On Duality in the Born-Infeld Theory

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The $SL(2,R)$ duality symmetric action for the Born-Infeld theory in terms of two potentials, coupled with non-trivial backgroud fields in four dimensions is established. This construction is carried out in detail by analysing the hamiltonian structure of the Born-Infeld theory. The equivalence with the usual Born-Infeld theory is shown.

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Nowadays the concept of duality is widely recognized by its unifying role in physics. The five known different superstring theories are now unified by duality in the framework of M Theory (see for example [1]). The simplest case where duality appears are the Maxwell's equations without sources, interchanging the equations of motions and the Bianchi identities. Schwarz and Sen [2] have developed a method to raise duality symmetry at the level of the action but at the price of losing the explicit Lorentz invariance. The classical and quantum equivalence with electromagnetism has been well established [3], [4], [5]. Earlier, Deser and Teitelboin [6] noticed that the Maxwell theory in its hamiltonian formulation is invariant under non-local dualty transformations. Moreover, several attempts have been made at conciliating duality symmetry with Lorentz invariance [7]. On the other hand, the Born-Infeld theory [8], initially conceived as an alternative for electromagnetism, has recently received considerable attention because the world volume action of a D-brane is described by a kind of non-linear Born-Infeld action [9]. Several aspects of the duality symmetry in the Born-Infeld theory have been developed recently [10], [11]. In particular, Perry and Schwarz [12] proposed a non-manifestly Lorentz invariant Born-Infeld action for a self-interacting self-dual antisymmetric tensor field in $D = 6$. Afterward, Pasti, Sorokin and Tonin presented a manifestly covariant formulation of this action [13], from which Berman [14], after dimensional reduction to four dimensions and breaking the Lorentz symmetry, obtained an action for the Born-Infeld theory coupled with the axion and dilaton fields with a $Z(2)$ symmetry.

In this article, we will study the Born-Infeld theory both pure and coupled with the axion and dilaton fields from the point of view of their hamiltonian structures in four dimensions, setting up the formulation *a la* Schwarz-Sen of the $SL(2,R)$ duality invariant Born-Infeld action. Our results extend those obtained by Berman when the axion field is involved, making evident the $SL(2,R)$ duality invariance. We will show that our results lead to the Born-Infeld theory after the elimination of one of two potentials.

We will start with the Born-Infeld theory without non-trivial background fields. Despite its highly non-linear character, the abelian gauge Born-Infeld theory whose action is given by

$$I = \int d^4x \left[1 - \sqrt{-det(\eta_{mn} + F_{mn})}\right], \tag{1}$$

describes the casual and ghost free propagation of massless spin-1 fotons like Maxwell's theory. In order to perform the hamiltonian analysis it will be convenient to introduce an auxiliary field v [15] and rewrite down the Born-Infeld action as

$$I = \int d^4x \left[1 - \frac{1}{2}v(1 + \frac{1}{2}F^{mn}F_{mn} - \frac{1}{16}(\mathcal{F}^{mn}F_{mn})^2) - \frac{1}{2v}\right]. \tag{2}$$

We have used the fact that in four dimensions $\mid det(\eta_{mn} + F_{mn}) \mid = (1 + \frac{1}{2}F^{mn}F_{mn} - \frac{1}{16}(\mathcal{F}^{mn}F_{mn})^2)$ and $\mathcal{F}^{mn} \equiv \frac{1}{2}\epsilon^{mnpq}F_{pq}$ is the dual of the field strength $F_{mn} = \partial_m A_n - \partial_n A_m$. The canonical momenta are found to be

$$\pi_i = \frac{\delta\mathcal{L}}{\delta\dot{A}_i} = vD_{ij}E_j, \quad \pi^0 = \frac{\delta\mathcal{L}}{\delta\dot{A}_0} = 0, \quad p = \frac{\delta\mathcal{L}}{\delta\dot{v}} = 0, \tag{3}$$

where $E_i \equiv F_{0i}$, $B_i \equiv \epsilon^{ijk}\partial_j A_k$ and $D_{ij} \equiv \delta_{ij} + B_iB_j$. Our flat metric signature is $(-1,+1,+1,+1)$. $\phi^1 \equiv \pi^0 \approx 0$ and $\phi^2 \equiv p \approx 0$ are the primary constraints. Since D is a non-singular matrix ($detD = 1 + \vec{B}.\vec{B} \neq 0$) there exist the inverse of D and from eq.(3) we obtain that

$$\dot{A}_i = \frac{1}{vdetD}\left[\delta_{ij}detD - B_iB_j\right]\pi_j + \partial_i A_0. \tag{4}$$

