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A NOTE ON A PAPER OF McARTHUR

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ABSTRACT

This note fills the gap in the proof of Theorem 1 of the paper "On a theorem of Orlicz Pettis" by McArthur, Pacific J. Math. V. 22 (1967), 297-302.

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In [2], McArthur proved the theorem of Orlicz-Pettis about subseries convergence in locally convex Hausdorff spaces. In the proof of this theorem, the author makes an assertion (result (F)) without sufficient restrictions on the sequence $(f_n)_1^\infty$. The object of this note is to establish (F) when $(f_n)_1^\infty$ is equicontinuous. This additional hypothesis of $(f_n)_1^\infty$ is used in the statement of (D) in [2] and therefore, the rest of the proof of the theorem remains valid.

Following the notations in [2], E is a Hausdorff locally convex space with topology $\mathcal{T}$. E^* denotes the space of all $\mathcal{T}$-continuons linear functionals on E. $\overline{sp}\{x_i\}$ is the closed linear subspace generated by $\{x_i\}_1^\infty$ in $(E,\mathcal{T})$. (ℓ) denotes the sequence space ℓ_1.

In the proof of Theorem 1 of [2], the author proves that $\lim_n \sum \varepsilon_i f_n(x_i) = 0$ (1), for each sequence $(\varepsilon_i)_1^\infty$, where $\varepsilon_i = \pm 1, 0$ and $(f_n)_1^\infty$ is an arbitrary sequence in E^* such that $\lim_n f_n(x) = 0$ for all $x \in \overline{sp}\{x_i\}$. The author then asserts that by following Pettis [3],

$$\lim_n \sum_{i=1}^{\infty} |f_n(x_i)| = 0 \quad (F).$$

Since the argument of Pettis [3] is valid only for Banach spaces, a modification of the argument of Pettis has to be used to establish (F) and this needs the additional hypothesis that $(f_n)_1^\infty$ is equicontinuous.

Let $(f_n)_1^\infty$ be equicontinuous in E^*. Then there exists a balanced, convex, open neighborhood U of zero such that $|f_n(U)| < 1$ for all $n \in \mathbb{N}$. If μ_U is the Minkowski functional of U, then μ_U is a continuous semi-norm on E. Let τ_U be the locally convex topology induced by μ_U on E. Then (E,τ_U) is a locally convex space, not necessarily Hausdorff. E_U^* denotes the space of all τ_U-continuous linear functionals on E.

LEMMA 1. Let $\|f\|_U = \sup\limits_{\mu_U(x)\leq 1} |f(x)|$, $x \in E$, $f \in E_U^*$. Then $(E_U^*, \|\cdot\|_U)$ is a Banach space. Further, $\|f_n\|_U \leq 1$ for all $n \in \mathbb{N}$.

PROOF. To show that $\|\cdot\|_U$ is a norm in E_U^*, it is enough to verify that $f = 0$ if $\|f\|_U = 0$. Suppose that $\|f\|_U = 0$ and $x \in E$. As U is absorbing, there exists a $\lambda > 0$ such that $\frac{x}{\lambda} \in U$ and consequently, $\mu_U(\frac{x}{\lambda}) < 1$, from which it follows that $f(\frac{x}{\lambda}) = 0$. Hence $f = 0$. By a known argument, using the fact that U is absorbing, we can easily prove that $(E_U^*, \|\cdot\|_U)$ is complete. By the choice of U, it is clear that $\|f_n\|_U \leq 1$ for all n.

We observe that $E_U^* \subset E^*$, as the topology τ_U is weaker than $\mathcal{T}$.

LEMMA 2. The transformation $T: E_U^* \to (\ell)$, given by

$$Tf = \left\{f(x_i)\right\}_{i=1}^{\infty}$$

is linear and continuous.

PROOF. By the hypothesis (B) of [2], $\sum_{i=1}^{\infty} |f(x_i)| < \infty$ for all $f \in E^*$ and hence for all $f \in E_U^*$. Thus T has its range in (ℓ). By Lemma 1 and by the argument in the proof of Lemma 3.2.1 of [1], the result follows.

Using these lemmas, we shall now show that (1) $\Rightarrow$(F) when $(f_n)_1^{\infty}$ is equicontinuous in E^*. Let $\mathfrak{D}$ be the linear subspace generated in $(\ell)^*$ by all elements of the form $(\varepsilon_i)_1^{\infty}$, $\varepsilon_i = \pm 1, 0$. Clearly, $\mathfrak{D}$ is dense in $(\ell)^*$. From (1), it follows that $\lim_n \sum_{i=1}^{\infty} \beta_i f_n(x_i) = 0$ (2) for every $\beta = (\beta_i)_1^{\infty} \in \mathfrak{D}$.

Let $\alpha = (\alpha_i)_1^{\infty} \in (\ell)^*$. Given $\varepsilon > 0$, there exists a $\beta = (\beta_i)_1^{\infty} \in \mathfrak{D}$, such that $\|\alpha - \beta\|_{\infty} < \frac{\varepsilon}{2} \cdot \frac{1}{\|T\|}$, where $\|T\| < \infty$ by Lemma 2. By (2), there exists n_o such that $|\sum_{i=1}^{\infty} \beta_i f_n(x_i)| < \frac{\varepsilon}{2}$ for $n \geq n_o$. In other words, $|\beta(y_n)| < \frac{\varepsilon}{2}$, where $y_n = \left\{f_n(x_i)\right\}_{i=1}^{\infty} \in (\ell)$. Therefore, it follows that

$$|\alpha(y_n)| \leq |\alpha(y_n)-\beta(y_n)| + |\beta(y_n)| \leq \|\alpha-\beta\|_\infty \; \|Tf_n\| + \frac{\varepsilon}{2}$$

$$\leq \|\alpha-\beta\|_\infty \; \|T\| + \frac{\varepsilon}{2} < \varepsilon,$$

for $n \geq n_o$. Thus, $y_n \to 0$ weakly in (ℓ). Consequently, by a well known result, $y_n \to 0$ in the norm topology of (ℓ). This proves (F).

R E F E R E N C E S

1. E. Hille and R. S. Phillips, Functional Analysis and Semi-groups. Amer. Math. Soc. Colloq. Publ. V. 31, (1957).

2. C.W. McArthur, On a theorem of Orlicz and Pettis, Pacific. J. Math. V. 22 (1967), 297-302.

3. B.J. Pettis, On integration in vector spaces, Trans. Amer. Math. Soc. 44 (1938), 277-304.

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