geq 1$, and for $u>1$ we have

$$\int_0^u (u-s)dp(s) = u[p(0^+)-p(0)] + (u-1)[p(1^+)-p(1^-)] = u-1;$$

3°. If $\phi(u) = u$ for $0 \leq u \leq 1$ and u^2 for $u>1$, then $p(s)=1$ for $0 \leq s < 1$ and $2s$ for $s \geq 1$, and for $u>1$ we have

$$\int_0^u (u-s)dp(s) = u[p(0^+)-p(0)] + (u-1)[p(1^+)-p(1^-)] + \int_1^u (u-s)2ds$$
$$= u-1+2u^2 - u^2 - 2u+1 = u^2 - u.$$

In the sequel the space (Ω', Σ', ν) will be a σ-finite measure space

Theorem 1. Let $T: L_1(\mu) + L_\infty(\mu) \to L_1(\nu) + L_\infty(\nu)$ be an operator such that $T0=0$ and

(3) $$||Tx-Ty||_1 \leq M||x-y||_1 \quad \forall x,y \in L_1(\mu),$$

(4) $$||Tx||_\infty \leq M||x||_\infty \quad \forall x \in L_\infty(\mu).$$

Then

(5) $$I_\phi(Tx/M) \leq I_\phi(x) \quad \forall x \in L_\phi^0(\mu) \cap L_1(\mu)$$

and

(6) $$||Tx||_\phi \leq M||x||_\phi \quad \forall x \in L_\phi(\mu) \cap L_1(\mu).$$

First proof. By taking T/M instead of T, if necessary, we may assume that $M=1$.

First, we prove that if (4) holds and M is 1 then for each $x \in L_1(\mu)+L_\infty(\mu)$

(7) $$|Tx(t) - (Tx)^{(\alpha)}(t)| \leq |Tx(t)-T(x^{(\alpha)})(t)| \quad \nu\text{-a.e.}.$$

In fact, if $|Tx(t)| \leq \alpha$ then (7) is obvious; if on the other hand $|Tx(t)| > \alpha$ then since $||T(x^{(\alpha)})||_\infty \leq ||x^{(\alpha)}||_\infty \leq \alpha$, it follows that $|T(x^{(\alpha)})(t)| \leq \alpha$ ν-a.e.. Hence

$$|Tx(t) - (Tx)^{(\alpha)}(t)| = |Tx(t) - \alpha \operatorname{sgn} Tx(t)| = |Tx(t)| - \alpha$$

$$\leq |Tx(t)| - |T(x^{(\alpha)})(t)| \leq |Tx(t) - T(x^{(\alpha)})(t)| \ \nu\text{-a.e.} .$$

Now, if $x \in L^o_\phi(\mu) \cap L_1(\mu)$ then from the representation (2) of ϕ and the Fubini theorem

$$I_\phi(Tx) = \int_{\Omega'} \phi(|Tx(t)|)d\nu$$

$$= \int_{\Omega'} \Big[a|Tx(t)| + \int_0^\infty (|Tx(t)| - s)_+ \, dp(s)\Big] d\nu$$

$$= a||Tx||_1 + \int_0^\infty \int_{\Omega'} (|Tx(t)| - s)_+ \, d\nu dp(s)$$

$$= a||Tx||_1 + \int_0^\infty \int_{\Omega'} |Tx(t) - (Tx)^{(s)}(t)| \, d\nu dp(s).$$

Using property (7) of T and the assumption (3) we have

$$I_\phi(Tx) \leq a \, ||Tx||_1 + \int_0^\infty ||Tx - T(x^{(s)})||_1 \, dp(s)$$

$$\leq a||x||_1 + \int_0^\infty ||x - x^{(s)}||_1 \, dp(s)$$

$$= a||x||_1 + \int_0^\infty \int_\Omega (|x(t)| - s)_+ d\mu dp(s).$$

Again, from the Fubini theorem and representation (2) of ϕ

$$I_\phi(Tx) \leq \int_\Omega \Big[a|x(t)| + \int_0^\infty (|x(t)| - s)_+ dp(s)\Big] d\mu$$

$$= \int_\Omega \phi(|x(t)|)d\mu = I_\phi(x).$$

Hence $Tx \in L^o_\phi(\nu)$ and $I_\phi(Tx) \leq I_\phi(x)$.