It is straightforward to compute the canonical hamiltonian

$$H_o = \frac{1}{2v}(\vec{\pi}.\vec{\pi}+1) + \frac{v}{2}(\vec{B}.\vec{B}+1) - \frac{1}{2vdetD}(\vec{\pi}.\vec{B})^2 - 1 - A_0\partial_i\pi_i. \tag{5}$$

Requiring consistency of the primary constraint ϕ^1, the Gauss's constraint $\phi^3 \equiv \partial_i\pi_i \approx 0$ follows inmediately, while the consistency of the constraint ϕ^2 lead us to determine the value of the auxiliary field v:

$$v = \frac{1}{detD}\sqrt{(\vec{\pi}.\vec{\pi}+1)(\vec{B}.\vec{B}+1) - (\vec{\pi}.\vec{B})^2}. \tag{6}$$

It is worth recalling that the canonical hamiltonian is invariant under non-local duality transformation [6]

$$\delta A_i = \theta\nabla^{-2}\epsilon^{ijk}\partial_j\pi_k, \quad \delta\pi_i = \theta\epsilon^{ijk}\partial_j A_k, \tag{7}$$

whenever the auxiliary field transforms as

$$\delta v = \frac{2\theta}{detD}v(\vec{\pi}.\vec{B}). \tag{8}$$

Eliminating the auxiliary field v using eq.(6), the well known non-linear canonical hamiltonian [11], [16] is recovered

$$H_o = \sqrt{1 + \vec{\pi}.\vec{\pi} + \vec{B}.\vec{B} + (\vec{\pi}\mathbf{x}\vec{B}).(\vec{\pi}\mathbf{x}\vec{B})} - 1 - A_0\partial_i\pi_i. \tag{9}$$

The set of constraints are first class ones and we can write the BRST invariant generating functional (choosing the Coulomb gauge) for the Born-Infeld theory as

$$Z = \int DA_i D\pi_i DcD\bar{c}\delta(\partial_i\pi_i)\delta(\partial_i A_i)exp(iI_{eff}), \tag{10}$$

where c and $\bar{c}$ are the pair of ghost-antighost fields associated to the first-class constraint ϕ^3 and the effective action is

$$I_{eff} = \int d^4x[\pi_i\dot{A}_i - H_o - i\bar{c}\nabla^2 c], \tag{11}$$

which is invariant under duality transformations (7). c and $\bar{c}$ are inert under duality and since they are decoupled from the pair π_i and A_i, we will omit their presence from now on. Moreover, we will assume that the Gauss's constraint is solved, then both $\vec{\pi}$ and $\vec{A}$ are transverse.

Following the ideas developed in references [17], [5] we introduce two new fields (renaming $A_i^1 \equiv A_i$ and $\pi_i^1 \equiv \pi_i$)

$$A_i^2 \equiv \epsilon^{ijk}\nabla^{-2}\partial_j\pi_k^1, \quad \pi_i^2 \equiv \epsilon^{ijk}\partial_j A_k^1, \tag{12}$$

with the goal to achieve the duality transformations ($\mathcal{L}_{12} = 1 = -\mathcal{L}_{21}$):

$$\delta A_i^\alpha = \mathcal{L}_{\alpha\beta}A_i^\beta, \quad \delta\pi_i^\alpha = \mathcal{L}_{\alpha\beta}\pi_i^\beta, \tag{13}$$

generated by an abelian Chern-Simons term: $G = \int d^3x\epsilon^{ijk}, A_i^\alpha\partial jA_k^\alpha$. The action (11) is rewriten down as

$$I = \int d^4x\left[\frac{1}{2}\pi_i^\alpha\dot{A_i}^\alpha - \sqrt{1 + B_i^\alpha B_i^\alpha + \frac{1}{4}(\epsilon^{ijk}B_j^\alpha\mathcal{L}_{\alpha\beta}B_k^\beta)^2} + 1\right], \tag{14}$$

