Parte II

Dualidad y D-branas

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Capítulo

7

Bosonic String

7.1 Classical Theory

Let us start considering classical bosonic string theory in flat Minkowski spacetime[α]. The dynamics is obtained from the condition of minimizing the area of the sheet embedded in Minkowski spacetime, obtained as the string evolves. The Nambu-Goto action [5] is then simply proportional to the area of the worldsheet,

$$S = -T \int d\sigma d\tau \sqrt{- \det \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}}. \qquad (7.1)$$

However, (7.1) is extremely nonlinear; a more convenient action can be obtained if, in addition to $X^\mu(\sigma,\tau)$, a new variable $h^{\alpha\beta}$, which will be the metric tensor on the string worldsheet, is introduced. Then, the physical system is described by [6], [7]

$$S = -\frac{T}{2} \int d\sigma d\tau \sqrt{h} h^{\alpha\beta} \partial_\alpha X \partial_\beta X, \qquad (7.2)$$

The equations of motion, with respect to $h^{\alpha\beta}$, imply that

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}} = 0. \qquad (7.3)$$

[α]A number of excellent books and reviews is available on the subject. See for instance references [1] to [4].

The parameter T in (7.1) has units of squared mass, and can be identified with the string tension,

$$T = \frac{1}{2\pi\alpha'}. \tag{7.4}$$

The action (7.2) is invariant under the reparametrizations

$$\begin{aligned} \delta X^\mu &= \xi^\alpha \partial_\alpha X^\mu, \\ \delta h^{\alpha\beta} &= \xi^\gamma \partial_\gamma h^{\alpha\beta} - \partial_\gamma \xi^\alpha h^{\gamma\beta} - \partial_\gamma \xi^\beta h^{\alpha\gamma}, \\ \delta(\sqrt{h}) &= \partial_\alpha(\xi^\alpha \sqrt{h}), \end{aligned} \tag{7.5}$$

and the Weyl scaling

$$\delta h^{\alpha\beta} = \Lambda h^{\alpha\beta}. \tag{7.6}$$

Using the Weyl and the reparametrization invariances of (7.2), three independent elements of $h^{\alpha\beta}$ can be fixed, to choose the gauge

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.7}$$

In this gauge, the Euler-Lagrange equations of motion for (7.2) become

$$\Box X = 0. \tag{7.8}$$

Defining light cone coordinates,

$$\begin{aligned} \sigma^- &= \tau - \sigma, \\ \sigma^+ &= \tau + \sigma, \end{aligned} \tag{7.9}$$

the generic solution to the wave equation (7.8) can be written as a sum of right and left moving modes,

$$X^\mu = X^\mu_R(\sigma^-) + X^\mu_L(\sigma^+). \tag{7.10}$$

In this coordinates vanishing of the energy-momentum tensor becomes

$$\begin{aligned}\dot{X}^\mu X'_\mu &= 0, \\ \dot{X}^2 + X'^2 &= 0,\end{aligned} \qquad (7.11)$$

where $\dot{X} \equiv \frac{\partial X}{\partial \tau}$ and $X' \equiv \frac{\partial X}{\partial \sigma}$.
However, choosing the conformal gauge (7.7) does not completely fix the gauge freedom. There is yet a residual invariance with respect to a combination of reparametrizations and Weyl scaling, satisfying

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda \eta^{\alpha\beta}. \qquad (7.12)$$

In terms of the light cone coordinates on the worldsheet defined in (7.9), the residual gauge transformations (7.12) become equivalent to generic reparametrizations, of the form

$$\begin{aligned}\sigma^+ &\to \tilde{\sigma}^+(\sigma^+), \\ \sigma^- &\to \tilde{\sigma}^-(\sigma^-)\end{aligned} \qquad (7.13)$$

or, equivalently,

$$\begin{aligned}\tau &\to \tilde{\tau} = \frac{1}{2}[\tilde{\sigma}^+(\tau+\sigma) + \tilde{\sigma}^-(\tau-\sigma)], \\ \sigma &\to \tilde{\sigma} = \frac{1}{2}[\tilde{\sigma}^+(\tau+\sigma) - \tilde{\sigma}^-(\tau-\sigma)].\end{aligned} \qquad (7.14)$$

¿From (7.14), it is clear that once we fix $\tilde{\tau}$, the coordinate $\tilde{\sigma}$ is completely determined. Moreover, from the first equation in (7.14), $\tilde{\tau}$ is a solution to the free massless wave equation,

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2}\right) \tilde{\tau} = 0, \qquad (7.15)$$

which is simply the equation of motion for the coordinate X^μ. Thus, a possible way to fix the residual gauge is identifying the time coordinate τ with some of the X^μ coordinates. In order to do that, let us introduce the light-cone coordinates,

$$\begin{aligned} X^\pm &= \frac{X^0 \pm X^1}{\sqrt{2}}, \\ X^\perp &= X^i, \end{aligned} \tag{7.16}$$

where $i = 2, \ldots, d-1$, with d the dimension of spacetime. In this coordinate system, X^+ is playing the role of a time, so that we can impose

$$X^+ = \mathcal{P}^+ \tau, \tag{7.17}$$

where $\mathcal{P}^+$ is some constant, with the physical meaning of momentum density in the + direction. This choice is known as the *light-cone gauge* (see, for instance, [8], [9]) and implies, as X^+ is independent of σ, that every point on the string is at the same value of the time, X^+. The constant momentum density $\mathcal{P}^+$ can be conveniently defined, in string units, as $\frac{1}{2\pi\sqrt{\alpha'}}$. Then, if p^+ is the total momentum in the + direction,

$$\int_0^{\sigma^{max}} \mathcal{P}^+ = p^+, \tag{7.18}$$

so that a condition on σ arises, $\sigma^{max} = 2\pi\sqrt{\alpha'}p^+$. Hence, the surface of evolution for open string processes in the light-cone is conveniently parametrized through

$$\begin{aligned} \tau_i &\leq \tau \leq \tau_f, \\ 0 &\leq \sigma \leq p_i^+ 2\pi\sqrt{\alpha'}; \end{aligned} \tag{7.19}$$

thus, if we have two incoming strings, with momentum p_1^+ and p_2^+, the σ interval must be divided into two pieces of length

$p_1^+ \ 2\pi\sqrt{\alpha'}$ and $p_2^+ \ 2\pi\sqrt{\alpha'}$, as shown in Figure 7.1. In this diagram, at a given value of time τ_0, there is only a single string, resulting from the joining of the two incoming strings.

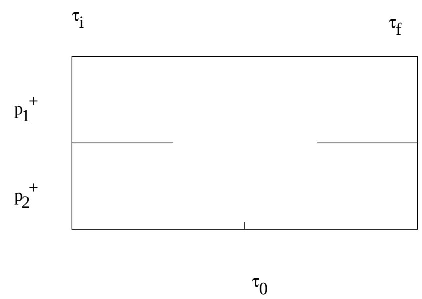

Figura 7.1: Diagram of two incoming strings

In the light-cone gauge, the constraint equations from the vanishing energy-momentum tensor, (7.11), become

$$\begin{aligned} X'^{-} &= \frac{1}{\mathcal{P}^{+}}(\dot{X}^{\perp}.\, X'^{\perp}), \\ \dot{X}^{-} &= \frac{1}{2\mathcal{P}^{+}}((\dot{X}^{\perp})^2 + (X'^{\perp})^2), \end{aligned} \tag{7.20}$$

which reduce the physical degrees of freedom to the transverse fluctuations of the string.

7.2 Closed Bosonic String

We will first work out the case of the closed bosonic string; in this case, we impose periodic boundary conditions,

$$X^{\mu}(\tau, \sigma) = X^{\mu}(\tau, \sigma + \pi). \tag{7.21}$$

The solution to (7.8), compatible with these boundary conditions, becomes

$$\begin{aligned} X_R^\mu &= \frac{1}{2}x^\mu + \frac{1}{2}(2\alpha')p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)}, \\ X_L^\mu &= \frac{1}{2}x^\mu + \frac{1}{2}(2\alpha')p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}. \end{aligned} \tag{7.22}$$

Using this Fourier decomposition we get, for the hamiltonian,

$$H = \frac{1}{2}\left[\sum_{-\infty}^{\infty}\alpha_{-n}\alpha_n + \sum_{-\infty}^{\infty}\tilde{\alpha}_{-n}\tilde{\alpha}_n\right], \tag{7.23}$$

where we have used the notation

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu. \tag{7.24}$$

In the light-cone coordinates (7.9), the constraint (7.11) becomes $T_{++} = T_{--} = 0$.

