Symmetries of distributional domain wall geometries

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Generalizing the Lie derivative of smooth tensor fields to distribution-valued tensors, we examine the Killing symmetries and the collineations of the curvature tensors of some distributional domain wall geometries. The chosen geometries are rigorously the distributional thin wall limit of self gravitating scalar field configurations representing thick domain walls and the permanence and/or the rising of symmetries in the limit process is studied. We show that, for all the thin wall spacetimes considered, the symmetries of the distributional curvature tensors turns out to be the Killing symmetries of the pullback of the metric tensor to the surface where the singular part of these tensors is supported. Remarkably enough, for the non-reflection symmetric domain wall studied, these Killing symmetries are not necessarily symmetries of the ambient spacetime on both sides of the wall.

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I. INTRODUCTION

Consider a family of spacetimes $(\mathcal{M}, {}^\gamma\mathbf{g})$, where ${}^\gamma\mathbf{g}$ is C^∞ metric tensor which depends on a parameter γ. An isometry ψ on $(\mathcal{M}, {}^\gamma\mathbf{g})$ is defined to be a diffeomorphism $\psi : \mathcal{M} \to \mathcal{M}$ for which $\psi^*\mathbf{g} = \mathbf{g}$. The infinitesimal generator of a one-parameter group ψ_λ of local isometries is the C^∞ vector field $\mathbf{V}$ on $\mathcal{M}$ that satisfies

$$\mathcal{L}_{\mathbf{V}}\, {}^\gamma\mathbf{g} = 0, \tag{1}$$

and $\mathbf{V}$ is called a Killing vector field on $(\mathcal{M}, {}^\gamma\mathbf{g})$ relative to this group. To every one-parameter family of Killing symmetries there is an associated conserved quantity along the geodesics of the spacetime and these conserved quantities are useful for integrating the geodesic equation [1].

Although isometries are the most important transformations on $(\mathcal{M}, {}^\gamma\mathbf{g})$, geometric symmetries other than Killing symmetries may also be considered. Let ${}^\gamma\mathbf{Ric}$ be the Ricci curvature tensor of ${}^\gamma\mathbf{g}$. A vector field $\mathbf{V}$ on $\mathcal{M}$ that satisfies

$$\mathcal{L}_{\mathbf{V}}\, {}^\gamma\mathbf{Ric} = 0, \tag{2}$$

is called a Ricci collineation. It is well known that for every vector field $\mathbf{V}$ such that (1) is satisfied, i.e. for every Killing vector field, (2) is also satisfied and the Ricci tensor inherits the symmetries of the metric. However, other vector fields that are not Killing vectors may exist for which (2) is satisfied and these are called proper Ricci collineations. Since $\mathbf{Ric}$ is obtained by contracting the Riemann curvature tensor, Ricci collineations have a natural geometrical significance and it is believed that they can be useful to understand the interplay between geometry and physics in general relativity (for more details on these and other geometric symmetries, see references [2, 3, 4, 5]).

Now, consider a spacetime $(\mathcal{M}, \mathbf{g})$ of low differentiability, revealing itself through a lack of smoothness of the metric $\mathbf{g}$ and its curvature tensors. Up to what extent the classical concept of a geometric symmetry (in the smooth case) can be carried over to this situation? Let us assume that the metric tensors $\mathbf{g}$ and ${}^\gamma\mathbf{g}$ are distribution-valued tensors that satisfy

$$\mathbf{g} \equiv \lim_{\gamma\to 0} {}^\gamma\mathbf{g} \tag{3}$$

(in the sense of distributions) and that their corresponding curvature tensors have a well defined distributional meaning. Further, assume that the following diagram holds

$$\begin{array}{ccccccc} {}^\gamma\mathbf{g} & \longrightarrow & {}^\gamma\mathbf{Riem} & \longrightarrow & {}^\gamma\mathbf{Ric} & \longrightarrow & {}^\gamma\mathbf{G} \\ \downarrow{\scriptstyle \gamma\to 0} & & \downarrow{\scriptstyle \gamma\to 0} & & \downarrow{\scriptstyle \gamma\to 0} & & \downarrow{\scriptstyle \gamma\to 0} \\ \mathbf{g} & \longrightarrow & \mathbf{Riem} & \longrightarrow & \mathbf{Ric} & \longrightarrow & \mathbf{G} \end{array} \tag{4}$$

where ${}^\gamma\mathbf{g}$, ${}^\gamma\mathbf{Riem}$,${}^\gamma\mathbf{Ric}$, ${}^\gamma\mathbf{G}$ and $\mathbf{g}$, $\mathbf{Riem}$, $\mathbf{Ric}$, $\mathbf{G}$ are the distribution valued metric, Riemann, Ricci and Einstein tensors of the smooth and the distributional geometries, respectively. Although distributional curvatures are in general ill-defined due to the nonlinearities of general relativity, there is a class of distributional metrics for which the Riemann curvature tensor and its contractions can be interpreted as distributions [6]. Metrics for thin shells [7] are included into this class [6, 8]. Furthermore, for such a class an appropriate notion of convergence of metrics has been stated which ensures the convergence of the respective curvatures [6], in the sense that the diagram (4) holds.

Since the derivative of a distribution is a distribution, for a distributional metric whose curvature tensors make sense as distributions, it makes sense to consider their geometric symmetries. Obviously, this situation should be considered also from the distributional point of view. For example, equation (2) contains products of the Ricci tensor and the vector field which generates the symmetry, so that for a distribution-valued Ricci tensor such equation restricts the vector field to be C^∞. With this proviso, we can consider geometric symmetries in cases in which the curvature tensors are zero almost everywhere (when obtained within standard differential geometry). Furthermore, for distributional geometries such that the above diagram holds, we can study the permanence and/or the rising of symmetries in the limit process. Thus, from the study of these geometries we expect to get further insight about the nature of distributional curvatures in general relativity.

Killing symmetries of distributional metrics have been considered previously in reference [9], where it is shown that the Killing fields of the Schwarzschild metric are also Killing of its ultra-relativistic limit, the last one being a pp-wave with a distributional δ profile [10]. In addition, based on the analysis of the adjoint orbits of normal-form-preserving diffeomorphisms, Killing symmetries of impulsive pp-waves with distributional profiles have been analyzed [11] and the existence of non-smooth Killing vectors put forward [12].

In this paper, adopting a different approach for a rigorous definition of symmetries of distributional geometries, we are concerned with symmetries of domain wall spacetimes. Such spacetimes have been the subject of intense investigation, after it was realized in reference [13] that our four dimensional universe might be a thin (codimension one) distributional domain wall embedded in a five dimensional spacetime. Thin wall geometries have distribution-valued curvature tensor fields whose singular parts are proportional to a Dirac distribution supported on the surface where the wall is localized. All the metrics representing the possible backgrounds of an infinitely thin domain wall [14] have been found and classified [15, 16], these being joined at a common boundary, the surface where the wall is localized, following the Darmois-Israel formalism [7]. On the other hand, smooth domain wall geometries can be obtained as solutions to the coupled Einstein-scalar field system with a suitable symmetry breaking potential $V(\phi)$ [14, 17, 18, 19, 20]. The behavior of gravity in some of these models has been also investigated [21, 22, 23]. Recently, following the convergence criteria of reference [6], the distributional thin wall limit of some classes of domain wall spacetimes has been rigourously analyzed [24, 25] showing that the diagram (4), in which ${}^\gamma\mathbf{g}$ and $\mathbf{g}$ are the distribution valued metric tensors of the thick and thin domain wall spacetimes, respectively, holds. Specifically, we will study the symmetries of the metric and its curvature tensors of these distributional domain wall geometries. Domain walls have drastic gravitational effects in the ambient space and, due to its role in brane-world models, we are interested in their geometric symmetries besides those defining the plane-parallel symmetry since a larger group may exists for various particular models.