where $\vec{B}^1 = \vec{\nabla}\mathbf{x}\vec{A}^1$ and $\vec{B}^2 = \vec{\nabla}\mathbf{x}\vec{A}^2 = \vec{\pi}^1$. This action is just the canonical form for the following action, in terms of two potentials *a la* Schwarz-Sen, for the Born-Infeld theory

$$I = -\int d^4x\left[\frac{1}{2}B_i^\alpha\mathcal{L}_{\alpha\beta}E_i^\beta + \sqrt{1 + B_i^\alpha B_i^\alpha + \frac{1}{4}(\epsilon^{ijk}B_j^\alpha\mathcal{L}_{\alpha\beta}B_k^\beta)^2} - 1\right], \tag{15}$$

clearly invariant under duality transformations. This result coincides with the previously obtained by Berman [14] in the absence of non-trivial background fields. In the weak approximation, where we neglected the higher terms in $(B_i^\alpha)^2$, the Schwarz-Sen action for Maxwell theory is obtained.

Now, we will consider the Born-Infeld theory coupled with an axion and a dilaton, described by the following action:

$$I = \int d^4x \left[1 - \sqrt{-det(\eta_{mn} + e^{-\frac{1}{2}\phi} F_{mn})} + \frac{1}{4} \psi \mathcal{F}^{mn} F_{mn} \right]. \tag{16}$$

It is well known that the equations of motion derived from this action are $SL(2,R)$ duality invariant [10] (we have omitted the kinetic terms for the dilaton and the axion which are $SL(2,R)$ duality invariant). Besides this action is $SL(2,R)$ self-dual [15]. Introducing the auxiliary field v, we rewrite down the action as

$$I = \int d^4x \left[1 - \frac{1}{2} v (1 + \frac{1}{2} e^{-\phi} F^{mn} F_{mn} - \frac{1}{16} e^{-2\phi} (\mathcal{F}^{mn} F_{mn})^2) - \frac{1}{2v} + \frac{1}{4} \psi \mathcal{F}^{mn} F_{mn} \right]. \tag{17}$$

In that case, the canonical momentum associated to A_i is

$$\pi_i = \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = v e^{-\phi} D_{ij} E_j + \psi B_i, \tag{18}$$

where now $D_{ij} \equiv \delta_{ij} + e^{-\phi} B_i B_j$, its inverse given by $D^{-1}_{ij} \equiv detD^{-1}[\delta_{ij} detD - e^{-\phi} B_i B_j]$ and $detD = 1 + e^{-\phi} \vec{B}.\vec{B}$. The canonical hamiltonian is found to be

$$\begin{aligned} H_o = & \frac{1}{2v}(e^{\phi} \vec{\pi}.\vec{\pi} + 1) + \frac{v}{2}(e^{-\phi} \vec{B}.\vec{B} + 1) - \frac{1}{2v detD}(\vec{\pi}.\vec{B})^2 \\ & - \frac{1}{v detD} \psi e^{\phi} (\vec{\pi}.\vec{B}) + \frac{1}{2v detD} \psi^2 e^{\phi} \vec{B}.\vec{B} - 1 - A_0 \partial_i \pi_i. \end{aligned} \tag{19}$$

Preserving the primary constraint $p \approx 0$ the value of the auxiliary field v is determined:

$$v = \frac{1}{detD} \sqrt{(e^{\phi} \vec{\pi}.\vec{\pi} + 1)(e^{-\phi} \vec{B}.\vec{B} + 1) - [(\vec{\pi}.\vec{B})^2 + 2\psi e^{\phi} (\vec{\pi}.\vec{B}) - \psi^2 e^{\phi} \vec{B}.\vec{B}]}. \tag{20}$$

Putting this back into H_o, we obtain the non-linear canonical hamiltonian in four dimensions for the Born-Infeld coupled with the dilaton and the axion fields.

$$\begin{aligned} H_o = & \sqrt{1 + e^{\phi} \vec{\pi}.\vec{\pi} + e^{-\phi} \vec{B}.\vec{B} + (\vec{\pi} \mathbf{x} \vec{B}).(\vec{\pi} \mathbf{x} \vec{B}) - 2\psi e^{\phi} (\vec{\pi}.\vec{B}) + \psi^2 e^{\phi} \vec{B}.\vec{B}} \\ & - 1 - A_0 \partial_i \pi_i. \end{aligned} \tag{21}$$