Let us now introduce the Virasoro generators L_n, $\tilde{L}_n$ as the Fourier modes of T_{++} and T_{--}, respectively,

$$\begin{aligned} L_n &= \int T_{--}e^{-2\pi in\sigma}d\sigma = \frac{1}{2}\sum_{m=-\infty}^{\infty}\alpha_{m-n}\alpha_n, \\ \tilde{L}_n &= \int T_{++}e^{2\pi in\sigma}d\sigma = \frac{1}{2}\sum_{m=-\infty}^{\infty}\tilde{\alpha}_{m-n}\tilde{\alpha}_n. \end{aligned} \tag{7.25}$$

In terms of these operators, we get

$$H = L_0 + \tilde{L}_0. \tag{7.26}$$

Using now the constraints (7.11), relation (7.26), and $p^\mu p_\mu = -M^2$, we can get the classical mass formula

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (\alpha_{-n}\alpha_n + \tilde{\alpha_{-n}}\tilde{\alpha_n}). \qquad (7.27)$$

The constraint (7.3) also implies that the left and right contributions to (7.27) are equal. Using the standard quantization rules,

$$\begin{aligned} [\alpha_m^\mu, \tilde{\alpha}_n^\nu] &= 0, \\ [\alpha_m^\mu, \alpha_n^\nu] &= m\delta_{m+n}\eta^{\mu\nu}, \\ [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] &= m\delta_{m+n}\eta^{\mu\nu}, \\ [x^\mu, p^\nu] &= i\eta^{\mu\nu}, \end{aligned} \qquad (7.28)$$

we can promote L_n and $\tilde{L}_n$ to operators on the Fock-Hilbert space of oscillators. The only ambiguity related to normal ordering appears in the definition of L_0, leading to a free additive constant, a. The constraints $T_{++} = T_{--} = 0$ can be implemented at the quantum level by imposing

$$\begin{aligned} L_n|\psi> &= \tilde{L}_n|\psi>= 0, \\ (L_0 - a)|\psi> &= (\tilde{L}_0 - a)|\psi>= 0. \end{aligned} \qquad (7.29)$$

The quantum mass formula becomes

$$M^2 = -\frac{4}{\alpha'}a + \frac{4}{\alpha'}\sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n = -\frac{4a}{\alpha'} + \frac{4}{\alpha'}\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n. \quad (7.30)$$

Two things are left free in deriving (7.30), the constant a, defining the zero point energy, and the dimension of the target space. We can fix these constants by consistency of the quantum theory or, equivalently, imposing positivity on the metric for the Hilbert space of states satisfying (7.29). A different way to fix these constants is imposing Lorenz invariance in the light cone gauge, where physical degrees of freedom are reduced to transversal oscillations. The result, for the closed bosonic string, is that a should equal one and the number of dimensions should be 26. The origin of the critical dimension, 26, is clear in the modern covariant approach [6]. In fact, the algebra of Virasoro generators is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}D(m^3 - m), \qquad (7.31)$$

with D the dimension of spacetime. The integration over the worldsheet metric, in the covariant approach, can be done using the Faddeev-Popov trick [10], in terms of a ghost (b, c) system. The Virasoro algebra for the ghost system is

$$[L_m^g, L_n^g] = (m-n)L_{m+n}^g + \frac{1}{6}(m - 13m^3)\delta_{m+n}. \qquad (7.32)$$

The Virasoro generators for the ghost and matter system can be defined as

$$\tilde{L}_m = L_m + L_m^g - a\delta_m, \qquad (7.33)$$

where we have explicitly substracted the normal ordering constant. The algebra for $\tilde{L}_n$ is now given by

$$\begin{aligned}[\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{m+n} \\ &\quad + \left[\frac{D}{12}(m^3 - m) + \frac{1}{6}(m - 13m^3) + 2am\right] \\ &\quad \times \delta_{m+n}, \end{aligned} \qquad (7.34)$$

so that for $D = 26$ and $a = 1$ we recover the classical algebra of diffeomorphisms on the circle, Diff (S^1), with vanishing central extension.

¿From (7.30), we can easily deduce the spectrum of string states. First of all, we have a tachyon with no oscillator modes, and squared mass negative ($-\frac{4}{\alpha'}$). The massless modes are of the type

$$\alpha^{\mu}_{-1}\alpha^{\mu}_{-1}|0> . \qquad (7.35)$$

To discover the meaning of these modes, we can see the way they transform under $SO(24)$ in the light cone gauge; then, we get three different types of particles: gravitons, as the symmetric and traceless part in (7.35) transforms under $SO(24)$ as a spin two particle; a massless scalar for the trace part, called dilaton; and an antisymmetric second rank tensor, coming from the antisymmetric part.
In order to define string amplitudes we need to introduce vertex operators [11] for the emission or absorption of string states. For a generic string state, $|\Lambda>$, the corresponding vertex operator is given by

$$V_{\Lambda} = \int d^2\sigma \sqrt{h} F_{\Lambda}(\sigma) e^{ik^{\mu}X_{\mu}(\sigma)}, \qquad (7.36)$$

where $k^2 = -M^2$ the mass of the string state $|\Lambda>$, and with $F_{\Lambda}(\sigma)$ restricted by imposing invariance of V_{Λ} under scale transformations on the worldsheet. This amounts to requiring $F_{\Lambda}(\sigma)e^{ik.X}$ having conformal dimension equal two (it transforms under $\sigma \to \lambda\sigma$ like $V \to \lambda^{-2}V$). The vertex operators for the tachyon, graviton and antisymmetric tensor are, respectively, defined by

$$\begin{aligned} F_{tachyon} &= 1, \\ F_{graviton} &= \partial X^{\mu}\partial X_{\mu}, \\ F_{antisymmetric} &= \epsilon_{\alpha\beta}\partial^{\alpha}X^{\mu}\partial^{\beta}X_{\mu}. \end{aligned} \qquad (7.37)$$

A string vertex, representing the splitting of a closed string in two, will contribute to a three graviton amplitude, and should therefore be related to the gravitational constant κ. A generic closed string amplitude, with N external lines, and arbitrary genus g will then be proportional to

$$\kappa^N \kappa^{2g-2}. \tag{7.38}$$

The formal expression for these amplitudes is

$$< V_{\Lambda_1,k_1} \dots V_{\Lambda_N,k_N} >= \kappa^N \kappa^{2g-2} \int [DX]\,[Dh] \prod_{i=1}^{N} V_{\Lambda_i,k_i}$$

$$\exp(-\frac{1}{2\pi\alpha'} \int_{\Sigma_g} d^2\sigma \sqrt{h} h_{\alpha\beta} \partial^\alpha X^\mu \partial^\beta X^\mu), \tag{7.39}$$

where Σ_g is a Riemann surface of genus g.
To conclude this short review on the closed bosonic string, let us say a few words on the closed string propagator. We will start with the hamiltonian (7.26). The proper time representation of the propagator will be

$$\Delta = \int \frac{d\rho}{\rho} \rho^{L_0 + \tilde{L}_0} = \frac{1}{L_0 + \tilde{L}_0}. \tag{7.40}$$

¿From the representation (7.40) with a normal ordering constant and a projector on physical states satisfying the constraint $L_0 = \tilde{L}_0$,

$$\Delta = \int_{|z|\leq 1} \frac{dz\, d\bar{z}}{(z\bar{z})^2} z^{L_0} \bar{z}^{\tilde{L}_0}, \tag{7.41}$$

where $z = \rho e^{i\phi}$. It is finally convenient to introduce a complex variable, $z = e^{2\pi i \alpha' \tau}$, such that $\rho = e^{-2\pi\alpha' \,\mathrm{Im}\, \tau}$, which allows us to interpret $2\pi\alpha'$ Im τ as an imaginary time in $1+1$ dimensions.

7.3 Open Bosonic String

Repeating previous comments on closed strings for the open case is straightforward. The only crucial point is deciding the type of boundary conditions to be imposed. From (7.2), we get boundary terms of the form

$$\frac{T}{2}\int \partial X^{\mu}\partial_{n}X_{\mu}, \tag{7.42}$$

with ∂_n the normal boundary derivative. In order to avoid momentum flow away from the string, it is natural to impose Neumann boundary conditions,

$$\partial_{n}X_{\mu}=0. \tag{7.43}$$

Using these boundary conditions the mode expansion (7.22) becomes, for the open string,

$$X^{\mu}(\sigma,\tau)=x^{\mu}+2\alpha' p^{\mu}\tau+i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_{n}^{\mu}e^{-in\tau}\cos n\sigma, \tag{7.44}$$

and the quantum mass formula (7.30) is

$$M^{2}=-\frac{4}{\alpha'}+\frac{4}{\alpha'}\sum_{n=1}^{\infty}\alpha_{-n}\alpha_{n}. \tag{7.45}$$

¿From (7.45) it is immediate reading the spectrum of states described by the oscillations of the string. As in the closed string, there is again a negative squared mass tachyon, while now the massless states are the 24 polarization states of the massless vector boson.

Vertex operators for open string states $|\Lambda>$ are defined as

$$V_{\Lambda}=\oint_{C} ds\sqrt{h}F_{\Lambda}(s)e^{ik.\,X(s)}, \tag{7.46}$$

with C the boundary of the open string worldsheet. Scale invariance of V_Λ implies now that $F_\Lambda(s)e^{ik.X(s)}$ is of conformal dimension equal one. The vertex operator for the massless vectors is

$$\oint_C ds\dot{X}^\mu(s)e^{ik.X(s)}. \tag{7.47}$$

For the second excited level, corresponding to massive states, the vertex operator becomes

$$\oint_C ds\sqrt{h}\dot{X}^\mu(s)\dot{X}^\nu(s)e^{ik.X(s)}. \tag{7.48}$$

In the same spirit as in the closed bosonic string, open string vertices can be related to a gauge coupling constant, g. Thus, for a generic open string amplitude with N external lines and L holes, the dependence on g should be

$$g^N g^{2L-2}. \tag{7.49}$$

In order to relate g and κ we can simply consider a closed string amplitude at tree level, with one external closed string. According to (7.38), this amplitudes will be proportional to κ^{-1}. However, this amplitude can also be interpreted as a tree level amplitude for open strings with no external line, i. e., a disc, that, from (7.49), will behave as g^{-2}. Identifying both coefficients, we get the desired relation,

$$\kappa \sim g^2. \tag{7.50}$$

It must be stressed that closed an open strings are naturally coupled. For instance, nothing prevents us from inserting closed string vertex operators in an open string amplitude. The insertion of a closed string state is topologically equivalent to creating a hole, and therefore increasing the power of g, according to (7.49), by a power of two, which is equivalent to (7.50).