In section II, after an overview on the subject of distribution-valued tensors, we give the definition of the Lie derivative of a tensor distribution along a C^∞ vector field. In the next three sections, we examine the Killing symmetries and the collineations of the singular Ricci and Einstein curvature tensors associated to the distributional thin wall limit of some thick domain wall spacetimes for which the diagram (4) holds [24]. The last section is devoted to summarize and discuss the results.

II. MATHEMATICAL FRAMEWORK

We first recall some fundamental results about distribution-valued tensors on a C^∞ paracompact n-dimensional manifold $\mathcal{M}$ [26, 27, 28].

Let $\mathcal{D}_p(\mathcal{M})$ be the space of C^∞ p-tensor fields on $\mathcal{M}$ with compact support endowed with its Schwartz topology, i.e., the space of test p-tensor fields. The space of p-cotensor distributions on $\mathcal{M}$, $\mathcal{D}^{*'}_p(\mathcal{M})$, is defined as the dual of $\mathcal{D}_p(\mathcal{M})$. Now, in order to keep things simple, we endow the C^∞ paracompact manifold $\mathcal{M}$ with a C^∞ metric η. However, it should be noted that tensor distributions and their derivatives can be described without assuming the presence of a metric [29] and that, by using de Rham currents [30] and replacing test tensors by test n-forms, the introduction of a volume form can also be avoided. Here we shall follow the approach of [26].

Let $\mathbf{U}$ be a test p-tensor field on $\mathcal{M}$. The identification of a locally integrable p-cotensor field $\mathbf{T}$ with a distribution-valued tensor is defined by

$$\mathbf{T}[\mathbf{U}] \equiv \int_{\mathcal{M}} \mathbf{T}\cdot\mathbf{U}\,\boldsymbol{\omega}_\eta, \tag{5}$$

where $\mathbf{T}\cdot\mathbf{U}$ denotes the scalar product of $\mathbf{T}$ and $\mathbf{U}$, and $\boldsymbol{\omega}_\eta$ is the volume element of $\boldsymbol{\eta}$. Since (5) is the integral of an n-form with compact support, we have

$$\mathbf{T}[\mathbf{U}] \equiv \int_{\varphi(\mathcal{M})} T_{i_1\ldots i_p} U^{i_1\ldots i_p} |\det\eta|^{\frac{1}{2}} dx^1 \ldots dx^n \tag{6}$$

in the domain $\varphi(\mathcal{M})$ of the chart $(x^1,\ldots,x^n)$. Obviously, (6) is independent of the choice of coordinate system covering the corresponding domain.

Next, let us define the Lie derivative of a tensor field in the sense of distributions. Let $\mathbf{V}$ be a C^∞ vector field and $\mathbf{T}$ be a C^1 p-cotensor field on $\mathcal{M}$. The Lie derivative of $\mathbf{T}$ is the p-cotensor field $\mathcal{L}_{\mathbf{V}}\mathbf{T}$ such that, for every test p-tensor field $\mathbf{U}$

$$\mathcal{L}_{\mathbf{V}}\mathbf{T}[\mathbf{U}] \equiv \int_{\mathcal{M}} \mathcal{L}_{\mathbf{V}}\mathbf{T}\cdot\mathbf{U}\,\boldsymbol{\omega}_\eta = \int_{\mathcal{M}} \mathcal{L}_{\mathbf{V}}(\mathbf{T}\cdot\mathbf{U}\,\boldsymbol{\omega}_\eta) - \int_{\mathcal{M}} \mathbf{T}\cdot\mathcal{L}_{\mathbf{V}}(\mathbf{U}\,\boldsymbol{\omega}_\eta)\,. \tag{7}$$

Now, since $\mathbf{T}\cdot\mathbf{U}\,\boldsymbol{\omega}_\eta$ is an n-form with compact support, we have

$$\int_{\mathcal{M}} \mathcal{L}_{\mathbf{V}}(\mathbf{T}\cdot\mathbf{U}\,\boldsymbol{\omega}_\eta) = \int_{\mathcal{M}} \mathbf{d}\,\mathbf{i}_{\mathbf{V}}(\mathbf{T}\cdot\mathbf{U}\,\boldsymbol{\omega}_\eta) = \int_{\partial\mathcal{M}} \mathbf{i}_{\mathbf{V}}(\mathbf{T}\cdot\mathbf{U}\,\boldsymbol{\omega}_\eta) = 0, \tag{8}$$

where $\mathbf{i}_{\mathbf{V}}$ denotes interior product, in the last step we have used Stokes' theorem and the surface term vanishes because $\mathbf{U}$ has compact support. Therefore

$$\mathcal{L}_{\mathbf{V}}\mathbf{T}[\mathbf{U}] = -\int_{\mathcal{M}} \mathbf{T}\cdot\mathcal{L}_{\mathbf{V}}(\mathbf{U}\,\boldsymbol{\omega}_\eta) = -\int_{\mathcal{M}} (\mathbf{T}\cdot\mathcal{L}_{\mathbf{V}}\mathbf{U})\boldsymbol{\omega}_\eta + (\mathbf{T}\cdot\mathbf{U})\mathcal{L}_{\mathbf{V}}\boldsymbol{\omega}_\eta\,. \tag{9}$$

On the other hand, we have

$$\mathcal{L}_{\mathbf{V}}\boldsymbol{\omega}_\eta = (\nabla\cdot\mathbf{V})\boldsymbol{\omega}_\eta \tag{10}$$

where ∇ is the derivative in $\boldsymbol{\eta}$ and

$$\nabla\cdot\mathbf{V} \equiv \nabla_j V^j. \tag{11}$$

It follows that

$$\mathcal{L}_{\mathbf{V}}\mathbf{T}[\mathbf{U}] = -\int_{\mathcal{M}} \mathbf{T}\cdot(\mathcal{L}_{\mathbf{V}}\mathbf{U} + \mathbf{U}(\nabla\cdot\mathbf{V}))\boldsymbol{\omega}_\eta\,. \tag{12}$$

Therefore, to have a definition which coincides with the usual one when $\mathbf{T}$ is C^1, we define for $\mathbf{T}$ an arbitrary p-cotensor distribution and $\mathbf{V}$ a C^∞ vector field on $\mathcal{M}$ the Lie derivative of $\mathbf{T}$ as the p-cotensor distribution given by

$$\mathcal{L}_{\mathbf{V}}\mathbf{T}[\mathbf{U}] \equiv -\mathbf{T}[\mathcal{L}_{\mathbf{V}}\mathbf{U} + (\nabla\cdot\mathbf{V})\mathbf{U}]. \tag{13}$$

Note that (13) makes sense since both $\mathcal{L}_{\mathbf{V}}\mathbf{U}$ and $(\nabla\cdot\mathbf{V})\,\mathbf{U}$ are test p-tensor fields if $\mathbf{V}$ is a C^∞ vector field.