Introducing two new field variables like eq.(12), after solving the Gauss's constraint, we arrive to the following hamiltonian in terms of two potentials

$$H_o = \sqrt{1 + B_i^{\alpha} (\mathcal{L}^T M \mathcal{L})_{\alpha\beta} B_i^{\beta} + \frac{1}{4} (\epsilon^{ijk} B_j^{\alpha} \mathcal{L}_{\alpha\beta} B_k^{\beta})^2} - 1, \tag{22}$$

where we have introduced the symmetric $SL(2,R)$ matrix

$$M = \begin{pmatrix} e^{\phi} & \psi e^{\phi} \\ \psi e^{\phi} & e^{-\phi} + \psi^2 e^{\phi} \end{pmatrix}, \tag{23}$$

satisfying

$$M = M^T, \quad M \mathcal{L} M^T = \mathcal{L}. \tag{24}$$

This is the hamiltonian for the following non-manifestly Lorentz invariant Born-Infeld action coupled to the axion and dilaton fields, in terms of two potentials

$$I = - \int d^4x \left[\frac{1}{2} B_i^{\alpha} \mathcal{L}_{\alpha\beta} E_i^{\beta} + \sqrt{1 + B_i^{\alpha} (\mathcal{L}^T M \mathcal{L})_{\alpha\beta} B_i^{\beta} + \frac{1}{4} (\epsilon^{ijk} B_j^{\alpha} \mathcal{L}_{\alpha\beta} B_k^{\beta})^2} - 1 \right]. \tag{25}$$

We propose this action as the generalization of the Born-Infeld theory in terms of two potentials, invariant under the $SL(2,R)$ duality transformations

$$M \to \omega^T M \omega, \quad A_i^{\alpha} \to \omega^T_{\alpha\beta} A_i^{\beta}, \tag{26}$$

with $\omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R)$ ($det\omega = +1$). Now, we are going to show that after eliminating A_i^2 from the action (25), we will obtain the Born-Infeld theory coupled with the dilaton and axion fields in the axial gauge $A_0^\alpha = 0$. Introducing an auxiliary variable v, the action (25) is rewritten down as

$$I = -\int d^4x[\frac{1}{2}B_i^\alpha \mathcal{L}_{\alpha\beta}E_i^\beta + \frac{1}{2}v(1 + e^{-\phi}B_i^1B_i^1 + e^{\phi}B_i^2B_i^2 + (B_i^1B_i^1)(B_i^2B_i^2) - (B_i^1B_i^2)^2 - 2\psi e^{\phi}B_i^1B_i^2 + \psi^2 e^{\phi}B_i^1B_i^1) \quad (27)$$
$$+ \frac{1}{2v} - 1].$$

Independent variations in A_i^2, lead us to the following equations of motion

$$\epsilon^{ijk}\partial_j[ve^{\phi}(\delta_{kl}(1 + e^{-\phi}B_i^1B_i^1) - e^{-\phi}B_k^1B_l^1)B_l^2 - E_k^1 - ve^{\phi}\psi B_k^1] = 0. \quad (28)$$

After using the freedom of gauging the gauge symmetry: $\delta\vec{A}^1 = \vec{\nabla}\xi$, we can solve eq. (28) for B_i^2

$$B_i^2 = (detD)^{-1}D_{ij}[v^{-1}e^{-\phi}E_j^1 + \psi B_j^1]. \quad (29)$$

Substituting this value of B_i^2 into the action (27), we obtain

$$I = -\int d^4x[\frac{1}{2}ve^{-\phi}B_i^1B_i^1 - \frac{1}{2vdetD}e^{-\phi}E_i^1E_i^1 - \frac{1}{2vdetD}e^{-2\phi}(E_i^1B_i^1)^2 + \frac{1}{2}v + \frac{1}{2v} - \psi E_i^1B_i^1 - 1]. \quad (30)$$

Finally, we can eliminate the auxiliary field v through its equation of motion:

$$v = (detD)^{-1}\sqrt{1 + e^{-\phi}(B_i^1B_i^1 - E_i^1E_i^1) - e^{-2\phi}(E_i^1B_i^1)^2} \quad (31)$$

and putting this back into eq. (30), the Born-Infeld action (eq. (16)) is recovered.

It would be interesting to analyze the quantum aspects of this theory as well as find out its supersymmetric extension. Moreover, it would be enlightening to try to construct the covariant version of this formulation.

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