7.4 Background Fields

The simplest generalization of the worldsheet lagrangian (7.2) in background fields should naturally include all the massless states of the closed string. The obvious is the $g^{\mu\nu}$ metric of the target spacetime,

$$S_1 = -\frac{T}{2}\int d^2\sigma \sqrt{h} h_{\alpha\beta} g^{\mu\nu}(X)\partial_\alpha X_\mu \partial_\beta X_\nu . \tag{7.51}$$

However, not any background $g^{\mu\nu}$ is allowed, since we want to preserve Weyl invariance on the worldsheet. Scale invariance, for the two dimensional system defined by (7.51) is equivalent, from the quantum field theory point of view, to requiring a vanishing β-function. At one loop, the β-function [12] for (7.51) is given by

$$\beta = -\frac{1}{2\pi\alpha'}R, \tag{7.52}$$

where R is the Ricci tensor of the target spacetime. Therefore, the first condition we require on allowed spacetime backgrounds is to be Ricci flat manifolds.

The antisymmetric tensor field, $B_{\mu\nu}$, and the dilaton field, Φ, can also be added as extra backgrounds to (7.51),

$$S = S_1 - \frac{T}{2}\int d^2\sigma \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) + \frac{1}{4}\int d^2\sigma \sqrt{h}\Phi(X)R^{(2)}, \tag{7.53}$$

with $R^{(2)}$ in (7.53) the worldsheet curvature. α' does not appear in the last term due to dimensional reasons (the first two terms in (7.53) contain the X^μ field, which has length units). The term $\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}$ can be interpreted as the minimal coupling of the string to the $B_{\mu\nu}(X)$ field,with the gauge transformations of $B_{\mu\nu}$ defined by

$$\delta B^{\mu\nu} = \partial^\mu \Lambda^\nu - \partial^\nu \Lambda^\mu . \tag{7.54}$$

This can be interpreted as the string being a source for the 2-form gauge field $B_{\mu\nu}$.
Once the background fields in (7.53) have been added, the condition of Weyl invariance generalizes to vanishing β - functions for g, B and Φ. At one loop, they are

$$\begin{aligned} R_{\mu\nu} + \frac{1}{4} H_{\mu}^{\lambda\rho} H_{\nu\lambda\rho} - 2 D_\mu D_\nu \Phi &= 0, \\ D_\lambda H^\lambda_{\mu\nu} - 2(D_\lambda \Phi) H^\lambda_{\mu\nu} &= 0, \\ 4(D_\mu\Phi)^2 - 4 D_\mu D^\mu \Phi + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + (D-26) &= 0, \end{aligned} \tag{7.55}$$

where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$. The set of equations (7.55) can be interpreted as the Euler-Lagrange equations for the action

$$S = -\frac{1}{2\kappa^2} \int d^{26} x \sqrt{g} e^{-2\Phi} (R - 4 D_\mu \Phi D^\mu \Phi + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}), \tag{7.56}$$

that describes the long wavelength limit of the interactions of the massless modes of the closed bosonic string. We will refer to this action as the one in the "string frame". When the factor $e^{-2\Phi}$ is absorbed in the metric through a suitable rescaling, what we get is the string effective action in the "Einstein frame".
It is important to stress that the computation leading to equations (7.55) is only taking into account short distance effects on the worldsheet, and is therefore independent of the topology. Notice that for a constant dilaton field, the last term in (7.53) is simply

$$\chi \cdot \Phi, \tag{7.57}$$

with χ the Euler number; in terms of the genus, g, for a generic

Riemann surface the Euler number is simply given by

$$\chi = 2 - 2g. \tag{7.58}$$

Therefore, the partition function at genus g behaves like

$$e^{-\Phi(2-2g)}.$$

The topological Euler number possesses a nice meaning in string theory: it is equal to the number of vertices joining three closed strings, needed to build up a Riemann surface of genus g. In fact, any Riemann surface of genus $g \geq 2$ can be obtained sewing $3g-3$ tube propagators through $2g-2$ vertices. This naturally leads to a precise physical meaning of the dilaton background field: it is the string coupling constant,

$$g = e^{\Phi}. \tag{7.59}$$

In ordinary quantum field theory, the condition on the background fields to be solutions to the equations of motion amounts to requiring vanishing of tadpoles at zero momentum (one point functions) for the quantum fields representing small fluctuations around the background. Based on the previous relation between the vanishing beta functions and the equations of motion for the effective lagrangian (7.56), we can use as a criterion for defining good string backgrounds the condition of vanishing tadpoles [1],

$$< V >= 0, \tag{7.60}$$

for vertex operators V representing the small quantum fluctuations of the background fields, and where the expectation value in (7.60) is calculated in the corresponding background.

It is then easy to see that, at tree level, in string perturbation theory, and for closed strings, condition (7.60) is equivalent to the requirement of Weyl invariance on the worldsheet. In fact, V possesses conformal dimension equal two, and at tree level we can fix the insertion of V at the origin of the Riemann sphere. In that case, under scale transformations on the worldsheet,

$$< V > \to \lambda^{-2} < V >, \tag{7.61}$$

and, therefore, Weyl invariance requires $< V >= 0$. However, in general condition (7.60) is stronger than simply Weyl invariance at tree level. In fact, by $< V >= 0$ we mean the sum of all string contributions, which naturally include higher loop effects to the tadpole. If in addition we allow coupling to open strings, then in the computation of $< V >$ we should also take into account the open string contribution. However, in the open string case, Weyl invariance does not force, at tree level, the tadpole for closed string vertex operators to vanish. When considering simply the closed bosonic string, we get a contribution to the tadpole of the dilaton vertex, at one loop, even in a flat spacetime background metric. Imposing (7.60) at one loop gives rise to the formal relation depicted in Figure 7.4 [13].

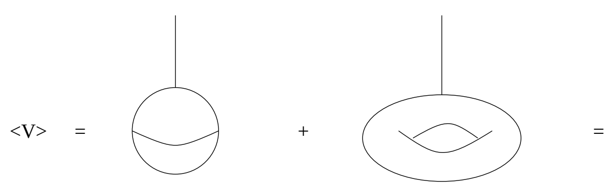

Figura 7.2: Tadpole cancellation for closed strings

The first term in Figure 7.4, corresponding to a Riemann sphere, is given by the value of the dilaton beta function in (7.55), for a background with constant dilaton field and vanishing antisymmetric tensor. Thus, from Figure 7.4, we formally get

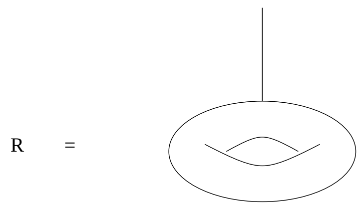

Figura 7.3: A vanishing tadpole implies a non vanishing cosmological constant

i. e., vanishing dilaton tadpole at one loop is equivalent to working in a background metric with a non vanishing cosmological constant, given by the one loop dilaton tadpole. This is known as the Fishler-Susskind mechanism [13].
For future convenience, we will include a brief discussion on the dilaton tadpole for the closed bosonic string. In order to define the one loop amplitude, we will start with the propagator Δ defined in (7.41), and compute the trace in the string Hilbert space,

$$\int_{|z|\leq 1} \frac{dz\, d\bar{z}}{(z\bar{z})^2} \text{ tr } (z^{L_0} \bar{z}^{\tilde{L}_0}). \tag{7.62}$$

z has a clear geometrical interpretation, as characterizing a genus one Riemann surface. The value of Im τ is the longitudinal length of the torus, while Re τ is the rotation performed before gluing the extremes of a cylinder, in order to build up the torus. This is the standard representation of the torus through the elliptic modulus τ. The integration in (7.62) should be restricted, by modular invariance, to the fundamental region of the $Sl(2, \mathbf{Z})$ group, acting on the upper half complex plane[α].

Now, the dilaton tadpole for a dilaton of zero momentum is given by

$$< V_D >= R \int \int_F \frac{dz\, d\bar{z}}{(z\bar{z})^2} \text{ tr } (z^{L_0} \bar{z}^{\tilde{L}_0} V_D(k=0))\, d^{26}p, \quad (7.63)$$

with $V_D(k=0) = \dot{X}_R^{\mu}(z)\dot{X}_{L\,\mu}(\bar{z})$. We can now easily interpret the dilaton tadpole (7.63) as a cosmological constant.

