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LETTER TO THE EDITOR

Self-similar static solutions admitting a 2-space of constant curvature

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Abstract. A recent result by Haggag and Hajj-Boutros is reviewed within the framework of self-similar spacetimes, extending, in some sense, their results and presenting a family of metrics consisting of all the static spherically symmetric perfect fluid solutions admitting a homothety.

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In a recent paper, Haggag and Hajj-Boutros [1] presented a static, spherically symmetric perfect fluid solution with a stiff-matter-type equation of state (i.e. $p = \mu$). By means of a few clever changes of coordinates, the authors reduce the problem to that of solving a non-linear, second-order differential equation, whose polynomic solutions they investigate showing that only three such solutions exist, two of them being vacuum (flat Minkowski spacetime and Schwarzschild solution) and the third one leading to the new metric referred to above, henceforth called the HHB solution.

The purpose of this letter is to give all the static, spherically symmetric perfect fluid solutions admitting a homothety. This family can be completely characterized by means of a real parameter γ (arising quite naturally from the equation of state for these fluids, see below), which must be in the interval $[1, 2]$ in order to satisfy energy conditions. The two limiting values of γ, namely $\gamma = 1$ and $\gamma = 2$ correspond to Minkowski flat spacetime and to the HHB solution, respectively.

A few remarks concerning the similarity group and its action are in order here. It is a well known fact that an r-parameter group of homotheties H_r (in which at least one proper homothety exists) always admits an $(r-1)$-parameter subgroup of isometries G_{r-1}. Now, the maximal dimension of the group of homotheties that a perfect fluid spacetime may admit is $r = 7$, in which case it is one of the special Robertson–Walker spacetimes [2], and therefore they are all known. The case $r = 6$ is not compatible with an energy–momentum tensor of the perfect-fluid type; thus, apart from the special Robertson–Walker solutions mentioned above, the highest dimension of the group of homotheties that a perfect fluid spacetime may admit is $r = 5$. In such case, the associated isometry subgroup G_4 has necessarily three-dimensional non-null orbits [2]. Notice that this is precisely the case we are interested in. We shall not treat here the case in full generality, namely; studying all perfect fluid spacetimes admitting an H_5 of homotheties, since this would be beyond the purpose of this letter, but we shall restrict ourselves to the case when the subgroup G_4 has time-like orbits T_3 and the subgroup G_3 that it necessarily contains [3] has two-dimensional orbits. Everything else follows from these assumptions and the field equations. For further information on groups of homotheties and related issues, we refer the reader to [4–6].

We start with a spacetime that contains a non-null 2-space of constant curvature (i.e. there exists a three-parameter isometry group G_3 acting on this 2-space). In this case the orbits V_2 admit orthogonal surfaces in M [7]. By performing a coordinate transformation in the 2-spaces orthogonal to the Killing orbits the spacetime metric can be put into diagonal form:

$$\mathrm{d}s^2 = A^2(r,t)\big(-\mathrm{d}t^2 + \mathrm{d}r^2\big) + B^2(r,t)\big(\mathrm{d}\theta^2 + f^2(\theta,k)\,\mathrm{d}\phi^2\big) \tag{1}$$

$$f(\theta,k) = \begin{cases} \sin\theta & k=+1 \\ \theta & k=0 \\ \sinh\theta & k=-1 \end{cases} \tag{2}$$

where we have restricted ourselves to the case of space-like Killing orbits, since perfect fluid and dust solutions cannot admit a group G_3 on two-dimensional time-like orbits [3].

Using the Jacobi identities and the fact that the Lie bracket of a proper homothetic vector field (HVF) and a Killing vector (KV) is a KV it can be easily shown that the HVF X must be of either one of the following forms:

$$\text{(i)}\quad X = X^t(r,t)\partial_t + X^r(r,t)\partial_r \qquad k=-1,0,1 \tag{3}$$

$$\text{(ii)}\quad X = X^t(r,t)\partial_t + X^r(r,t)\partial_r - \theta\partial_\theta \qquad k=0\,. \tag{4}$$

Now by using isotropic coordinates, one finds that static metrics can be expressed as [3]

$$\mathrm{d}s^2 = -A^2(r)\,\mathrm{d}t^2 + B^2(r)\big[\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + f^2(\theta,k)\,\mathrm{d}\phi^2\big)\big] \tag{5}$$

where ∂_t is the hypersurface orthogonal time-like KV.