In the next sections, (13) will be used to define and compute the Lie derivative of a distribution-valued metric tensor $\mathbf{g}$ with $\mathbf{V}$ a C^∞ vector field that generates a one parameter group of isometries of the spacetime $(\mathcal{M},\mathbf{g})$. We also make use of (13) to define and compute the Ricci and Einstein collineations of the distributional geometry associated to these spacetimes. Since all the distribution-valued metric tensors that we shall consider are regular metrics in the sense of reference [6], let us recall their definition. Suppose that $(\mathcal{M},\mathbf{g})$ are given such that

1. $\mathbf{g}$ and $(\mathbf{g}^{-1})$ exist everywhere and are locally bounded,

2. the first derivative $\nabla\mathbf{g}$ (in the sense of distributions) of $\mathbf{g}$ in some smooth derivative operator ∇ exists and is locally square-integrable, i.e. the outer product of the derivative with itself is locally integrable.

Following reference [6], these are the minimal conditions for the Riemann curvature tensor to be definable as a distribution by the usual coordinate formula and we shall say that $\mathbf{g}$ is a regular metric. Since for this class of metrics the outer product of any number of metrics and inverse metrics with the Riemann curvature tensor can be interpreted as distributions, the Ricci and Einstein curvature tensors of a regular metric make sense as distributions. Finally, it should be stressed that for such class an appropriate notion of convergence of metrics exists which ensures the convergence of the respective curvatures [6], in the sense that the diagram (4) holds (For details, see reference [6]).

III. A DOMAIN WALL WITH A DE SITTER EXPANSION

Consider the spacetime ($\mathbb{R}^4$, ${}^\gamma\mathbf{g}$) where the metric tensor ${}^\gamma\mathbf{g}$, in a particular coordinate basis, is given by

$$ {}^\gamma\mathbf{g} = \cosh^{-2\gamma}\frac{\beta x}{\gamma}\left(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}x\,\mathbf{d}x + e^{2\beta t}(\mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\right), \tag{14} $$

where β and γ are constants, with $0 < \gamma < 1$. This spacetime is generated by a thick domain wall, i.e. a solution to the coupled Einstein-scalar field equations

$$ {}^\gamma\mathbf{G} + {}^\gamma\mathbf{g}\,\Lambda \equiv {}^\gamma\mathbf{Ric} - \frac{1}{2}{}^\gamma\mathbf{g}\,{}^\gamma\mathrm{R} + {}^\gamma\mathbf{g}\,\Lambda = 8\pi\left[\mathbf{D}\phi\,\mathbf{D}\phi - \mathbf{g}\left(\frac{1}{2}(\mathbf{D}\phi\,|\,\mathbf{D}\phi) + V(\phi)\right)\right] \tag{15} $$

and

$$ \Box\phi - \frac{d}{d\phi}V(\phi) = 0, \tag{16} $$

where $\Box \equiv \Delta\mathbf{D} + \mathbf{D}\Delta$, with $\Lambda = 0$, and

$$ \phi = \phi_0\tan^{-1}(\sinh(\beta x/\gamma), \qquad \phi_0 \equiv \sqrt{\gamma(1-\gamma)/4\pi}, \tag{17} $$

$$ V(\phi) = \frac{1}{8\pi}\beta^2(2+\frac{1}{\gamma})\cos^{2(1-\gamma)}(\phi/\phi_0). \tag{18} $$

The scalar field takes values $\pm\pi\phi_0/2$ at $x\pm\infty$ corresponding to two consecutive minima of the potential and interpolates smoothly between these values [17, 18, 20], with γ playing the role of the wall's thickness [24]. The five-dimensional analogue of this geometry, considered as a thick brane-world model, has been studied in reference [23].

Note that (14) is C^∞ as are also all its curvature tensor fields. In particular, for the Ricci and Einstein tensor fields we obtain

$$ {}^\gamma\mathbf{Ric} = -(2+\frac{1}{\gamma})\,\beta^2\cosh^{-2}\frac{\beta x}{\gamma}\left(-\mathbf{d}t\,\mathbf{d}t + \frac{3}{1+2\gamma}\,\mathbf{d}x\,\mathbf{d}x + e^{2\beta t}(\mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\right) \tag{19} $$

and

$$ {}^\gamma\mathbf{G} = -(1+\frac{2}{\gamma})\,\beta^2\cosh^{-2}\frac{\beta x}{\gamma}\left(-\mathbf{d}t\,\mathbf{d}t + \frac{3\gamma}{2+\gamma}\,\mathbf{d}x\,\mathbf{d}x + e^{2\beta t}(\mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\right). \tag{20} $$

Now, from (1) and (14), we obtain six linearly independent Killing vector fields on ($\mathbb{R}^4$, ${}^\gamma\mathbf{g}$) given by

$$ \mathbf{V}_1 = \boldsymbol{\partial}_y, \qquad \mathbf{V}_2 = \boldsymbol{\partial}_z, \qquad \mathbf{V}_3 = z\boldsymbol{\partial}_y - y\boldsymbol{\partial}_z, \tag{21} $$

$$ \mathbf{V}_4 = \boldsymbol{\partial}_t - \beta(y\boldsymbol{\partial}_y + z\boldsymbol{\partial}_z), \tag{22} $$

$$ \mathbf{V}_5 = 2\beta y\mathbf{V}_4 + \left(\beta^2(y^2+z^2) - e^{-2\beta t}\right)\boldsymbol{\partial}_y, \tag{23} $$

$$ \mathbf{V}_6 = 2\beta z\mathbf{V}_4 + \left(\beta^2(y^2+z^2) - e^{-2\beta t}\right)\boldsymbol{\partial}_z\,. \tag{24} $$

The Killing vectors (21) are generic to the plane parallel symmetry, two spatial translations $\mathbf{V}_1$, $\mathbf{V}_2$ and one spatial rotation $\mathbf{V}_3$. The six vector fields (21-24) are also Killing vectors on the $(2+1)$-dimensional de Sitter spacetime $(\mathbb{R}^3, \bar{\mathbf{g}})$ with

$$ \bar{\mathbf{g}} = -\mathbf{d}t\,\mathbf{d}t + e^{2\beta t}(\mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z), \tag{25} $$

where $\mathbf{V}_4$ is a quasi-time translation and $\mathbf{V}_5$, $\mathbf{V}_6$ are quasi-Lorentz rotations. Note that (21-24) are all independent of the thickness γ of the wall. It is straightforward to show that for (21-24) we have

$$\mathcal{L}_{\mathbf{V}}\,{}^{\gamma}\mathbf{Ric} = 0, \qquad \mathcal{L}_{\mathbf{V}}\,{}^{\gamma}\mathbf{G} = 0. \tag{26}$$

One can also show that there are no other vector fields for which (26) is satisfied. Hence, the Ricci (19) and Einstein (20) curvature tensors only admit improper collineations.