The second part of proof follows inmediately from the above and the definitions of L_ϕ space and $||\cdot||_\phi$ norm.

<u>Second proof</u> (when $\nu(\Omega')\leq\mu\Omega$ and modulo some facts about rearrangement). As in the first proof, let M=1. Given u>0, let for $x\in L_1(\mu)$ be $\alpha=x^*(u)$ and $x_1=x^{(\alpha)}$. Then by properties (iv) and (iii) of rearrangement

$$\int_0^u (Tx)_\nu^*(t)dt \leq \int_0^u (Tx-Tx_1)_\nu^*(t)dt + \int_0^u (Tx_1)_\nu^*(t)dt$$

$$\leq \int_0^\infty (Tx-Tx_1)_\nu^*(t)dt + u||Tx_1||_\infty$$

$$\int_{\Omega'} |Tx(t)-Tx_1(t)|d\nu + u||Tx_1||_\infty.$$

Using assumptions (3) and (4) on T, property (iii) again and the fact that $d_x(\alpha)\leq u$ we get

$$\int_0^u (Tx)_\nu^*(t)dt \leq \int_\Omega |x(t)-x_1(t)|d\mu+u||x_1||_\infty$$

$$= \int_\Omega (|x(t)|-\alpha)_+ d\mu+u\alpha \leq \int_0^{d_x(\alpha)} (x_\mu^*(t)-\alpha)dt + u\alpha$$

$$\leq \int_0^u (x_\mu^*(t)-\alpha)dt+u\alpha = \int_0^u x_\mu^*(t)dt.$$

Hence, if $x\in L_1(\mu)$ then

$$(8) \qquad \int_0^u (Tx)_\nu^*(t)dt \leq \int_0^u x_\mu^*(t)dt \quad \text{for all } 0\leq u\leq\mu\Omega.$$

Now, we prove that if (8) holds and $\nu(\Omega')\leq\mu\Omega$ then for $x\in L_\phi^0(\mu)\cap L_1(\mu)$ we have

$$\int_{\Omega'} \phi(|Tx(t)|)d\nu \leq \int_\Omega \phi(|x(t)|)d\mu.$$

In fact, let $A_s=\{t>0 : x_\mu^*(t)>s\}$, $B_s=\{t>0 : (Tx)_\nu^*(t)>s\}$ and $a_s=mA_s$, $b_s=mB_s$. Then from property (iii) of rearrangement, representation (2) of ϕ and the Fubini theorem we have that

$$I_\phi(Tx) = \int_{\Omega'} \phi(|Tx(t)|)d\nu = \int_0^{\nu(\Omega')} \phi((Tx)_\nu^*(t))dt$$

$$= a \int_0^{\nu(\Omega')} (Tx)_\nu^*(t)dt + \int_0^\infty \int_0^{\nu(\Omega')} [(Tx)_\nu^*(t)-s]_+ \, dtdp(s)$$

$$\leq a \int_0^{\nu(\Omega')} (Tx)_\nu^*(t)dt + \int_0^\infty \int_0^{b_s} [(Tx)_\nu^*(t)-s] \, dtdp(s),$$

and from the assumption (8),

$$\leq a \int_0^{\mu\Omega} x_\mu^*(t)dt + \int_0^\infty (\int_0^{b_s} x_\mu^*(t)dt - sb_s)dp(s).$$

But now, if $b_s \leq a_s$ then

$$\int_0^{b_s} x^*(t)dt - sb_s = \int_0^{a_s} x^*(t)dt - \int_{b_s}^{a_s} x^*(t)dt - sb_s$$

$$\leq \int_0^{a_s} x^*(t)dt - x^*(a_s)(a_s-b_s) - sb_s$$

$$\leq \int_0^{a_s} x^*(t)dt - s(a_s-b_s) - sb_s = \int_0^{a_s} x^*(t)dt - sa_s,$$

and if $b_s > a_s$ then

$$\int_0^{b_s} x^*(t)dt - sb_s = \int_0^{a_s} x^*(t)dt + \int_{a_s}^{b_s} x^*(t)dt - sb_s,$$

and $t > a_s$ implies $x^*(t) \leq s$,

$$\leq \int_0^{a_s} x^*(t)dt + s(b_s-a_s) - sb_s = \int_0^{a_s} x^*(t)dt - sa_s.$$