Let us then first recall the way to calculate the cosmological constant in ordinary (D-dimensional) quantum field theory. The contribution of particles of mass m comes from

$$\Lambda = \pm \int \frac{d^D p}{(2\pi)^D} \ln(p^2 + m^2), \quad (7.64)$$

with the sign depending on the boson or fermion nature of the particles. Through a proper time representation,

$$\Lambda = \mp \frac{1}{2} \int_0^{\infty} \frac{dt}{2\pi\alpha' t} \int \frac{d^D p}{(2\pi)^D} \exp[-2\pi\alpha' t(p^2 + m^2)]. \quad (7.65)$$

The generalization of (7.65) to string theory can be implemented identifying t and $\pi \text{Im}\, \tau$, and taking into account the

[α]The modular group, $Sl(2, \mathbf{Z})$ is defined as the set of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbf{Z},$$

with determinant equal one. The generators of the group are

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and satisfy the relation $(ST)^3 = \mathbf{I}$. Tori with different values of τ, related by an element of $Sl(2, \mathbf{Z})$, share the same complex structure.

mass formula. The result, once the projection on states with $L_0 = \tilde{L}_0$ has been performed, is

$$\Lambda_{string} = -\int \frac{d^{26}p}{(2\pi)^{26}} \int \frac{d^2\tau}{\mathrm{Im}\ \tau} e^{-2\pi\alpha' \mathrm{Im}\ \tau (p^2+m^2)} \mathrm{tr}\ (z^N \bar{z}^{\bar{N}}), \tag{7.66}$$

with N and $\bar{N}$ the operator numbers. It is now easy to check that (7.66) is proportional to the dilaton tadpole, (7.63). Thus, the condition $< V_D >= 0$, at one loop, implies that we should work on a target spacetime with non vanishing cosmological constant, i. e., an Anti-deSitter spacetime.

Capítulo

8

Toroidal Compactifications

8.1 T-Duality

8.1.1 Closed Bosonic String Compactifications

A torus is a Ricci flat manifold that can be used as target spacetime. Let us consider the simplest case, $\mathbf{R}^{25} \times S^1$, where the compact dimension, S^1, is taken to be of radius R. Then, the coordinate x^{25}, living on this S^1, must satisfy

$$x^{25} \equiv x^{25} + 2\pi n R. \tag{8.1}$$

If we now include the identification (8.1) in the mode expansion (7.22) we get, for the right and left momenta,

$$\begin{aligned} p_L &= \frac{m}{2R} - nR, \\ p_R &= \frac{m}{2R} + nR, \end{aligned} \tag{8.2}$$

while the mass formula becomes

$$M^2 = \left(\frac{m}{R} - \frac{nR}{\alpha'}\right)^2 + \frac{4}{\alpha'}(N-1) = \left(\frac{m}{R} + \frac{nR}{\alpha'}\right)^2 + \frac{4}{\alpha'}(\bar{N}-1), \tag{8.3}$$

with N and $\bar{N}$ the total level of left and right moving excitations, respectively. From (8.3) we get the relation

$$(N - \bar{N}) = mn. \tag{8.4}$$

The first thing to be noticed, from (8.2), is the invariance under the transformation [14]

$$\begin{aligned} T : R &\to \frac{\alpha'}{R}, \\ m &\to n. \end{aligned} \tag{8.5}$$

A nice way to represent (8.2) is using a lattice of $(1,1)$ type, which will be referred to as $\Gamma^{1,1}$. This is an even lattice, as can be observed from (8.2):

$$p_L^2 - p_R^2 = 2mn. \tag{8.6}$$

The transformation (8.5) acts on the lattice momenta as

$$\begin{aligned} p_L &\to -p_L, \\ p_R &\to p_R. \end{aligned} \tag{8.7}$$

Hence, (8.5) amounts to the change [17]-[20]

$$X^{25} \to X'^{25} = X_R - X_L. \tag{8.8}$$

In fact, string theory in X'^{25} variables, compactified on a circle of radius R, possesses the spectrum of winding and momentum of string theory defined on X^{25} variables, but compactified on a circle of radius $\frac{\alpha'}{R}$.
If Π is the spacelike 1-plane where p_L lives, then $p_R \in \Pi^\perp$. In fact, p_L forms a θ [α] angle with the positive axis of the $\Gamma^{1,1}$

[α] θ is chosen as the coordinate parametrizing the radius of the compact dimension.

lattice, while p_R forms a negative angle, $-\theta$, and changes in R, which are simply changes in θ (or Lorentz rotations in the $\Gamma^{1,1}$ hyperbolic space), are changes in the target space preserving the $\beta = 0$ condition, and therefore are what can be called the *moduli of the σ-model* (7.51). Of course, no change arises in the spectrum upon rotations of the Π and $\Pi^{\perp}$ planes. We have now obtained a good characterization of the moduli space for the string σ-model on a simple S^1 torus. However, in addition to rotations in Π and $\Pi^{\perp}$, we should also take into account the symmetry (8.5), representing rotations of the $\Gamma^{1,1}$ lattice. The previous discussion can be generalized to compactifications on higher dimensional tori, T^d (i. e., working in a background spacetime $\mathbf{R}^{26-d} \times T^d$). In this case, (p_L, p_R) will belong to a lattice $\Gamma^{d,d}$, and the moduli space will be given by [15]

$$O(d,d;\mathbf{Z})\backslash O(d,d)/O(d) \times O(d), \tag{8.9}$$

where the $O(d,d;\mathbf{Z})$ piece generalizes the transformations (8.5) to T^d. From now we will call these transformations T-duality [16]. Notice also that the dimension of the moduli (8.9) is $d \cdot d$, which is the number of massless degrees of freedom that have been used to define the background fields of the σ-model (7.53). The manifold (8.9) is the first example of moduli of a σ-model we find; these moduli spaces will be compared, in next section, to moduli spaces arising upon $K3$ compactifications. The T-duality transformation (8.5) is not only a symmetry of the spectrum of the string, but also a symmetry of the interactions.

However, in order for it to be a symmetry of the interactions, we should also change the string coupling constant as [21]

$$g \to g' = \frac{g\sqrt{\alpha'}}{R}. \tag{8.10}$$

This change in the string coupling constant is such that the Newton constant in the compactified space remains constant.

8.2 Discrete Light Cone: Compactification of Light

A modification of the standard light-cone quantization has been recently proposed [?]. The idea is compactifying the light-like direction, X^-,

$$X^- \sim X^- + 2\pi R \tag{8.11}$$

or, equivalently,

$$\begin{pmatrix} X^0 \\ X^1 \end{pmatrix} \sim \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} + \begin{pmatrix} 2\pi R/\sqrt{2} \\ -2\pi R/\sqrt{2} \end{pmatrix}. \tag{8.12}$$

When this light-like compactification is performed, the momentum p^+ is quantized through

$$p^+ = \frac{N}{2\pi R}. \tag{8.13}$$

After imposing condition (8.11), the Hilbert space of the string is divided into superselection sectors, characterized by the value of N. The quantization condition (8.13), together with equation (7.18), implies that the minimum length of the σ axis is given by $\frac{\sqrt{\alpha'}}{R}$. This minimum length string can be used to define something like a "string parton". In the simplest sector, which is the one with $N = 1$, the dynamics is trivial, as there is only a single string of minimal length. The sector with $N = 2$ contains processes as those represented in Figure 7.1, describing the joining and splitting of strings,

with $p_1^+ = p_2^+ = \frac{\sqrt{\alpha'}}{R}$. Thus, the dynamics becomes more complicated as N grows.
Strings can wind around the compact direction, X^-. The winding number ν is defined, as usual, by

$$2\pi R\nu = \int \frac{dX^-}{d\sigma}\, d\sigma. \tag{8.14}$$

Using (7.20), we get

$$2\pi R\nu = \int \frac{1}{\mathcal{P}^+}\frac{\partial X^i}{\partial\sigma}\frac{\partial X^i}{\partial\tau} d\sigma \tag{8.15}$$

that, from (7.18), leads to

$$2\pi R\nu = \frac{(N-\bar{N})}{p^+}, \tag{8.16}$$

which implies the condition

$$(N-\bar{N}) = N\nu. \tag{8.17}$$

This relation is the analog, for compactifications in the light-like direction X^-, of (8.4). If we also compactify some transversal direction, X^i, when combining (8.17) and (8.4), we get

$$(N-\bar{N}) = \nu N + mn. \tag{8.18}$$

The discrete light-cone construction immediately raises a number of questions. The simplest one is naturally whether the theory, restricted to a particular value of N, is consistent, and which is then the critical dimension. It is easy realizing, when repeating the usual no ghost theorem, that for a fixed value of N, and $p^+ = \frac{N}{2\pi R}$, the quantum consistency of the theory requires, as in the standard bosonic case, $D_{critical} = 26$. The second comment, concerning compactification in a light-like direction, is that now the value of the compactification radius, R, is not playing the role of a moduli, as the target space-time metric is independent of R.