Next, consider the Lie derivative of (14) in the sense of distributions as given by (13). Let $\boldsymbol{\eta}$ be the ordinary four-dimensional Minkowski metric tensor in cartesian coordinates

$$\boldsymbol{\eta} = -\mathbf{d}t\,\mathbf{d}t + \mathbf{d}x\,\mathbf{d}x + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z, \tag{27}$$

and let ∇ be the derivative operator in $\boldsymbol{\eta}$. Let $\mathbf{U}$ be a test tensor on $\mathbb{R}^4$. From (13), it is straightforward to verify that for all Killing vectors (21-24), we have

$$\mathcal{L}_{\mathbf{V}}\,{}^{\gamma}\mathbf{g}[\mathbf{U}] = 0\,, \tag{28}$$

as expected. Indeed, it is to make such things true that the Lie derivative is defined as in (13). For the sake of completeness, let us to show explicitly the computation of this derivative for the Killing vector given by (22). We have

$$\begin{aligned}
\mathcal{L}_{\mathbf{V_4}}\,{}^{\gamma}\mathbf{g}[\mathbf{U}] &\equiv -{}^{\gamma}\mathbf{g}[\mathcal{L}_{\mathbf{V_4}}\mathbf{U} + (\nabla\cdot\mathbf{V_4})\mathbf{U}] \\
&= -\int_{\mathbb{R}^4} {}^{\gamma}\mathbf{g}\cdot(\mathcal{L}_{\mathbf{V_4}}\mathbf{U} + \mathbf{U}(\nabla\cdot\mathbf{V_4}))\boldsymbol{\omega}_\eta \\
&= -\int_{\mathbb{R}^4} \cosh^{-2\gamma}\frac{\beta x}{\gamma}\Big((\partial_t - \beta y\partial_y - \beta z\partial_z - 2\beta)(-U^{tt} + U^{xx}) \\
&\qquad\qquad + e^{2\beta t}(\partial_t - \beta y\partial_y - \beta z\partial_z)(U^{yy} + U^{zz})\Big)\, dt\, dx\, dy\, dz \\
&= 0,
\end{aligned} \tag{29}$$

where in last step we have integrated by parts and used the fact that $\mathbf{U}$ is of compact support. For all the Killing vector fields (21-24), analogous computations show that

$$\mathcal{L}_{\mathbf{V}}\,{}^{\gamma}\mathbf{Ric}[\mathbf{U}] = 0, \qquad \mathcal{L}_{\mathbf{V}}\,{}^{\gamma}\mathbf{G}[\mathbf{U}] = 0, \tag{30}$$

as expected.

We turn now to consider the $\gamma \to 0$ limit of this geometry. In reference [24], it has been proved that (14) provides a sequence of metrics that satisfies the convergence conditions of [6] such that the limit of the Riemann curvature tensor exists and is the Riemann tensor of the limit metric. The same holds for the other curvature tensors. We have

$$\mathbf{g} \equiv \lim_{\gamma\to 0}{}^{\gamma}\mathbf{g} = (\Theta_x^-\, e^{2\beta x} + \Theta_x^+\, e^{-2\beta x})\left(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}x\,\mathbf{d}x + e^{2\beta t}(\mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\right), \tag{31}$$

$$\mathbf{Ric} \equiv \lim_{\gamma\to 0}{}^{\gamma}\mathbf{Ric} = 2\beta\,\delta_0\left(\mathbf{d}t\,\mathbf{d}t - 3\mathbf{d}x\,\mathbf{d}x - e^{2\beta t}(\mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\right), \tag{32}$$

$$\mathbf{G} \equiv \lim_{\gamma\to 0}{}^{\gamma}\mathbf{G} = 4\beta\,\delta_0\left(\mathbf{d}t\,\mathbf{d}t - e^{2\beta t}(\mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\right), \tag{33}$$

where Θ_x^- and Θ_x^+ are the Heaviside distributions with support on $x < 0$ and $x > 0$ respectively and δ_0 is the Dirac measure with support on the surface $x = 0$. Note that $\mathbf{g}$ is piecewise smooth and that the pullback of $\mathbf{g}$ to the surface $x = 0$ is the same from both sides. Note also that this pullback is given by (25). Indeed, the above expressions should be understood in the sense of distributions. Thus, $\mathbf{g} \equiv \lim_{\gamma\to 0}{}^{\gamma}\mathbf{g}$ means

$$\mathbf{g}[\mathbf{U}] \equiv \lim_{\gamma\to 0}{}^{\gamma}\mathbf{g}[\mathbf{U}]. \tag{34}$$

In fact, ${}^{\gamma}\mathbf{g}$, $({}^{\gamma}\mathbf{g})^{-1}$ and $\nabla\,{}^{\gamma}\mathbf{g}$ converge locally (in square integral) to $\mathbf{g}$, $(\mathbf{g})^{-1}$ and $\nabla\mathbf{g}$ respectively and $\mathbf{Ric}$ and $\mathbf{G}$ are the distribution-valued Ricci and Einstein curvatures of $\mathbf{g}$ [24]. It follows that the diagram (4), in which ${}^{\gamma}\mathbf{g}$ and

$\mathbf{g}$ are the distribution valued metric tensors of the thick and thin domain wall spacetimes, respectively, holds in the sense of distributions.

Now, as follows from the distributional convergence (4) proved in reference [24] and the fact that the vectors (21-24) are all smooth vector fields independent of the wall's thickness, these are also Killing vectors of $(\mathbb{R}^4, \mathbf{g})$ in the sense that $\mathcal{L}_{\mathbf{V}}\, \mathbf{g}$ is the zero distribution on $\mathbb{R}^4$. As an example, consider the Killing vector given by (22). For (31) we have

$$\begin{aligned} \mathcal{L}_{\mathbf{V_4}}\, \mathbf{g}[\mathbf{U}] \;&\equiv\; -\mathbf{g}[\mathcal{L}_{\mathbf{V_4}} \mathbf{U} + (\nabla\cdot\mathbf{V_4})\mathbf{U}] \\ &= -\int_{\mathbb{R}^4} \mathbf{g}\cdot(\mathcal{L}_{\mathbf{V_4}}\mathbf{U} + \mathbf{U}(\nabla\cdot\mathbf{V_4}))\boldsymbol{\omega}_\eta \\ &= -\int_{x<0} e^{2\beta x}\big((\partial_t - \beta y\partial_y - \beta z\partial_z - 2\beta)(-U^{tt} + U^{xx}) \\ &\qquad\qquad + e^{2\beta t}(\partial_t - \beta y\partial_y - \beta z\partial_z)(U^{yy} + U^{zz})\big)\big)\, dt\, dx\, dy\, dz \\ &\quad -\int_{x>0} e^{-2\beta x}\big((\partial_t - \beta y\partial_y - \beta z\partial_z - 2\beta)(-U^{tt} + U^{xx}) \\ &\qquad\qquad + e^{2\beta t}(\partial_t - \beta y\partial_y - \beta z\partial_z)(U^{yy} + U^{zz})\big)\big)\, dt\, dx\, dy\, dz \\ &= 0, \end{aligned} \tag{35}$$

where we have integrated by parts and used the fact that $\mathbf{U}$ is of compact support.

Indeed, (21-24) are also improper Ricci collineations. For (32) and (22) we have

$$\begin{aligned} \mathcal{L}_{\mathbf{V_4}}\, \mathbf{Ric}[\mathbf{U}] \;&\equiv\; -\mathbf{Ric}[\mathcal{L}_{\mathbf{V_4}} \mathbf{U} + (\nabla\cdot\mathbf{V_4})\mathbf{U}] \\ &= -\beta\int_{x=0} \big((\partial_t - \beta y\partial_y - \beta z\partial_z - 2\beta)(2U^{tt} - 3U^{xx}) \\ &\qquad\qquad -2e^{2\beta t}(\partial_t - \beta y\partial_y - \beta z\partial_z)(U^{yy} + U^{zz})\big)\big)\, dt\, dy\, dz \\ &= 0, \end{aligned} \tag{36}$$

where in last step we have integrated by parts and used again the fact that $\mathbf{U}$ is of compact support. Analogous computations show that

$$\mathcal{L}_{\mathbf{V}}\, \mathbf{Ric}[\mathbf{U}] = 0, \qquad \mathcal{L}_{\mathbf{V}}\, \mathbf{G}[\mathbf{U}] = 0, \tag{37}$$