Hence, from the above, Fubini theorem, representation (2) of ϕ and property (iii) we get

$$I_\phi(Tx) \leq a \int_0^{\mu\Omega} x^*(t)dt + \int_0^\infty (\int_0^{a_s} x^*(t)dt - sa_s)dp(s)$$

$$= a \int_0^{\mu\Omega} x^*(t)dt + \int_0^\infty \int_0^{\mu\Omega} [x^*(t)-s]_+ \, dtdp(s)$$

$$= \int_0^{\mu\Omega} \phi(x^*(t))dt = \int_\Omega \phi(|x(t)|)d\mu = I_\phi(x).$$

In the case when T is a linear operator and $\phi(u) = u^p$ we have a simple proof of the particular case of M. Riesz interpolation theorem, i.e., for $p_0=1$, $p_1=\infty$ and $1<p<\infty$ (cf. also [6], where it is assumed that T is also positive). In the nonlinear case with $\phi(u) = u^p$ the above theorem follows also from Lions [9] and Peetre [15]. Brezis and Strauss [1] used in the proof of Theorem 1 the additional condition of T being positive . Moreover, their theorem is for convex lower semi-continuous function ϕ on $\mathbb{R}$ such that $\min\phi=\phi(0)=0$. In [11] it is proved that if both measure spaces are the same and either nonatomic or counting then (3) and (4) with M=1 imply (8). Then the Calderón-Mitjagin theorem (see [7], p. 105) and the fact that Orlicz space $L_\phi(\mu) \cap L_1(\mu)$ has the Fatou property imply (6).

Corollary 1. Let $T: L_1+L_\infty \to L_1+L_\infty$ be a linear operator which is bounded in L_1 and L_∞. If either $L_\phi \subset (L_1+L_\infty)^a$ or $L_\phi = L_{\phi_0}+L_\infty$ hold with $L_{\phi_0} \subset (L_1+L_\infty)^a$, then T is bounded in L_ϕ.

Proof. For any $x \in L_\phi$ the sequences $x_n = x1_{A_n}$ is in $L_\phi \cap L_1$, where $A_n \nearrow \Omega$ and $\mu A_n < \infty$ (such a sequences of sets exists because μ is σ-finite). If $L_\phi \subset (L_1+L_\infty)^a$ then $||x-x_n||_{L_1+L_\infty} \to 0$ and the boundedness of T in L_1+L_∞ implies $||Tx-Tx_n||_{L_1+L_\infty} \to 0$.

Thus $Tx_{n_k} \to Tx$ μ-a.e. By Fatou Lemma and Theorem 1 we have

$$||Tx||_\phi \leq \liminf_{k\to\infty} ||Tx_{n_k}||_\phi \leq M \liminf_{k\to\infty} ||x_{n_k}||_\phi \leq M||x||_\phi .$$

If $L_\phi = L_{\phi_0}+L_\infty$ with $L_{\phi_0} \subset (L_1+L_\infty)^a$ then from the above T is bounded in L_{ϕ_0} . Therefore T, as linear operator, is bounded in $L_{\phi_0}+L_\infty = L_\phi$.

It is natural to ask: is it true that if L_ϕ is an Orlicz space then either $L_\phi \subset (L_1+L_\infty)^a$ or $L_\phi = L_{\phi_0}+L_\infty$ with $L_{\phi_0} \subset (L_1+L_\infty)^a$?. We remark that if $\mu\Omega<\infty$ then $L_\phi \subset L_1 = L_1^a$ and if μ is a counting measure then either $\ell_\phi \subset \ell_\infty^a = c_0$ or $\ell_\phi = \ell_\infty = \ell_1 + \ell_\infty$ with $\ell_1 \subset c_0$.

3. MAIN THEOREMS

Using Theorem 1 and considerations from Orlicz paper [14] we prove a more general version of Orlicz theorem about interpolation of Lipschitz operators in L_1 and L_∞.