In this coordinate chart, the HVF in (3) and (4) takes the following forms:

$$\text{(i)}\quad X = nt\partial_t + R(r)\partial_r \qquad k=-1,0,1 \tag{6}$$

$$\text{(ii)}\quad X = nt\partial_t + R(r)\partial_r - \theta\partial_\theta \qquad k=0 \tag{7}$$

where n is a constant.

The homothetic equation $\mathcal{L}_X g_{ab} = 2g_{ab}$ specified to the components rr and $\theta\theta$ of the metric (5), gives

$$R_{,r} - \frac{R}{r} - X^\theta{}_{,\theta} = 0 \tag{8}$$

and integrating, one gets

$$\text{(i)}\quad X = nt\partial_t + qr\partial_r \qquad k=-1,0,1 \tag{9}$$

$$\text{(ii)}\quad X = nt\partial_t + (-r\ln r + cr)\partial_r - \theta\partial_\theta \qquad k=0 \tag{10}$$

where c and q ($\neq 0$) are constants.

Case (i). By means of the coordinate transformation $\hat{r} = r^{1/q}$, the HVF and the metric can be written as

$$X = nt\partial_t + \hat{r}\partial_{\hat{r}} \tag{11}$$

$$\mathrm{d}s^2 = -\hat{A}^2(\hat{r})\,\mathrm{d}t^2 + \hat{B}^2(\hat{r})\big[q^2\,\mathrm{d}\hat{r}^2 + \hat{r}^2\big(\mathrm{d}\theta^2 + f^2(\theta,k)\,\mathrm{d}\phi^2\big)\big]\,. \tag{12}$$

The metric functions can be determined via the homothetic equations, which gives

$$\hat{B} = \text{constant} \qquad \hat{A} \propto \hat{r}^{1-n}\,. \tag{13}$$

Defining a new radial coordinate r as $r = \hat{r}\hat{B}$, one can come to the following simple forms for X and the metric

$$X = nt\partial_t + r\partial_r \tag{14}$$

$$ds^2 = -r^{2(1-n)}dt^2 + q^2dr^2 + r^2\left(d\theta^2 + f^2(\theta, k)\, d\phi^2\right). \tag{15}$$

Case (ii). Imposition of the homothetic equations specified to the metric (5) and to the HVF (10), leads directly to

$$ds^2 = -(-\ln r + c)^{2(n-1)}\, dt^2 + \frac{b^2}{r^2(-\ln r + c)^4}\left[dr^2 + r^2\left(d\theta^2 + \theta^2\, d\phi^2\right)\right] \tag{16}$$

where b and c are constants.

From the expressions (15) and (16) of the metric, it is immediate to see that the components tt of their respective Einstein tensors are negative for $k = -1$ and 0 (i.e. hyperbolic and flat 2-spaces) and therefore cannot verify energy conditions. Thus, it only remains to study the spherically symmetric case.

In this latter case, the field equations for a perfect fluid matter content lead to the metric

$$ds^2 = -r^{4-4/\gamma}\, dt^2 + \left(\frac{\gamma^2 + 4\gamma - 4}{\gamma^2}\right) dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\phi^2\right). \tag{17}$$

These metrics have already been found, following a completely different approach, by Ibañez and Sanz [9] and particular cases of them can also be found in Misner and Zapolsky [10] (which are particular cases of Tolman class VI solutions). Some particular cases (when the HVF is orthogonal to the fluid flow) were also studied by Herrera *et al* [11]. The matter variables are

$$\mu = \frac{1}{r^2}\left(\frac{4\gamma - 4}{\gamma^2 + 4\gamma - 4}\right) \tag{18}$$

$$p = (\gamma - 1)\mu \tag{19}$$

as one would have expected from p and μ being functions of r alone (and therefore, by the implicit function theorem, the fluid has a barotropic equation of state) and the spacetime being self-similar [8]. The HVF takes then the form

$$X = \frac{2-\gamma}{\gamma}t\partial_t + r\partial_r\,. \tag{20}$$

These are *all the static, spherically-symmetric self-similar perfect fluid solutions*. They are shear-free and have null volume expansion since the 4-velocity u of the fluid is parallel to the time-like KV, the vorticity is also zero (since u is orthogonal to the orbits S_2 of the G_3 they contain); and the fluid has non-geodesic flow.

The particular case, $\gamma = 2$, is the HHB solution [1] and in this case the HVF X becomes orthogonal to the fluid 4-velocity, and for $\gamma = 1$ the spacetime is obviously flat.

It is interesting to notice that static and self-similar solutions admitting a 2-space of constant curvature can only be spherically symmetric (irrespectively of the matter content) and that an H_5 static spacetime with a G_3 acting on space-like orbits, necessarily contains an H_4.

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