for all the Killing symmetries of (14), a result that obviously can not be proved outside the distributional setting. It follows that for this spacetime the diagram

$$\begin{array}{ccccc} \mathcal{L}_{\mathbf{V}}\, {}^\gamma\mathbf{g} = 0 & \longrightarrow & \mathcal{L}_{\mathbf{V}}\, {}^\gamma\mathbf{Ric} = 0 & \longrightarrow & \mathcal{L}_{\mathbf{V}}\, {}^\gamma\mathbf{G} = 0 \\ \big\downarrow{\scriptstyle \gamma\to 0} & & \big\downarrow{\scriptstyle \gamma\to 0} & & \big\downarrow{\scriptstyle \gamma\to 0} \\ \mathcal{L}_{\mathbf{V}}\, \mathbf{g} = 0 & \longrightarrow & \mathcal{L}_{\mathbf{V}}\, \mathbf{Ric} = 0 & \longrightarrow & \mathcal{L}_{\mathbf{V}}\, \mathbf{G} = 0 \end{array} \tag{38}$$

holds in the sense of distributions. Besides (21-24), there is no other C^∞ vector field for which (37) are satisfied. It should be noted that both diagrams, (4) and (38), hold under γ-dependent smooth diffeomorphisms, whenever in the limit $\gamma \to 0$ these remain bounded in order to avoid different identifications of points in the manifold under these diffeomophisms.

IV. A DOMAIN WALL EMBEDDED IN AN ANTI-DE SITTER SPACETIME

Consider the spacetime $(\mathbb{R}^4,\, {}^\gamma\mathbf{g})$ where the metric tensor is given by

$${}^\gamma\mathbf{g} = \cosh^{-2\gamma}\frac{\beta x}{\gamma}\,(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z) + \mathbf{d}x\,\mathbf{d}x, \tag{39}$$

with β and γ constants and $\gamma > 0$. This is a thick domain wall spacetime, solution to the coupled Einstein-scalar field equations (15-16) with $\Lambda = -3\beta^2$ and

$$\phi = \phi_0 \tan^{-1}(\sinh(\beta x/\gamma)), \qquad \phi_0 \equiv \sqrt{\gamma/4\pi} \tag{40}$$

and

$$V(\phi) = \frac{1}{8\pi}\beta^2 \left(3 + \frac{1}{\gamma}\right) \cos^2(\phi/\phi_0). \tag{41}$$

This spacetime behaves asimptotically (i.e. far away of the wall) as an anti de Sitter spacetime [24] and its five dimensional analogue provides a thick domain wall version [22] of the original Randall-Sundrum scenario [13]. Here, γ plays the role of the wall's thickness and the distributional $\gamma \to 0$ limit of this geometry has been analyzed in reference [24].

There are six Killing vector fields on ($\mathbb{R}^4$, ${}^\gamma\mathbf{g}$), given by

$$\mathbf{V}_1 = \boldsymbol{\partial}_y, \qquad \mathbf{V}_2 = \boldsymbol{\partial}_z, \qquad \mathbf{V}_3 = z\boldsymbol{\partial}_y - y\boldsymbol{\partial}_z, \tag{42}$$

$$\mathbf{V}_4 = \boldsymbol{\partial}_t, \qquad \mathbf{V}_5 = t\boldsymbol{\partial}_y + y\boldsymbol{\partial}_t, \qquad \mathbf{V}_6 = z\boldsymbol{\partial}_t + t\boldsymbol{\partial}_z, \tag{43}$$

where $\mathbf{V}_4$ is a time translation and $\mathbf{V}_5$, $\mathbf{V}_6$ are Lorentz rotations.

The Ricci and Einstein tensor fields of (39) are given by

$${}^\gamma\mathbf{Ric} = \Lambda \tanh^2 \frac{\beta x}{\gamma}\, {}^\gamma\mathbf{g} + \frac{\beta^2}{\gamma} \cosh^{-2} \frac{\beta x}{\gamma} \left(\cosh^{-2\gamma} \frac{\beta x}{\gamma} \left(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z\right) + 3\mathbf{d}x\,\mathbf{d}x \right), \tag{44}$$

$${}^\gamma\mathbf{G} = -\Lambda \tanh^2 \frac{\beta x}{\gamma}\, {}^\gamma\mathbf{g} - \frac{2\beta^2}{\gamma} \cosh^{-2(\gamma+1)} \frac{\beta x}{\gamma} \left(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z\right). \tag{45}$$

The distributional thin wall limits of (39) and (44,45) are given by

$$\mathbf{g} \equiv \lim_{\gamma\to 0} {}^\gamma\mathbf{g} = \left(\Theta_x^- e^{2\beta x} + \Theta_x^+ e^{-2\beta x}\right) \left(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z\right) + \mathbf{d}x\,\mathbf{d}x\,, \tag{46}$$

$$\mathbf{Ric} \equiv \lim_{\gamma\to 0} {}^\gamma\mathbf{Ric} = \Lambda\,\mathbf{g} + 2\beta\,\delta_0 \left(-\mathbf{d}t\,\mathbf{d}t + 3\mathbf{d}x\,\mathbf{d}x + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z\right), \tag{47}$$

$$\mathbf{G} + \mathbf{g}\,\Lambda \equiv \lim_{\gamma\to 0} ({}^\gamma\mathbf{G} + {}^\gamma\mathbf{g}\,\Lambda) = 4\beta\,\delta_0 \left(\mathbf{d}t\,\mathbf{d}t - \mathbf{d}y\,\mathbf{d}y - \mathbf{d}z\,\mathbf{d}z\right), \tag{48}$$

where **Ric** and **G** are the distribution-valued Ricci and Einstein curvatures of **g** [24]. The diagram (4), in which ${}^\gamma\mathbf{g}$ and **g** are given by (39) and (46), respectively, holds in the sense of distributions.

Note that **g** is piecewise smooth and that the pullback of **g** to the surface $x = 0$ is the same from both sides of this surface. Note also that the six vector fields (42-43) are also Killing vectors on the $(2+1)$-dimensional Minkowski spacetime $(\mathbb{R}^3, \bar{\mathbf{g}})$ with $\bar{\mathbf{g}}$ the Minkowski metric which appears as this pullback.

Next, let $\boldsymbol{\eta}$ be the Minkowski metric tensor (27) and let ∇ be the derivative operator in $\boldsymbol{\eta}$. Let **U** be a test 2-tensor field on $\mathbb{R}^4$. From (13), it is straightforward to verify that for the vector fields given by (42,43), we have

$$\mathcal{L}_\mathbf{V}\, {}^\gamma\mathbf{g}[\mathbf{U}] = 0\,, \qquad \mathcal{L}_\mathbf{V}\, \mathbf{g}[\mathbf{U}] = 0\,, \tag{49}$$

as expected. Hence, (42,43) are Killing vectors of $(\mathbb{R}^4, \mathbf{g})$ in the sense that $\mathcal{L}_\mathbf{V}\,\mathbf{g}$ along these vectors is the zero distribution on $\mathbb{R}^4$. Analogous computations show that

$$\mathcal{L}_\mathbf{V}\, {}^\gamma\mathbf{Ric}[\mathbf{U}] = 0\,, \qquad \mathcal{L}_\mathbf{V}\, {}^\gamma\mathbf{G}[\mathbf{U}] = 0 \tag{50}$$

and

$$\mathcal{L}_\mathbf{V}\, \mathbf{Ric}[\mathbf{U}] = 0\,, \qquad \mathcal{L}_\mathbf{V}\, \mathbf{G}[\mathbf{U}] = 0\,. \tag{51}$$

It follows that for this spacetime the diagram (38) holds in the sense of distributions. Besides (42,43), there is no other C^∞ vector field for which (50, 51) are satisfied on $\mathbb{R}^4$.