Theorem 2. Let $T: L_1(\mu)+L_\infty(\mu) \to L_1(\nu)+L_\infty(\nu)$ be an operator such that

(3) $$||Tx-Ty||_1 \leq M||x-y||_1 \qquad \forall\, x,y \in L_1(\mu),$$

(4') $$||Tx-Ty||_\infty < M||x-y||_\infty \qquad \forall\, x,y \in L_\infty(\mu).$$

Then

(6') $$||Tx-Ty||_\phi \leq M||x-y||_\phi \qquad \forall\, x,y \in L_\phi(\mu) \cap L_1(\mu).$$

Proof. For any fixed $x_0 \in L_1(\mu) \cap L_\infty(\mu)$ and for $x \in L_1(\mu)+L_\infty(\mu)$ let

$$T_1x := T(x+x_0) - Tx_0.$$

Then $T_1 0 = 0$ and

$$||T_1x-T_1y||_1 = ||T(x+x_0)-T(y+x_0)||_1 \leq M||x-y||_1 \qquad \forall x,y \in L_1(\mu),$$

$$||T_1x||_\infty = ||T(x+x_0)-Tx_0||_\infty \leq M||x||_\infty \qquad \forall x \in L_\infty(\mu).$$

From Theorem 1 we get

$$||T_1x||_\phi \leq M||x||_\phi \qquad \forall\, x \in L_\phi(\mu) \cap L_1(\mu)$$

this means that

$$||T(x+x_0)-Tx_0||_\phi \leq M||x||_\phi \qquad \forall\, x \in L_\phi(\mu) \cap L_1(\mu)$$

or

(9) $$||Tx-Tx_0||_\phi \leq M||x-x_0||_\phi \quad \forall\, x \in L_\phi(\mu) \cap L_1(\mu),\ x_0 \in L_1(\mu) \cap L_\infty(\mu).$$

For arbitrary $x,y \in L_\phi(\mu) \cap L_1(\mu)$ we consider the truncations $x^{(k)}, y^{(k)}$. Then $z_k := T(x^{(k)})-T(y^{(k)})$ converges to $Tx-Ty$ in the $L_1(\nu)$-norm, because (3) and $||\ \ ||_1$-norm is continuous.

Consequently, the same convergence holds in the measure ν.

Therefore, for a property chosen sequence k_n, the sequence z_{k_n} converge ν-a.e. to Tx-Ty. Then $x_n := x^{(k_n)}$ and $y_n := y^{(k_n)}$ have the following property

(10) $x_n, y_n \in L_1(\mu) \cap L_\infty(\mu)$, $|x_n - y_n| \leq |x-y|$ μ-a.e. and $Tx_n - Ty_n \to Tx - Ty$ ν-a.e.

Now, by (10), Fatou property of the norm and (9) we get

$$||Tx-Ty||_\phi \leq \liminf_{n\to\infty} ||Tx_n - Ty_n||_\phi$$

$$\leq M\liminf_{n\to\infty} ||x_n - y_n||_\phi \leq M||x-y||_\phi.$$

Corollary 2. (Orlicz theorem). If $T: L_1(0,1) \to L_1(0,1)$ is a Lipschitz operator in $L_1(0,1)$ and $L_\infty(0,1)$ the T is also Lipschitz in $L_\phi(0,1)$.

Corollary 3. If the operator $T: L_1(\mu) \cap L_\infty(\mu) \to L_1(\nu) \cap L_\infty(\nu)$ satisfy (3) and (4') for $x,y \in L_1(\mu) \cap L_\infty(\mu)$ then (6') holds for $x,y \in L_1(\mu) \cap L_\infty(\mu)$.

Corollary 3 with additional assumption that $L_1(\mu) \cap L_\infty(\mu)$ is dense in both $L_\phi(\mu)$ and $L_\infty(\mu)$ (this means that $\mu\Omega<\infty$ and $L_\phi = L_\phi^a$) is a particular case of a general theorem of Browder [2].

Remark 1. If $\mu\Omega=\infty$ and for arbitrary $x,y \in L_\phi(\mu)$, it is possible to construct sequences x_n, y_n with properties (10) then it is easy to see that (6') holds even for $x,y \in L_\phi(\mu)$. On the other hand, if $\mu\Omega=\infty$ and $L_\phi(\mu) \cap L_1(\mu)$ is dense in $L_\phi(\mu)$ then, by continuity, (6') holds also for $x,y \in L_\phi(\mu)$.
We prove now that density of $L_\phi \cap L_1$ in L_ϕ is equivalent to condition δ_2 for small u of ϕ.