8.3 Open Bosonic String Compactifications: D-Branes

By introducing the complex coordinate

$$z = \sigma^2 + i\sigma, \tag{8.19}$$

with $\sigma^2 \equiv i\tau$, the expansion in oscillator modes (7.44) can be rewritten as

$$X^\mu(\sigma,\tau) = x^\mu - i\alpha' p^\mu \ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu(z^{-n}+\bar{z}^{-n}). \tag{8.20}$$

Let us now consider the open string moving in $\mathbf{R^{25}} \times S^1$. Neumann boundary conditions in the compactified direction are

$$\partial_n X^{25} = 0. \tag{8.21}$$

Now, we will work out the way these boundary conditions modify under the $R \to \frac{\alpha'}{R}$ transformation [17]-[20]. To visualize the answer, we will consider the cylinder swept out by a time evolving closed string, both from the closed and open string pictures (in the open string picture the cylinder can be understood as an open string with both ends at the S^1 edges of the cylinder, Figure 8.3). In fact, from the closed string point of view, the propagation of the string is at tree level, while the open string approach is a one loop effect.
We will now see what Neumann boundary conditions for the open string mean, from the point of view of closed strings. This, naturally leads to $\partial_\tau X^{25} = 0$, which in terms of the complex variable $z = e^{\tau - i\sigma}$ reads

$$z\partial_z X_R^{25} + \bar{z}\partial_{\bar{z}} X_L^{25} = 0. \tag{8.22}$$

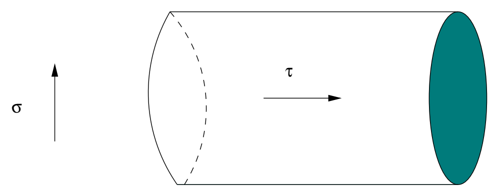

Figura 8.1: Open string, with its extreme points wrapping around the 25^{th} (compact) direction

If we now perform a T-duality transformation, in the sense of (8.8) we get, from (8.22),

$$z\partial_z X'^{25}_R - \bar{z}\partial_{\bar{z}} X'^{25}_L = 0, \tag{8.23}$$

or, equivalently, $\partial_\sigma X'^{25} = 0$. This last condition is equivalent to imposing Dirichlet boundary conditions for the open string. Thus, what T-duality does on the open string is exchanging Neumann and Dirichlet boundary conditions. Dirichlet boundary conditions for the open string mean that the value of the coordinate X^{25} is fixed, and does not change in the open string time. To visualize this condition, we can introduce the hypersurface in spacetime defined by X^{25} = constant. This is a $24 + 1$ dimensional space, where the end points of the open string are forced to move. We will call this hyperplane a D-24brane, and the $24+1$ dimensional space its worldvolume (see Figure 8.3). It is important to observe that the so defined D-brane is simply a mathematical plane of zero thickness.
Notice that open strings in the presence of this D-24brane are enforced to end on the D-brane worldvolume; in this sense, ordinary open strings with Neumann boundary conditions for all coordinates can be interpreted as open strings in the presence of a D-25brane, whose worldvolume is the whole spacetime. Then, the T-duality argument above can be interpreted as a

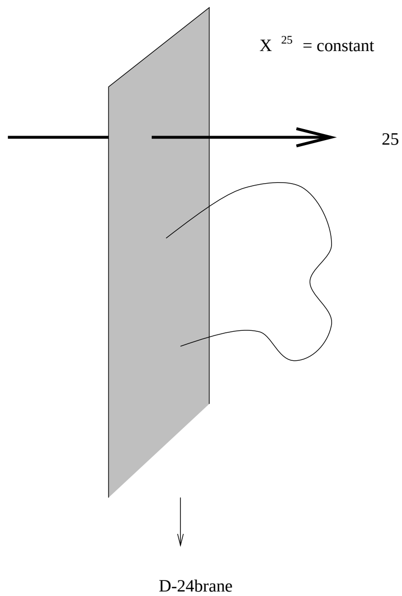

Figura 8.2: An open string with end points constrained to move in the worldvolume of a D-24brane

transformation, under T-duality, of the D-25brane into a D-24brane. This will in general be the rule, and we will pass from a D-$p$brane to a D-$(p-1)$brane, through T-duality on some of the coordinates of the D-$p$brane worldvolume. Reciprocally, we can also go from a D-$(p-1)$brane to a D-$p$brane through T-duality on some of the coordinates transversal to the D-$p$brane worldvolume.

Massless string states can propagate on the D-brane worldvolume. In fact, we know that for open strings, states of the form $\alpha_{-1}^{\mu}|k>$ (with k a momentum in the $p+1$ dimensional worldvolume) are massless, so that for a D-$p$brane we will have vectors moving in the worldvolume defined by $\alpha_{-1}^{\mu}|k>$, with $\mu = 0, \ldots, p$, and a set of scalars $\alpha_{-1}^{I}|k>$,

with $I = p+1, \ldots, 26$. This is, in fact, the spectrum we expect from dimensional reduction of the gauge theory defined by the massless modes of the open string in 26 dimensions. Thus, on a D-$p$brane, we have, at least, a $U(1)$ gauge theory with photons the string states $\alpha^{\mu}_{-1}|0>$, where $\mu = 0, \ldots, p$.
Now, once D-branes have entered the theory, a natural step is considering configurations with more than one D-brane. The simplest configurations are those with parallel D-branes, as in Figure 8.3. Then, a new sector in the spectrum appears, as strings are now allowed to stretch between the parallel D-branes. The end points of this strings are characterized by the positions C_1 and C_2 of the D-branes in the transversal directions. This reminds of the old Chan-Paton factors of open string theory: now, the extremal quantum numbers of the Chan-Paton model are promoted to the locations of the D-branes in transversal space.

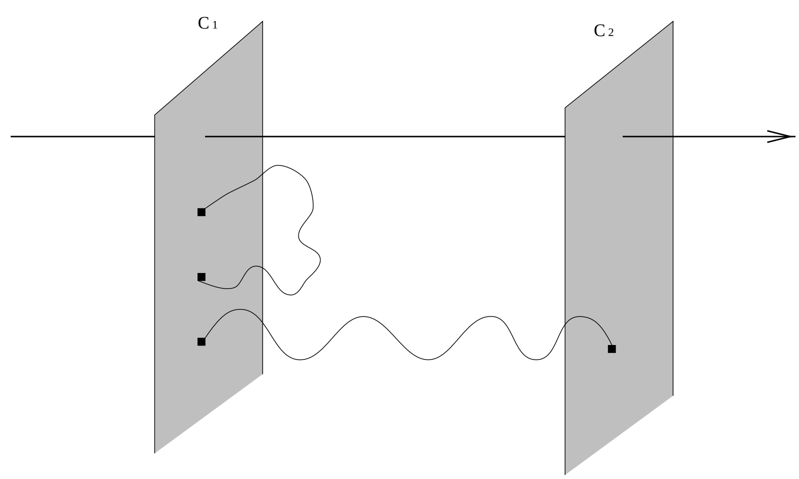

Figura 8.3: A parallel branes configuration

Including these stretching strings increases the spectrum of the D-brane worldvolume. In fact, now we have $\alpha^{\mu}_{-1}[C_1C_2]|k>$ string modes, with $\mu = 0, \ldots, p$, which are vector bosons, with a mass equal $\frac{[C_2-C_1]}{\alpha'}$. In addition, we have new scalars, $\alpha^{I}_{-1}[C_1C_2]|k>$, with $I = p+1, \ldots, 26$. This is the usual spectrum of a Higgs model with $U(2)$ gauge invariance broken down to $U(1)^2$, and scalar Higgs fields in the adjoint representation. If we accept the previous qualitative picture, we land into a new and very geometrical understanding of the Higgs mechanism and enhancement of gauge invariance. In fact, the enhancement of gauge symmetry will take place whenever the two parallel D-branes join.

As an amusing comment on some magical numerical coincidences, we will mention an interesting relation between the critical dimension of bosonic strings, and the beta function for $N = 0$ Yang-Mills theories. The beta function for Yang-Mills with scalar fields transforming in the adjoint representation is

$$\beta = -\frac{g^3}{16\pi^2}\frac{c_2(G)}{6}(22 - v), \tag{8.24}$$

with $c_2(G)$ the dual Coxeter number of the gauge group, G, and v the number of scalar fields in the adjoint representation. Notice that for a D-3brane in the bosonic string, we precisely get 22 saclar fields, corresponding to the transversal directions. This implies that, at first order in α', the worldvolume dynamics on the D-brane possesses vanishing beta function.

8.3.1 D-Brane Dynamics

Parallel D-branes interact dynamically through the exchange of closed strings. When interpreting Figure 8.3.1 from the point of view of the field theory on the D-brane worldvolume, the exchange of the closed string amplitude, at tree level, or an open string at one loop, is nothing but a computation of the cosmological constant, where the particles contributing to Λ are now the strings extending between the parallel D-branes.