Next, let $\mathcal{M}^+ \equiv \{(t,x,y,z) \in \mathbb{R}^4,\, x>0\}$ and let $\mathbf{U}^+$ be a test tensor of compact support $\mathcal{K} \subset \mathcal{M}^+$. From (12) we find that

$$\mathcal{L}_{\mathbf{V}}\mathbf{g}[\mathbf{U}^+] = 0 \tag{52}$$

is satisfied along the C^∞ vector fields on $\mathbb{R}^4$ given by

$$\begin{aligned}
\mathbf{V}_7 &= \boldsymbol{\partial}_x + \beta(t\boldsymbol{\partial}_t + y\boldsymbol{\partial}_y + z\boldsymbol{\partial}_z), && (53)\\
\mathbf{V}_8 &= \beta t\mathbf{V}_7 + \left(\frac{1}{2}\beta^2(-t^2+y^2+z^2) + \frac{1}{8}\,e^{2\beta x}\right)\boldsymbol{\partial}_t, && (54)\\
\mathbf{V}_9 &= \beta y\mathbf{V}_7 - \left(\frac{1}{2}\beta^2(-t^2+y^2+z^2) + \frac{1}{8}\,e^{2\beta x}\right)\boldsymbol{\partial}_y, && (55)\\
\mathbf{V}_{10} &= \beta z\mathbf{V}_7 - \left(\frac{1}{2}\beta^2(-t^2+y^2+z^2) + \frac{1}{8}\,e^{2\beta x}\right)\boldsymbol{\partial}_z. && (56)
\end{aligned}$$

It follows that $\mathcal{L}_{\mathbf{V}}\mathbf{g}$ is the zero distribution on $\mathcal{M}^+$, along these vector fields.

Now, let $\mathbf{U}$ be a test tensor on $\mathbb{R}^4$ and let us consider the Lie derivatives (in the sense of distributions) of (39) and (46) along the vector fields (53-56). Since ${}^\gamma\mathbf{g}$ and $\nabla\,{}^\gamma\mathbf{g}$ converge locally to $\mathbf{g}$ and $\nabla\mathbf{g}$ for $\gamma \to 0$ [24], it follows directly that $\mathcal{L}_{\mathbf{V}}\,{}^\gamma\mathbf{g}$ also converge locally to $\mathcal{L}_{\mathbf{V}}\mathbf{g}$ along the smooth vector fields (53-56). Thus, for $\mathbf{V}_7$ we have

$$\mathcal{L}_{\mathbf{V_7}}\,{}^\gamma\mathbf{g} = 2\beta\cosh^{-2\gamma}\frac{\beta x}{\gamma}(1-\tanh\frac{\beta x}{\gamma})(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z), \tag{57}$$

$$\mathcal{L}_{\mathbf{V_7}}\mathbf{g} = \Theta_x^-\,4\beta\,e^{2\beta x}(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z), \tag{58}$$

that satisfy

$$\lim_{\gamma\to 0}\mathcal{L}_{\mathbf{V_7}}\,{}^\gamma\mathbf{g}[\mathbf{U}] = \mathcal{L}_{\mathbf{V_7}}\mathbf{g}[\mathbf{U}]. \tag{59}$$

The fact that $\mathcal{L}_{\mathbf{V_7}}\mathbf{g}$ is the zero distribution on $\mathcal{M}^+$ may be interpreted naturally as a Killing symmetry of $\mathbf{g}$ on $\mathcal{M}^+$ generated by $\mathbf{V}_7$. Since this symmetry arises in the limit $\gamma \to 0$ of ${}^\gamma\mathbf{g}$, $\mathbf{V}_7$ is an *asymptotic* Killing vector field on $(\mathbb{R}^4,\,{}^\gamma\mathbf{g})$. The same considerations holds for the other C^∞ vector fields (54-56). Further, since $(\mathbb{R}^4, \mathbf{g})$ is reflection symmetric along the direction perpendicular to the wall, a symmetry inherited from $(\mathbb{R}^4,\,{}^\gamma\mathbf{g})$, the above considerations can be extended to $\mathcal{M}^- \equiv \{(t,x,y,z)\in\mathbb{R}^4,\, x<0\}$ under the replacement $x \to -x$. Indeed, these results are by no means unexpected, they simply put in a rigorous setting the emergence of additional symmetries in the $\gamma \to 0$ limit of the spacetime $(\mathbb{R}^4,\,{}^\gamma\mathbf{g})$, where we have an absolute control over what is going on.

For the sake of completeness, let us analyze the action of $\mathcal{L}_{\mathbf{V}}$ on the distribution-valued curvature tensors of the metric (46) along the vector fields (53-56). Let us consider again the vector field $\mathbf{V}_7$. We find

$$\mathcal{L}_{\mathbf{V_7}}\mathbf{Ric} = 2\beta(\delta_0' - 2\beta\delta_0)\,(-\mathbf{d}t\,\mathbf{d}t + 3\mathbf{d}x\,\mathbf{d}x + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z) - \Theta_x^-\,12\beta^3\,e^{2\beta x}(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z) \tag{60}$$

and

$$\mathcal{L}_{\mathbf{V_7}}(8\pi\mathbf{T}) \equiv \mathcal{L}_{\mathbf{V_7}}(\mathbf{G} + \Lambda\mathbf{g}) = 4\beta(\delta_0' + 2\beta\delta_0)\,(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\,. \tag{61}$$

Indeed, $\mathbf{V}_7$ is neither a Ricci collineation nor a matter collineation and the same conclusion extends to the vector fields (54-56). In particular, this shows explicitly that the distributional energy momentum tensor of the brane does not inherit all the symmetries of the bulk.

V. AN ASYMMETRIC DOMAIN WALL SPACETIME

Let us now to consider the spacetime $(\mathbb{R}^4,\,{}^\gamma\mathbf{g})$ where the C^∞ metric tensor ${}^\gamma\mathbf{g}$ is given by

$${}^\gamma\mathbf{g} = \cosh^{-2\gamma/3}\frac{\beta x}{\gamma}e^{-4\beta x/3}\left(-\mathbf{d}t\,\mathbf{d}t + e^{2\beta x}(\mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\right) + \cosh^{-2\gamma}\frac{\beta x}{\gamma}\mathbf{d}x\,\mathbf{d}x, \tag{62}$$

with β and γ constants and $0<\gamma<1$. This represents a two-parameter family of plane symmetric static domain wall spacetimes in which the reflection symmetry along the direction perpendicular to the wall has been relaxed, being asymptotically (i.e far away of the wall) flat for $x>0$ and behaving asymptotically as the Taub spacetime for $x<0$ [20]. The metric (62) is solution to the coupled Einstein-scalar field equations (15-16) with $\Lambda=0$ and

$$\phi=\phi_0\tan^{-1}(\sinh(\beta x/\gamma), \qquad \phi_0\equiv\frac{1}{6}\sqrt{3\gamma(1-\gamma)/\pi} \tag{63}$$

and

$$V(\phi)=\frac{1}{24\pi}\beta^2\frac{1}{\gamma}\cos^{2(1-\gamma)}(\phi/\phi_0), \tag{64}$$

where γ plays the role of the wall's thickness. The distributional $\gamma\to 0$ limit of this geometry has been analyzed in reference [25]. It should be noted that (62) does not inherit the Z_2 symmetry of $V(\phi)$, a fact that makes very interesting the analysis of the symmetries of this spacetime.