Proposition 1. Let μ be a σ-finite measure and $\mu\Omega=\infty$. Assume that Ω contains a nonatomic part of infinite measure or there are atoms $\{A_n\}_{n=1}^\infty$ such that $0<\inf_n \mu A_n \leq \sup_n \mu A_n < \infty$. Then the following conditions are equivalent:

(a) $L_\phi(\mu) \cap L_1(\mu)$ is dense in $L_\phi(\mu)$,

(b) $L_\phi(\mu) \cap L_\infty(\mu) = L_\phi^a(\mu) \cap L_\infty(\mu)$,

(c) ϕ satisfies condition δ_2 : $\limsup_{u\to 0^+} \frac{\phi(2u)}{\phi(u)} < \infty$.

Proof. Let us prove that (a) implies (b).

If $0\le x\in L_\phi \cap L_\infty$ then from the assumption there is a sequence $0\le x_n\in L_\phi \cap L_1$ such that $||x-x_n||_\phi \to 0$. Let $y_n = \min(x_n, ||x||_\infty)$. Then $y_n \in L_1 \cap L_\infty$ and

$$|x-y_n| = |\min(x,||x||_\infty) - \min(x_n,||x||_\infty)| \le |x-x_n| \ \mu\text{-a.e.},$$

i.e., $||x-y_n||_\phi \to 0$. If we prove that $y_n \in L_\phi^a$ then $x\in L_\phi^a$ (because L_ϕ^a is closed in L_ϕ) and so $x\in L_\phi^a \cap L_\infty$.

For any fixed $r>0$ let $c = \phi(r||y_n||_\infty)/(r||y_n||_\infty) \le \phi(r||x||_\infty)/(r||x||_\infty) < \infty$. Then $I_\phi(ry_n) \le cr\int_\Omega y_n(t)d\mu<\infty$ and so $y_n\in L_\phi^a$.

Now, let $x\in L_\phi$ and $r>0$ be such that $I_\phi(rx) < \infty$.

If $A_n = \{t\in\Omega : r|x(t)| > n^{-1}\}$ then

$$\phi(n^{-1})\mu A_n = \int_{A_n} \phi(n^{-1})d\mu \le \int_{A_n} \phi(r|x(t)|)d\mu \le I_\phi(rx) < \infty$$

for any natural number n. Hence $\mu A_n < \infty$ and so $x1_{A_n} \in L_\phi \cap L_1$.

On the other hand, there is a sequence (s_n) of simple functions such that $s_n \to x$. Of course, $s_n \in L_\phi \cap L_1$. Then putting

$$x_n = s_n 1_{\Omega\setminus A_n} + x1_{A_n}$$

We have that $x_n \in L_\phi \cap L_1$. Moreover, $x1_{\Omega\setminus A_n} - s_n 1_{\Omega\setminus A_n} \downarrow 0$, $x-x1_{A_n} = x1_{\Omega\setminus A_n} \downarrow 0$ and $x1_{\Omega\setminus A_n} \in L_\phi \cap L_\infty = L_\phi^a \cap L_\infty$. Therefore

$$||x-x_n||_\phi \le ||(x-s_n)1_{\Omega\setminus A_n}||_\phi + ||x-x1_{A_n}||_\phi \to 0$$

as $n\to\infty$. This proves that (b) implies (a).

Let us prove the equivalence of (b) and (c).

Assume that $\limsup_{u\to 0^+} \frac{\phi(2n)}{\phi(u)} = \infty$. Let (x_n) $\ell_\phi \setminus \ell_\phi^a$. If μ is nonatomic then there are pairwise disjoint sets B_n such that $\mu B_n=1$ for each natural number n. Let

$$x = \sum_{n=1}^{\infty} x_n 1_{B_n} \text{ (convergence in } \mu).$$

Then $x \in L_\phi \cap L_\infty \setminus (L_\phi^a \cap L_\infty)$. In the second case, let

$$x = \sum_{n=1}^{\infty} x_n 1_{A_n} \text{ (convergence in } \mu),$$

then also $x \in L_\phi \cap L_\infty \setminus (L_\phi^a \cap L_\infty)$. That is, (b) implies (c). The last implication which we should prove is that (c) implies (b). Let $x \in L_\phi \cap L_\infty$ and $I_\phi(rx) < \infty$ for some $r>0$. Put

$$c_n = \sup \left\{ \frac{\phi(2u)}{\phi(u)} : 0 \le u \le 2^{n-1} r ||x||_\infty \right\}.$$

Then $c_n < \infty$ (because ϕ satisfies δ_2 condition) and

$$I_\phi(2^n rx) \le c_n I_\phi(2^{n-1} rx) \le \ldots \le \prod_{k=1}^{n} c_k I_\phi(rx) < \infty,$$

i.e., $rx \in L_\phi^a$ and so $x \in L_\phi^a \cap L_\infty$.