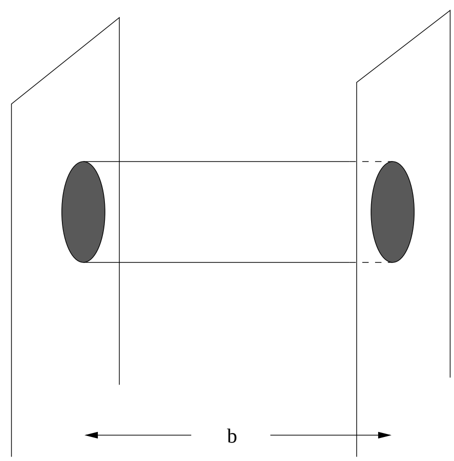

Figura 8.4: Tree level interaction through closed strings between D-branes

This one loop vacuum amplitude (in the open string channel) is obtained when summing up all the zero point energies of all the modes. Through Schwinger's proper time formalism,

$$\Lambda = \int_0^\infty \frac{dt}{2t} \sum_{k,\mathrm{oscillators}} e^{-t(k^2+M^2)} =$$

$$\int_0^\infty \frac{dt}{2t} \sum_i \int \frac{d^{p+1}k}{(2\pi)^{p+1}} e^{-t(k^2+M^2)}, \qquad (8.25)$$

where the mass spectrum is

$$M^2 = \frac{Y^j Y_j}{4\pi^2\alpha'} + \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} n\alpha_{-n}^i \alpha_n^i - 1\right), \qquad (8.26)$$

with Y^j measuring the background distance between the D-branes. Performing the momentum integrals and the oscillator sums, the amplitude, in terms of the string time $t = 2\pi\alpha' t_s$, in a spacetime of dimension D, becomes

$$\Lambda = 2\int_0^\infty \frac{dt_s}{2t_s}(8\pi^2\alpha' t_s)^{-\frac{p+1}{2}} e^{-\frac{Y.Y}{2\pi\alpha'}t_s} q^{-2}\sum_{n=1}^{\infty}(1-q^{2n})^{-D+2}, \tag{8.27}$$

where the definition $q \equiv e^{-\pi t}$ has been introduced, and the factor 2 is taking into account the two possible orientations of the open string extending between the D-branes. The $t \to 0$ limit can be easily obtained from Dedekind's eta function,

$$\eta(\tau) = e^{i\pi\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i\tau n}), \tag{8.28}$$

which satisfies $\eta(-\frac{1}{\tau}) = (-i\tau)^{1/2}\eta(\tau)$. In critical dimension,

$$\lim_{t\to 0} q^{-2}\prod_{n=1}^{\infty}(1-q^{2n})^{-24} = t^{12}[e^{2\pi/t} + 24 + \cdots]. \tag{8.29}$$

The dilaton (Figure 8.3.1) and graviton contribute through the massless pole $t = 0$ (the leading divergence is from the tachyon), and lead to the amplitude

$$\mathcal{A} = V_{p+1}\frac{24}{2^{12}}(4\pi^2\alpha')^{11-p}\pi^{p-23}\Gamma((23-p)/2)|b|^{p-23}. \tag{8.30}$$

¿From (8.30), it is clear that the eleven dimensional D-brane, with $p = 11$, has no α' dependence, which is certainly the result to be expected for the self dual brane in 26 dimensions. A natural way to interpret (8.30) is as the graviton and dilaton exchange between the two D-branes. We can thus include a coupling T_p of the D-$p$brane to the graviton, and represent (8.30) as $T_p^2 G(b)$, with $G(b)$ the massless graviton Green

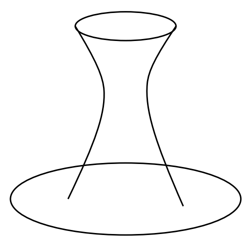

Figura 8.5: Dilaton contribution

function in transversal space. Introducing the gravitational constant κ, the amplitude will behave as κ^{-2}, so that, from (8.30),

$$T_p = \frac{\sqrt{\pi}}{16\kappa}(4\pi^2\alpha')^{\frac{11-p}{2}}, \tag{8.31}$$

which is consistent with a dilaton tadpole of order κ^{-1}. Using now the identification between κ and the string coupling constant, we find a D-brane tension behaving as $\frac{1}{g_{string}}$, which will be the main label of D-branes.

As the D-brane is a dynamical object, we can look for its worldvolume lagrangian in the presence of closed string backgrounds, $G_{\mu\nu}$ and $B_{\mu\nu}$. The worldvolume lagrangian should also contain the vector field on the worldvolume, which is an open string photon. As was shown in [22], the open string photon interacts with itself through a nonlinear Born-Infeld lagrangian. Taking this fact into account, and the value for the D-brane tension, we get, as the worldvolume lagrangian for the D-brane,

$$S = T_p \int d^{p+1}\xi \sqrt{\det\ (\tilde{G}_{\mu\nu} + \tilde{B}_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}, \tag{8.32}$$

with $\tilde{G}_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$ the induced metric and antisymmetric tensor on the worldvolume. The coefficient α' in front of $F_{\mu\nu}$

already indicates the stringy origin of the Born-Infeld string photon interactions. In principle, the check of conformal invariance for the worldvolume action can be done using standard beta function techniques [23], [24][α]. Invariance under (7.54) transformations requires transforming the vector field as

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ A_\mu &\to A_\mu - \Lambda_\mu. \end{aligned} \tag{8.33}$$

8.4 Orientifolds

Worldsheet parity transformations for open strings are defined by

$$\sigma \to \pi - \sigma. \tag{8.34}$$

In terms of the oscillators, this transformation becomes

$$\Omega : \alpha_n^\mu \to (-1)^n \alpha_n^\mu. \tag{8.35}$$

For the closed string, worldsheet parity is

$$\sigma \to -\sigma, \tag{8.36}$$

exchanging left and right moving oscillators,

$$\Omega : \alpha_n^\mu \to \tilde{\alpha}_n^\mu. \tag{8.37}$$

Invariance under worldsheet parity is equivalent to reducing the spectrum of states to those which are even with respect to Ω.

[α]Perturbative beta function analysis on the worldvolume lagrangian is essentially perturbative in α'; thus, the Born-Infeld form of the lagrangian should be induced from the first perturbative orders in α'.

In the case of the closed string, the T-duality transformation and worldsheet parity can be combined. The result is going from the string coordinate X^{μ} to $-X'^{\mu}$. Thus, for unoriented strings the description of the T-dual must be performed using string coordinates X'^{μ}, moded now by the spacetime parity transformation $X'^{\mu} \to -X'^{\mu}$. In other words, the theory on $\mathbf{R}^{24,1} \times S^1/\mathbf{Z}_2$, with radius R, described in terms of X'^{25}, is equivalent to the theory on $\mathbf{R}^{24,1} \times \hat{S}^1$ with radius $\frac{\alpha'}{R}$ and described in terms of the original variable, X^{25}. The orbifold space $S^1/\mathbf{Z}_2$ is topologically equivalent to a segment of length πR, with two *orientifold* 24 dimensional planes, at 0 and πR. If we consider now open and closed strings coupled, the effect of T-duality, as above described, will generate D-25branes, located at some fixed value of the compactified X^{25} coordinate. The difference with the oriented case is that now, for any D-24brane, we must also consider its image with respect to the orientifold plane. What this means is that the spectrum of open strings, ending on the D-brane, contains as a subsector the strings stretching between the D-brane and its orientifold image (see Figure 8.4).

It is important to stress that open strings are not ending on the orientifold plane; thus, we have no dynamics on the orientifold worldvolume. However, this does not prevent the existence of gravitational interactions between orientifold planes and D-branes. In fact, closed string states near the orientifold become cross cups (Figure 8.4), leading to an orientifold–orientifold interaction described by the Klein bottle, or an orientifold–D-brane interaction, described by the Moëbius strip. Therefore, the gravitational properties of orientifold planes will be given by the graviton and dilaton tadpoles for $\mathbf{RP}^2$, but this is a discussion we will postpone.

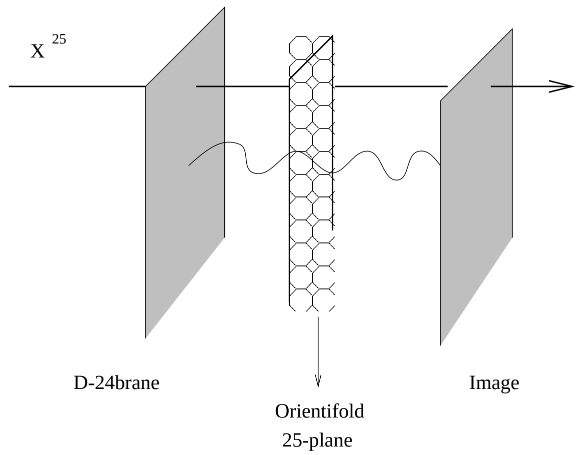

Figura 8.6: An orientifold plane

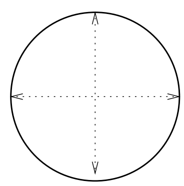

Figura 8.7: A cross cup arising as a consequence of the presence of an orientifold

Finally, let us see the kind of gauge group that is obtained on the D-brane worldvolumes in the presence of orientifold planes. If, for instance, we have $\frac{N}{2}$ D-branes to the left of an orientifold, we should naturally have $\frac{N}{2}$ image D-branes to the right. If the $\frac{N}{2}$ are joined and placed on the orientifold plane, the spectrum of massless particles obtained will be those defining the adjoint representation of the gauge group, $SO(N)$.

Capítulo

9

σ-Model Geometry

In this section we will work out the moduli spaces for the string σ-models defined on the worldsheet, with target space $K3$ manifolds.