The Ricci and Einstein tensor fields of (62) are given by

$${}^{\gamma}\mathbf{Ric}=\frac{\beta^2}{3\gamma}(\cosh\frac{\beta x}{\gamma})^{-2(1-2\gamma/3)}\left(-e^{-4\beta x/3}\mathbf{d}t\,\mathbf{d}t+e^{2\beta x/3}(\mathbf{d}y\,\mathbf{d}y+\mathbf{d}z\,\mathbf{d}z)\right)+\frac{\beta^2}{3\gamma}(3-2\gamma)\cosh^{-2}\frac{\beta x}{\gamma}\mathbf{d}x\,\mathbf{d}x, \tag{65}$$

$${}^{\gamma}\mathbf{G}=\frac{\beta^2}{3\gamma}(\cosh\frac{\beta x}{\gamma})^{-2(1-2\gamma/3)}(2-\gamma)\left(e^{-4\beta x/3}\mathbf{d}t\,\mathbf{d}t-e^{2\beta x/3}(\mathbf{d}y\,\mathbf{d}y+\mathbf{d}z\,\mathbf{d}z)\right)-\frac{\beta^2}{3}\cosh^{-2}\frac{\beta x}{\gamma}\mathbf{d}x\,\mathbf{d}x, \tag{66}$$

and the distributional thin wall limits of (62,65,66) are given by [25]

$$\begin{aligned}\mathbf{g}\equiv\lim_{\gamma\to 0}{}^{\gamma}\mathbf{g} \;&=\; \Theta_x^-\left(-e^{-2\beta x/3}\mathbf{d}t\,\mathbf{d}t+e^{2\beta x}\mathbf{d}x\,\mathbf{d}x+e^{4\beta x/3}(\mathbf{d}y\,\mathbf{d}y+\mathbf{d}z\,\mathbf{d}z)\right)\\ &\quad+\Theta_x^+\left(-e^{-2\beta x}\mathbf{d}t\,\mathbf{d}t+e^{-2\beta x}\mathbf{d}x\,\mathbf{d}x+\mathbf{d}y\,\mathbf{d}y+\mathbf{d}z\,\mathbf{d}z\right),\end{aligned} \tag{67}$$

$$\mathbf{Ric}\equiv\lim_{\gamma\to 0}{}^{\gamma}\mathbf{Ric}=\frac{1}{3}\beta\,\delta_0\left(-\mathbf{d}t\,\mathbf{d}t+3\mathbf{d}x\,\mathbf{d}x+\mathbf{d}y\,\mathbf{d}y+\mathbf{d}z\,\mathbf{d}z\right), \tag{68}$$

$$\mathbf{G}\equiv\lim_{\gamma\to 0}{}^{\gamma}\mathbf{G}=\frac{2}{3}\beta\,\delta_0\left(\mathbf{d}t\,\mathbf{d}t-\mathbf{d}y\,\mathbf{d}y-\mathbf{d}z\,\mathbf{d}z\right). \tag{69}$$

Note that $\mathbf{g}$ is piecewise smooth and that the pullback of $\mathbf{g}$ to the surface $x=0$ is the same from both sides and coincides with the ordinary $(2+1)$-dimensional Minkowski metric. Like its smoothed version (62), $\mathbf{g}$ is not reflection symmetric along the coordinate perpendicular to the wall. The spacetime $(\mathbb{R}^4,\mathbf{g})$ for $x>0$ is isometric to the ordinary $(3+1)$-dimensional Minkowski spacetime, while for $x<0$ it is the Taub spacetime [20, 25].

There are only four independent Killing vector fields on $(\mathbb{R}^4,\,{}^{\gamma}\mathbf{g})$ and these are given by

$$\mathbf{V}_1=\boldsymbol{\partial}_y, \qquad \mathbf{V}_2=\boldsymbol{\partial}_z, \qquad \mathbf{V}_3=z\boldsymbol{\partial}_y-y\boldsymbol{\partial}_z, \qquad \mathbf{V}_4=\boldsymbol{\partial}_t. \tag{70}$$

Let ∇ be the derivative operator in the ordinary Minkowski metric tensor (27). Let $\mathbf{U}$ be a test 2-tensor field on $\mathbb{R}^4$. From (13) it is straightforward to show that the four vectors given in (70) generate also isometries of (67) and are improper collineations for the distributional Ricci and Einstein curvature tensors. It follows that for this spacetime the diagram (38) holds in the sense of distributions.

Next, consider the C^∞ vector fields given by

$$\mathbf{V}_5=t\boldsymbol{\partial}_y+y\boldsymbol{\partial}_t, \qquad \mathbf{V}_6=z\boldsymbol{\partial}_t+t\boldsymbol{\partial}_z\,. \tag{71}$$

We have

$$\begin{aligned}\mathcal{L}_{\mathbf{V_5}}\,\mathbf{g}[\mathbf{U}] \;&\equiv\; -\mathbf{g}[\mathcal{L}_{\mathbf{V_5}}\mathbf{U}+(\nabla\cdot\mathbf{V_5})\mathbf{U}]\\ &=\; -\int_{\mathbb{R}^4}\mathbf{g}\cdot(\mathcal{L}_{\mathbf{V_5}}\mathbf{U}+\mathbf{U}(\nabla\cdot\mathbf{V_5}))\boldsymbol{\omega}_\eta\\ &=\; -\int_{x<0}(e^{-2\beta x/3}-e^{4\beta x/3})(U^{ty}+U^{yt})dt\,dx\,dy\,dz\\ &\quad\; -\int_{x>0}(e^{-2\beta x}-1)(U^{ty}+U^{yt})dt\,dx\,dy\,dz,\end{aligned} \tag{72}$$

where we have integrated by parts and used the fact that $\mathbf{U}$ is of compact support. It follows

$$\mathcal{L}_{\mathbf{V_5}}\,\mathbf{g} = 2\left(\Theta_x^- e^{\beta x/3} + \Theta_x^+ e^{-\beta x}\right)\sinh\beta x\,(\mathbf{d}t\,\mathbf{d}y + \mathbf{d}y\,\mathbf{d}t). \tag{73}$$

On the other hand, as can be guessed from the explicit form of $\mathbf{Ric}$, we have

$$\begin{aligned}\mathcal{L}_{\mathbf{V_5}}\,\mathbf{Ric}[\mathbf{U}] &\equiv -\mathbf{Ric}[\mathcal{L}_{\mathbf{V_5}}\,\mathbf{U} + (\nabla\cdot\mathbf{V_5})\mathbf{U}] \\ &= -\frac{1}{3}\beta\int_{x=0}(y\partial_t + t\partial_y)(U^{tt} + 3U^{xx} + U^{yy} + U^{zz})\,dt\,dy\,dz \\ &= 0,\end{aligned} \tag{74}$$

where in last step we have integrated by parts and used again the fact that $\mathbf{U}$ is of compact support. Analogous computations show that

$$\mathcal{L}_{\mathbf{V_6}}\,\mathbf{g} = 2\left(\Theta_x^- e^{\beta x/3} + \Theta_x^+ e^{-\beta x}\right)\sinh\beta x\,(\mathbf{d}t\,\mathbf{d}z + \mathbf{d}z\,\mathbf{d}t), \tag{75}$$

$$\mathcal{L}_{\mathbf{V_6}}\,\mathbf{Ric} = 0 \tag{76}$$

Hence, although V_5 and V_6 are not Killing vectors of $(\mathbb{R}^4, \mathbf{g})$, they are Ricci collineations. Further,

$$\mathcal{L}_{\mathbf{V_5}}\,\mathbf{G} = 0, \qquad \mathcal{L}_{\mathbf{V_6}}\,\mathbf{G} = 0. \tag{77}$$

and V_5 and V_6 are also Einstein collineations. Indeed, the above expressions should be understood in the sense of distributions. Besides (70,71), there is no other C^∞ vector field that generates symmetries of $\mathbf{Ric}$ and $\mathbf{G}$.