It may not be simple to check directly whether an operator is nonexpansive in L_ϕ. Then the following result may be useful:

<u>Corollary 4.</u> If $T: L_1(\mu)+L_\infty(\mu) \to L_1(\mu)+L_\infty(\mu)$ is nonexpansive in $L_1(\mu)$ and $L_\infty(\mu)$, then T is nonexpansive in $L_\phi(\mu)$ provided that $\mu\Omega<\infty$ or μ is such as in Proposition 1 and ϕ satisfies condition δ_2 .

<u>Remark 2</u>. If an operator T is such as in Theorem 2 then, with the same proof, $I_\phi(\frac{Tx-Ty}{M}) \le I_\phi(x-y)$ for $x,y \in L_1(\mu)$ such that $x-y \in L_\phi^o(\mu)$. Notice that in [1], this is proved with the additional assumption that T be positive. Moreover, if for the space X the inclusions $L_1(\mu)+L_\infty(\mu) \subset X \subset S(\mu)$ hold, and for any $x,y \in X$ it is possible to construct sequences x_n, y_n with property (10) then

(11) $$I_\phi\left(\frac{Tx-Ty}{M}\right) \leq I_\phi(x-y) \quad \text{for } x,y\in X.$$

If we assume a little more about operator than in Theorem 1 or Theorem 2 then the proof will be simpler.

<u>Theorem 3</u>. Let $L_1(\mu)+L_\infty(\mu)\subset X\subset S(\mu)$ and let $T:X\to S(\nu)$ be a positive monotonic C-sublinear operator, i.e., for any $x,y\in X$

(12) $$\begin{cases} x\geq 0 \Longrightarrow Tx\geq 0 \\ 0\leq x\leq y \Longrightarrow Tx\leq Ty \\ |T(\lambda x)| = |\lambda|\ |Tx| \\ |T(x+y)| \leq C(|Tx|+|Ty|) \end{cases}$$

and

(4) $$||Tx||_\infty \leq M||x||_\infty \quad \forall\ x\in L_\infty(\mu).$$

Then

(13) $$\phi(T(|x|)) \leq M^{-1}T(\phi(CM|x|)) \quad \nu\text{-a.e.}$$

In particular, if we assume further that

(3') $$||Tx||_1 \leq M||x||_1 \quad \forall x\in L_1(\mu),$$

then

(5') $$I_\phi(Tx/(CM)) \leq I_\phi(x) \quad \forall x\in L^0_\phi(\mu),$$

and

(6) $$||Tx||_\phi \leq CM||x||_\phi \quad \forall x\in L_\phi(\mu).$$

<u>Proof</u>. Let $\phi^*(v) = \sup_{u>0}(uv-\phi(u))$ and $\phi^{-1}(v) = \inf\{u>0:\phi(u)>v\}$. then $\phi^{**}=\phi$, $\phi(\phi^{-1}(u)) \leq u$ and by Young inequality

$$u\phi^{*-1}(v) \leq \phi(u)+\phi^*(\phi^{*-1}(v)) \leq \phi(u)+v\ .$$

Therefore, for any $v>0$,

$$|x| \leq \frac{\phi(|x|)+v}{\phi^{*-1}(v)} \quad \mu\text{-a.e.}$$

and from (12)

$$T(|x|) \leq C\ \frac{T\phi(|x|)+vT1}{\phi^{*-1}(v)} \qquad \nu\text{-a.e.}$$

Then from (4) we get

$$T(|x|) \leq C\ \frac{T\phi(|x|)+Mv}{\phi^{*-1}(v)} \qquad \nu\text{-a.e.}$$

Let us note that $\phi^{*-1}(u) = \inf_{v>0} \frac{u+v}{\phi^{-1}(v)}$; if ϕ' is an increasing function then this follows from the fact that for $v = \phi'(u)$ we have $u+v = \phi^{-1}(u)\ \phi^{*-1}(v)$ - equality in the Young theorem. For the general case- see [12], Lemma 2.

Thus

$$T(|x|) \leq CM\phi^{**-1}(T\phi(|x|)/M) = CM\phi^{-1}(T\phi(|x|)/M)$$

and (13) holds. The proof of the next part follows inmediately from (13) and the assumption (3').