9.1 $K3$ Geometry. A First Look at Quantum Cohomology

The concept of moduli space introduced in previous paragraph, for the σ-model (7.53), when the target space is a T^d torus, leading to manifold (8.9), can be generalized to more complicated spacetime geometries satisfying the constraints derived from conformal invariance, namely Ricci flat manifolds. This is a physical way to approach the theory of moduli spaces where, instead of working out the cohomology of the manifold, a string is forced to move on it, which allows to wonder about the moduli of the so defined conformal field theory. We will work out this question for the particular case of $K3$ surfaces.

Let us first recall the relation between supersymmetry and the number of complex structures. Let us think of a σ-model, with target space $\mathcal{M}$. Now, we want this σ-model to be invariant under some supersymmetry transformations. It turns out that in order to make the σ-model, whose bosonic part is given by

$$\eta^{\mu\nu} g_{ij}(\phi(x)) \partial_\mu \phi^i \partial_\nu \phi^j, \tag{9.1}$$

with η the metric on spacetime, and g the metric on the target, invariant under $N=2$ supersymmetry we have to require the manifold to be Kähler and, in order to be $N=4$ supersymmetry, to be hyperkähler.
Let us then begin with a description of $K3$ manifolds [25, 26, 27]. To characterize topologically $K3$, we will first obtain its Hodge diamond. The first property of $K3$ is that the canonical class,

$$K \equiv -c_1(T), \tag{9.2}$$

with $c_1(T)$ the first Chern class of the tangent bundle, T, is zero,

$$K = 0. \tag{9.3}$$

Equation (9.3) implies that there exists a holomorphic 2-form Ω, everywhere non vanishing. Using the fact that only constant holomorphic functions are globally defined, we easily derive, from (9.3), that

$$\dim H^{2,0} = h^{2,0} = 1. \tag{9.4}$$

In fact, if there are two different 2-forms Ω_1 and Ω_2, then Ω_1/Ω_2 will be holomorphic and globally defined, and therefore constant.
The second important property characterizing $K3$ is

$$\Pi_1 = 0, \tag{9.5}$$

so that

$$h^{1,0} = h^{0,1} = 0, \tag{9.6}$$

as $b_1 = h^{1,0} = h^{0,1} = 0$, because of (9.5).
The Euler number can be now derived using Noether-Riemann theorem, and property (9.3), and it turns out to be 24. Using now the decomposition of the Euler number as an alternating sum of Betti numbers, we can complete the Hodge diamond,

$$24 = b_0 - b_1 + b_2 - b_3 + b_4 = 1 - 0 + b_2 - 0 + 1, \qquad (9.7)$$

which implies that

$$\dim H^2 = 22, \qquad (9.8)$$

and therefore, from (9.4), we get

$$\dim H^{1,1} = h^{1,1} = 20, \qquad (9.9)$$

leading to the Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad (9.10)$$

Using Hirzebuch's pairing, we can give an inner product to the 22 dimensional space H^2. In homology terms, we have

$$\alpha_1 \cdot \alpha_2 = \#(\alpha_1 \cap \alpha_2), \qquad (9.11)$$

with $\alpha_1, \alpha_2 \in H^2(X, Z)$, and $\#(\alpha_1 \cap \alpha_2)$ the number of oriented intersections. From the signature complex,

$$\tau = \int_X \frac{1}{3}(c_1^2 - 2c_2) = -\frac{2}{3}\int_X c_2 = -\frac{2 \cdot 24}{3} = -16, \qquad (9.12)$$

we know that $H^2(X, Z)$ is a lattice of signature $(3, 19)$. The lattice turns out to be self dual, i. e., there exits a basis α_i^* such that

$$\alpha_i \cdot \alpha_j^* = \delta_{ij}, \qquad (9.13)$$

and even,

$$\alpha \cdot \alpha \in 2\mathbf{Z}, \qquad \forall \alpha \in H^2(X, \mathbf{Z}). \tag{9.14}$$

Fortunately, lattices with these characteristics are unique up to isometries. In fact, the $(3, 19)$ lattice can be represented as

$$E_8 \perp E_8 \perp \mathcal{U} \perp \mathcal{U} \perp \mathcal{U}, \tag{9.15}$$

with $\mathcal{U}$ the hyperbolic plane, with lattice $(1, 1)$, and E_8 the lattice of $(0, 8)$ signature, defined by the Cartan algebra of E_8. The appearance of E_8 in $K3$ will be at the very core of future relations between $K3$ and string theory, mainly in connection with the heterotic string.

Next, we should separately characterize the complex structure and the metric of $K3$. Concerning the complex structure, the proper tool to be used is Torelli's theorem, that establishes that the complex structure of a $K3$ marked surface[α] is completely determined by the periods of the holomorphic 2-form, Ω. Thus, the complex structure is fixed by

i) The holomorphic form Ω.

ii) A marking.

To characterize $\Omega \in H^{2,0}(X, \mathbf{C})$, we can write

$$\Omega = x + iy, \tag{9.16}$$

with x and y in $H^2(X, \mathbf{R})$, that we identify with the space $\mathbf{R}^{3,19}$. Now, we know that

$$\begin{aligned} \int_X \Omega \wedge \Omega &= 0, \\ \int_X \Omega \wedge \bar{\Omega} &> 0, \end{aligned} \tag{9.17}$$

[α]By a marked $K3$ surface we mean a specific map of $H^2(X, \mathbf{Z})$ into the lattice (9.15), that we will denote, from now on, $\Gamma^{3,19}$.

and we derive

$$\begin{aligned} x \cdot y &= 0, \\ x \cdot x &= y \cdot y. \end{aligned} \tag{9.18}$$

Therefore, associated with Ω, we define a plane of vectors $v = nx + my$ which, due to (9.17), is space-like, i. e.,

$$v \cdot v > 0. \tag{9.19}$$

The choice of (9.16) fixes an orientation of the two plane, that changes upon complex conjugation. Thus, the moduli space of complex structures of $K3$, will reduce to simply the space of oriented space-like 2-planes in $\mathbf{R}^{3,19}$. To describe this space, we can use a Grassmannian [26],

$$Gr = \frac{(O(3,19))^+}{(O(2) \times O(1,19))^+}, \tag{9.20}$$

where $(\ )^+$ stands for the part of the group preserving orientation. If, instead of working with the particular marking we have been using, we change it, the result turns out to be an isometry of the $\Gamma^{3,19}$ lattice; let us refer to this group by $O(\Gamma^{3,19})$. The moduli then becomes

$$\mathcal{M}^C = Gr/O^+(\Gamma^{3,19}). \tag{9.21}$$

The group $O(\Gamma^{3,19})$ is the analog to the modular group, when we work out the moduli space of complex structures for a Riemann surface ($Sl(2, \mathbf{Z})$ for a torus).
Once a complex structure has been introduced, we have a Hodge decomposition of H^2, as

$$H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}. \tag{9.22}$$

Thus, relative to a complex structure characterized by Ω, the Kähler form J in $H^{1,1}$ is orthogonal to Ω, and such that

$$\mathrm{Vol} = \int_X J \wedge J > 0, \tag{9.23}$$

which means that J is represented by a space-like vector in $\mathbf{R}^{3,19}$ and, therefore, together with Ω, spans the whole three dimensional space-like subspace of $\mathbf{R}^{3,19}$. Yau's theorem now shows how the metric is completely determined by J and Ω, i. e., by a space-like 3-plane in $\mathbf{R}^{3,19}$. Thus, we are in a similar position to the characterization of the moduli space of complex structures, and we end up with a Grassmannian manifold of three space-like planes in $\mathbf{R}^{3,19}$,

$$Gr = O(3,19)/O(3) \times O(19). \tag{9.24}$$

Now, we need to complete Gr with two extra ingredients. One is the volume of the manifold, that can change by dilatations, and the other is again the modular part, corresponding to isometries of $\Gamma^{3,19}$, so that finally we get

$$\mathcal{M}^M = O(\Gamma^{3,19})\backslash Gr \times \mathbf{R}^+. \tag{9.25}$$

Hence, the moduli of the σ-model (7.53), defined on a $K3$ surface, will contain the moduli of Einstein metrics on $K3$ (see equations (9.24) and (9.25)). Now, the dimension of manifold (9.25) is 58. For the σ-model (7.53) we must also take into account the moduli of B-backgrounds. In the string action, what we have is the integral, $\int B$, over the worldsheet, which now becomes a 2-cycle of $K3$; thus, the moduli of B-backgrounds is given by the second Betti number of the $K3$ manifold, which is 22. Finally, the dilaton field Φ has to be taken into account in (7.53). As mentioned, if Φ is constant, as we will require, it counts the number of loops in the perturbation series, so we will not consider it as an extra moduli. More precisely, we will probe the $K3$ geometry working at tree level in string theory. Under these conditions, the σ moduli space is of dimension [28]

$$58 + 22 = 80, \tag{9.26}$$

and the natural guess is the manifold

$$\mathcal{M}^\sigma = O(4,20)/O(4) \times O(20). \tag{9.27}$$

Naturally, this is not the final answer, as we have not divided yet by the equivalent to the T-duality trasnformations in the toroidal case, which are, for $K3$, isometries of the $H^2(X;\mathbf{Z})$ lattice, i. e.,

$$O(\Gamma^{3,19}). \tag{9.28}$$

However, the final answer is not the quotient of (9.27) by (9.28), as an important symmetry from the point of view of conformal field theory is yet being missed: mirror symmetry.