It should be noted that $\mathcal{L}_{\mathbf{V_5}}\,\mathbf{g}$ and $\mathcal{L}_{\mathbf{V_6}}\,\mathbf{g}$ are piecewise smooth and that the pullbacks of $\mathcal{L}_{\mathbf{V_5}}\,\mathbf{g}$ and $\mathcal{L}_{\mathbf{V_6}}\,\mathbf{g}$ to the surface $x = 0$ are the same from both sides of the wall and vanish. It follows that the symmetries of the distribution-valued curvature tensors $\mathbf{Ric}$ and $\mathbf{G}$ (which are supported on the surface $x = 0$) coincide with the Killing symmetries of the pullback of $\mathbf{g}$ to this surface. For the sake of completeness, let us show that the same result is also reached within the standard thin shell formalism [7, 31]. For $(\mathbb{R}^4, \mathbf{g})$ with $\mathbf{g}$ given by (67), the pullback $\bar{\mathbf{g}}$ of $\mathbf{g}$ to the surface $x = 0$ is the same from both sides and is given by

$$\bar{\mathbf{g}} = (-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\,. \tag{78}$$

The extrinsic curvature tensor $\mathbf{K}$ of the surfaces $x = x_0$ as submanifolds of $(\mathbb{R}^4, \mathbf{g})$, is the C^∞ regularly discontinuous tensor across the surface $x = 0$ given by

$$\mathbf{K} = -\frac{1}{3}\beta\Theta_x^-\left(e^{-2\beta x_0/3}\mathbf{d}t\,\mathbf{d}t + 2e^{-4\beta x_0/3}(\mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\right) - \beta\Theta_x^+ e^{-2\beta x_0}\mathbf{d}t\,\mathbf{d}t\,. \tag{79}$$

Thus, the discontinuity $[[\mathbf{K}]] \equiv \mathbf{K}|_{x_0=0^+} - \mathbf{K}|_{x_0=0^-}$ of $\mathbf{K}$ across the surface $x = 0$, which is declared as a purely intrinsic property of this surface, is the ordinary C^∞ tensor field defined on the surface $x = 0$ given by

$$[[\mathbf{K}]] = \frac{2}{3}\beta\,(-\mathbf{d}t\,\mathbf{d}t + \mathbf{d}y\,\mathbf{d}y + \mathbf{d}z\,\mathbf{d}z)\,. \tag{80}$$

Therefore, the symmetries of $[[\mathbf{K}]]$ are the symmetries of the pullback $\bar{\mathbf{g}}$ of $\mathbf{g}$ to the surface $x = 0$. Now, we have

$$\mathbf{Ric} = \frac{1}{2}\delta_0\left([[\mathbf{K}]] + (\bar{\mathbf{g}}^{-1}\cdot[[\mathbf{K}]])\,\mathbf{d}x\,\mathbf{d}x\right)\,, \tag{81}$$

from which it follows that the symmetries of the distributional Ricci tensor of $\mathbf{g}$ turns out to be the Killing symmetries of the induced metric $\bar{\mathbf{g}}$ on the surface $x = 0$ where the 2-brane is localized. It should be noted that this is not an example of the well known (trivial) symmetry inheritance. Although, the pullback $\bar{\mathbf{g}}$ of $\mathbf{g}$ to the surface $x = 0$ acts a metric on this surface (as follows from the fact that the surface $x = 0$ has a well-defined intrinsic geometry) and therefore the Killing vectors of this pullback are naturally collineations of the curvature tensors of this pullback, $\mathbf{Ric}$ given by (68) or (81) is not the Ricci tensor of $\bar{\mathbf{g}}$. The same conclusions can be extended also to all the thin domain wall geometries considered in the previous sections.

Remarkably enough, the asymmetric geometry considered here, explicitly shows that the Killing symmetries of the pullback of the metric tensor to the surface where the thin wall is localized may form a larger group than the group of Killing symmetries which are common to the ambient spacetime on both sides of the wall.

VI. SUMMARY AND DISCUSSION

In this work, by generalizing the Lie derivative of smooth tensor fields to distribution-valued tensors, we defined and computed the Killing symmetries and the Ricci and Einstein collineations of some distributional domain wall geometries for which the diagram (4), with ${}^{\gamma}\mathbf{g}$ and $\mathbf{g}$ the distribution valued metric tensors of the thick and thin domain wall spacetimes respectively, holds rigourously in the sense of distributions. For all the geometries considered, the distribution-valued curvature tensors of the thin wall limit have singular parts proportional to a Dirac distribution supported on the surface Σ where the thin wall is localized. We found that the Killing symmetries of the distributional geometry of the thin wall spacetime $(\mathbb{R}^4, \mathbf{g})$ are the Killing symmetries of the smooth thick domain wall spacetime $(\mathbb{R}^4, {}^{\gamma}\mathbf{g})$ and that, besides these, there are no other isometries. However, as expected, the thin wall geometry may shows additional symmetries on the open disjoint sets $\mathcal{M}^+$ and $\mathcal{M}^-$, that admit Σ as a boundary, which are not isometries inherited from the corresponding smooth geometry. These symmetries are the asymptotic (i.e. far away of the wall) Killing symmetries of $(\mathbb{R}^4, {}^{\gamma}\mathbf{g})$.

For the thin domain walls with reflection symmetry of sections III and IV, the Killing vectors of $(\mathbb{R}^4, \mathbf{g})$ are the only symmetries of the corresponding distribution-valued Ricci and Einstein tensors. Therefore the Ricci and Einstein collineations of these thin wall geometries are improper. For the asymmetric thin domain wall of section V, we found that the collineations of the distributional Ricci and Einstein tensors form a larger group than the one formed by the Killing symmetries of the corresponding spacetime $(\mathbb{R}^4, \mathbf{g})$. The additional symmetries are then proper Ricci and Einstein collineations. Finally, for all the thin wall spacetimes $(\mathbb{R}^4, \mathbf{g})$ considered, the symmetries of the distributional curvature tensors turns out to be the Killing symmetries of $(\Sigma, \bar{\mathbf{g}})$, where $\bar{\mathbf{g}}$ is the pullback of $\mathbf{g}$ to Σ.

Although we have restricted ourselves to consider four-dimensional domain wall spacetimes, these models are straightforwardly generalized to D-dimensional domain wall spacetimes [25]. On the other hand, the analysis presented here can be carried out, in principle, for all the distribution-valued curvature tensors of a spacetime $(\mathcal{M}, \mathbf{g})$, whenever the distribution-valued metric tensor $\mathbf{g}$ is a regular metric in the sense of reference [6]. This generalization and its implications will be discussed in a forthcoming paper.

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