Corollary 5. Positive and monotonic sublinear L_1-L_∞ contractions are also contractions in Orlicz spaces L_ϕ. This corollary holds also for positive and linear contractions. We wish to point out that this result generalizes the corresponding result in [8] for linear positive contractions on L_p spaces.

4. APPLICATIONS

We now consider some applications of the results of the last section.

a) Averaging operator and Jensen inequality

Let $A = (A_n)$ be a finite or countable disjoint collection of a measurable sets of Ω with $0 < A_n < \infty$. Define the averaging operator $P_A : L_1(\mu) + L_\infty(\mu) \to L_1(\mu) + L_\infty(\mu)$ by

$$P_A x(t) = \sum_n \left(\frac{1}{\mu A_n} \int_{A_n} x(s)d\mu\right) 1_{A_n}(t).$$

Then P_A is a linear positive operator bounded in $L_1(\mu)$ and in $L_\infty(\mu)$ with the respective norms equal to one. Hence from Theorem 3

$$\phi(P_A(|x|)) \leq P_A(\phi(|x|)) \quad \mu\text{-a.e.}$$

and, in particular, we have the following:

Theorem A (Jensen inequality). If $0<\mu A<\infty$ then

$$\phi\left(\frac{1}{\mu A}\int_A |x|\,d\mu\right) \leq \frac{1}{\mu A}\int_A \phi(|x|)\,d\mu .$$

Moreover, P_A is bounded in $L_\phi(\mu)$ and has norm equal to one.

b) Inequalities with rearrangement function

Let $T : S_0(\mu) \to S_0(0,\infty)$ be defined by $Tx=x^*_\mu$. Then from property (iii) of rearrangement $\int_0^\infty Tx(t)dt = \int_\Omega |x|d\mu$. We prove now that T is a nonexpansive map from $L_1(\mu)$ into $L_1(0,\infty)$ and from $L_\infty(\mu)$ into $L_\infty(0,\infty)$.

If $x,y\in L_1(\mu)$ then

$$||Tx-Ty||_1 = \int_0^\infty (Tx-Ty)_+\,dt + \int_0^\infty (Ty-Tx)_+dt$$

$$\leq \int_0^\infty [T(\max(|x|,\ |y|)) - Ty]\,dt + \int_0^\infty [T(\max(|x|,\ |y|) - Tx]\,dt$$

$$= \int_\Omega [\max(|x|,\ |y|) - |y|\,]\,d\mu + \int_\Omega [\max(|x|,\ |y|) - |x|\,]\,d\mu$$

$$= \int_\Omega \big||x| - |y|\big|d\mu \ \leq ||x-y||_1 .$$

If $x,y\in L_\infty(\mu)$ then $|x| \leq ||x-y||_\infty + |y|$, $|y| \leq ||x-y||_\infty + |x|$; thus $x^* \leq ||x-y||_\infty + y^*$ and $y^* \leq ||x-y||_\infty + x^*$. Hence

$$||Tx-Ty||_\infty = ||x^*-y^*||_\infty \leq ||x-y||_\infty .$$

Let us note that for $x,y \in S_0(\mu)$ it is possible to construct sequences x_n, y_n with property (10). In fact, let $x_n = x^{(n)} 1_{A_n}$ and $y_n = y^{(n)} 1_{A_n}$, where $A_n \nearrow \Omega$ and $\mu A_n < \infty$ (such a sequence of sets exists because μ is σ finite). Then $x_n, y_n \in L_1(\mu) \cap L_\infty(\mu)$ and from property (1) of truncation and property (i) of rearrangement we get $|x_n - y_n| \le |x^{(n)} - y^{(n)}| \le |x-y|$ and $x_n^* - y_n^* \longrightarrow x^* - y^*$ a.e..

From the above and remark 2 we have

<u>Theorem B</u>. For each Orlicz function ϕ and for any $x,y \in S_0(\mu)$

$$I_\phi(x^* - y^*) \le I_\phi(x-y).$$

This inequality is proved in [3] for the case $L_p(0,1)$ and in [4] for the case when $\Omega = \mathbb{R}^n$ with Lebesgue measure. Let us note that in [4] this inequality is assumed to hold for $x,y \in S(\mathbb{R}^n)$. This is a misunderstanding because if $x,y \in S \setminus S_0$, then $x^* = y^* = \infty$ and the left side of inequality is not defined.

<u>Theorem C</u> (Hardy, Littlewood and Pólya). Let $x,y \in L_1(\mu) + L_\infty(\mu)$. Then $\int_0^u x^*(s)ds \le \int_0^u y^*(s)\,ds$ for any $0 < u < \mu\Omega$ if and only if $I_\phi(x) \le I_\phi(y)$ for any Orlicz function ϕ.

The proof that the inequality with rearrangement functions implies that for I_ϕ is the same as the second proof of the Theorem 1. On the other hand, if $I_\phi(x) \le I_\phi(y)$ for any Orlicz function then $\int_\Omega (|x|-t)_+ d\mu \le \int_\Omega (|y|-t)_+ d\mu$ for any $t > 0$, because $\phi_t(u) = (u-t)_+$ is an Orlicz function.

Let $0 < u < \mu\Omega$ and $t = y^*(u)$. Then

$$\int_0^u [x^*(s)-t]\,ds \le \int_0^{\mu\Omega} [x^*(s)-t]\,ds = \int_0^\infty d_{(x^*-t)_+}(\lambda)d\lambda$$

$$= \int_0^\infty d_{x^*}(t+\lambda)d\lambda = \int_0^\infty d_x(t+\lambda)d\lambda = \int_0^\infty d_{(|x|-t)_+}(\lambda)d\lambda$$

$$= \int_\Omega (|x|-t)_+ d\mu \leq \int_\Omega (|y|-t)_+ d\mu = \int_0^{\mu\Omega} [y^*(s)-t]\, ds$$

$$= \int_0^u [y^*(s)-t]\, ds,$$

and the proof is complete.

From theorems B and C it follows easily the following

Theorem D (Lorentz-Shimogaki inequality [10]). If $x, y \in L_1(\mu) + L_\infty(\mu)$, then for $0<u<\mu\Omega$

$$\int_0^u (x^*-y^*)^*(s)ds \leq \int_0^u (x-y)^*(s)ds.$$

c) Modulus of continuity of equimeasurable functions

Let us now confine our attention to periodic functions on [0,1] with period 1. Given an $x \in L_\phi(0,1)$, the expression

$$\omega_\phi(t,x) = \sup_{0\leq h\leq t} ||x(\cdot+h) - x(\cdot)||_\phi$$

is called the integral modulus of continuity, in L_ϕ, of x. Using Brudnyĭ result in [3] for $L_p(0,1)$ spaces and Orlicz theorem we prove the following inequality

Theorem E (Brudnyĭ inequality). If $x \in L_\phi(0,1)$ then

$$\omega_\phi(t,x^*) \leq 3\omega_\phi(t,x).$$

Proof. Theorem B (or Orlicz theorem, i.e. the special case of our Theorem 2) implies that operator $T: L_1(0,1) \to L_1(0,1)$ defined by $Tx=x^*$ is nonexpansive in $L_\phi(0,1)$.

Pólya - Szegö inequality means that $||(Tx)'||_\phi \leq ||x'||_\phi$ for $x \in \dot{L}_\phi = \{x \in L_\phi : x \in AC,\ x' \in L_\phi\}$.

Now, let us consider, for $x \in L_\phi$ and $t>0$, the Steklov average function

$$x_t(u) = t^{-1} \int_u^{u+t} x(s)ds = t^{-1} \int_0^t x(s+u)ds.$$

Then

$$||x_t - x||_\phi \leq t^{-1} \int_0^t ||x(\cdot + s) - x||_\phi \, ds \leq t^{-1} \int_0^t \omega_\phi(s,x)ds \leq \omega_\phi(t,x),$$

$$x_t \in \overset{\bullet}{L}_\phi \text{ and } ||x_t'||_\phi = t^{-1} \, ||x(\cdot + t) - x||_\phi \leq t^{-1}\omega_\phi(t,x).$$

Hence

$$\omega_\phi(t,Tx) \leq \omega_\phi(t,Tx-Tx_t) + \omega_\phi(t,Tx_t)$$

$$\leq 2||Tx-Tx_t||_\phi + t||(Tx_t)'||_\phi \leq 2||x-x_t||_\phi + t||x_t'||_\phi$$

$$\leq 2\omega_\phi(t,x) + \omega_\phi(t,x) = 3\omega_\phi(t,x).$$

This proves the theorem.

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