9.2 Mirror Symmetry

In order to get a geometrical understanding of mirror symmetry [29], we need first to define the Picard lattice.
Let us then consider curves inside the $K3$ manifold. The Picard lattice is defined as

$$\mathrm{Pic}(X) = H^{1,1}(X) \cap H^2(S,\mathbf{Z}), \tag{9.29}$$

which means curves (i. e., 2-cycles) holomorphically embedded in X. By definition (9.29), $\mathrm{Pic}(X)$ defines a sublattice of $H^2(S;\mathbf{Z})$. This Picard lattice has signature $(18,t)$. Let us consider, as an example, an elliptic fibration where the base is a 2-cycle B, and F is the fiber. The Picard lattice defined by these two 2-cycles is given by

$$\begin{aligned} B\cdot B &= -2, \\ B\cdot F &= 1, \\ F\cdot F &= 0, \end{aligned} \tag{9.30}$$

which is a lattice of $(1,1)$ type. Self intersections are given by the general expression

$$C\cdot C = 2(g-1), \tag{9.31}$$

where g is the genus, so that for $g=0$, the base space, we get -2, and for the elliptic fiber, with $g=1$, we get 0 for the intersection. The intersection between the base and the fiber, $B\cdot F$, reflects the nature of the fibration. Notice that expression (9.31) is consistent with the even nature of the lattice $\Gamma^{3,19}$. Now, from (9.29), it is clear that the number of curves we have in $\mathrm{Pic}(X)$ depends on the complex structure. Taking this fact into account, we can wonder about the moduli space of complex structures preserving a given Picard sublattice; for instance, we can be interested in the moduli space of elliptic fibrations preserving the structure of the fibration. As $\mathrm{Pic}(X)$ are elements in $H^{1,1}(X)$, they should be orthogonal to Ω, so the moduli we are looking for will be defined in terms of the Grassmannian of space-like 2-planes in $\mathbf{R}^{2,19-t}$, i. e.,

$$Gr^P = O(2,19-t)/O(2)\times O(19-t), \tag{9.32}$$

where we should again quotient by the corresponding modular group. This modular group will be given by isometries of the lattice Λ, called the transcendental lattice, and is simply defined as the orthogonal complement to the Picard lattice. Thus, Λ is of $\Gamma^{2,19-t}$ type, and the moduli preserving the Picard group is

$$\mathcal{M}^P = Gr^P/O(\Lambda). \tag{9.33}$$

As is clear from (9.32), the dimension of the moduli space of complex structures preserving the Picard group, reduces in an amount given by the value of t for the Picard lattice. At this point of the discussion, a question at the core of mirror symmetry comes naturally to our mind, concerning the possibility to define a manifold X^* whose Picard group is the transcendental lattice Λ of X [30]. In these terms, the answer is clearly negative, as the Picard lattice is of signature $(1,t)$, and Λ is of signature $(2,19-t)$, so that we need either passing from Λ to a $(1,t')$ lattice, or generalize the concept of Picard lattice, admitting lattices of signature $(2,t)$. It turns out that both approaches are equivalent, but the second has a more physical flavor; in order to get from Λ a Picard lattice, what we can do is to introduce an isotropic vector f in Λ, and define the new lattice through

$$f^{\perp}/f, \tag{9.34}$$

which is of $(1,18-t)$ type; now, the mirror manifold X^* is defined as the manifold possessing as Picard lattice the one defined by (9.34). The moduli space of the mirror manifold is therefore given by the equivalent to expression (9.32),

$$Gr^{*P} = O(2,t+1)/O(2)\times O(t+1). \tag{9.35}$$

Then, we observe that the dimension of the two moduli spaces sums up to 20, and that the dimension of the moduli space of the mirror manifold is exactly given by the rank $t+1$ of the Picard of the original moduli space.

A different approach will consist in defining the so called quantum Picard lattice. Given a Picard lattice of signature $(1, t)$, we define its quantum analog as the lattice of signature $(2, t+1)$, obtained after multiplying by the hyperbolic lattice $\Gamma^{1,1}$. So, the question of mirror will be, given a manifold X, with transcendental lattice Λ, finding a manifold X^* such that its quantum Picard lattice is precisely Λ. Now, we observe that the quantum Picard lattices of X and X^* produce a lattice of signature $(4, 20)$. The automorphisms $O(\Gamma^{4,20})$ will result of composing the T-duality transformations and mirror symmetry. Coming back to (9.27), and including mirror symmetry, we get, as moduli space of the σ-model on $K3$,

$$O(4, 20; \mathbf{Z}) \backslash O(4, 20)/O(4) \times O(20). \tag{9.36}$$

This concludes our analysis of σ-models on $K3$.

9.3 Elliptic Fibrations

Let V be an elliptic fibration,

$$\Phi : V \to \Delta, \tag{9.37}$$

with Δ an algebraic curve, and $\Phi^{-1}(a)$, with a any point in Δ, an elliptic curve. Let us denote $\{a_\rho\}$ the finite set of points in Δ such that $\Phi^{-1}(a_\rho) = \mathcal{C}_\rho$ is a singular fiber. Each singular fiber $\mathcal{C}_\rho$ can be written as

$$\mathcal{C}_\rho = \sum_i n_{i\rho} \Theta_{i\rho}, \tag{9.38}$$

where $\Theta_{i\rho}$ are non singular rational curves, with $\Theta_{i\rho}^2 = -2$, and $n_{i\rho}$ are integer numbers. Different types of singularities are characterized by (9.38) and the intersection matrix

$(\Theta_{i\rho}.\Theta_{j\rho})$. All different types of Kodaira singularities satisfy the relation

$$\mathcal{C}_\rho^2 = 0. \tag{9.39}$$

Let $\tau(u)$ be the elliptic modulus of the elliptic fiber at the point $u \in \Delta$. For each path α in $\Pi_1(\Delta')$, with $\Delta' = \Delta - \{a_\rho\}$, we can define a monodromy transformation S_α, in $Sl(2, \mathbf{Z})$, acting on $\tau(u)$ as follows:

$$S_\alpha \tau(u) = \frac{a_\alpha \tau(u) + b_\alpha}{c_\alpha \tau(u) + d_\alpha}. \tag{9.40}$$

Each type of Kodaira singularity is characterized by a particular monodromy matrix.

In order to define an elliptic fibration [33], the starting point will be an algebraic curve Δ, that we will take, for simplicity, to be of genus zero, and a meromorphic function $\mathcal{J}(u)$ on Δ. Let us assume $\mathcal{J}(u) \neq 0, 1, \infty$ on $\Delta' = \Delta - \{a_\rho\}$. Then, there exists a multivalued holomorphic function $\tau(u)$, with $\mathrm{Im}\,\tau(u) > 0$, satisfying $\mathcal{J}(u) = j(\tau(u))$, with j the elliptic modular j-function on the upper half plane. As above, for each $\alpha \in \Pi_1(\Delta')$ we define a monodromy matrix S_α, acting on $\tau(u)$ in the form defined by (9.40). Associated to these data we will define an elliptic fibration, (9.37). In order to do that, let us first define the universal covering $\tilde{\Delta}'$, of Δ', and let us identify the covering transformations of $\tilde{\Delta}'$ over Δ', with the elements in $\Pi_1(\Delta')$. Denoting by $\tilde{u}$ a point in $\tilde{\Delta}'$, we define, for each $\alpha \in \Pi_1(\Delta')$, the covering transformation $\tilde{u} \to \alpha\tilde{u}$, by

$$\tau(\alpha\tilde{u}) = S_\alpha \tau(\tilde{u}); \tag{9.41}$$

in other words, we consider τ as a single valued holomorphic function on $\tilde{\Delta}'$. Using (9.40), we define

$$f_\alpha(\tilde{u}) = (c_\alpha \tau(\tilde{u}) + d_\alpha)^{-1}. \tag{9.42}$$

Next, we define the product $\tilde{\Delta}' \times \mathbf{C}$ and, for each (α, n_1, n_2), with $\alpha \in \Pi_1(\Delta')$, and n_1, n_2 integers, the automorphism

$$g(\alpha, n_1, n_2) : (\tilde{u}, \lambda) \to (\alpha\tilde{u}, f_\alpha(\tilde{u})(\lambda + n_1\tau(\tilde{u}) + n_2)). \quad (9.43)$$

Denoting by $\mathcal{G}$ the group of automorphisms (9.43), we define the quotient space

$$B' \equiv (\tilde{\Delta}' \times \mathcal{C})/\mathcal{G}. \quad (9.44)$$

This is a non singular surface, since g, as defined by (9.43), has no fixed points in $\tilde{\Delta}'$. From (9.43) and (9.44), it is clear that B' is an elliptic fibration on Δ', with fiber elliptic curves of elliptic modulus $\tau(u)$. Thus, by the previous construction, we have defined the elliptic fibration away from the singular points a_ρ.
Let us denote E_ρ a local neigbourhood of the point a_ρ, with local coordinate t, and such that $t(a_\rho) = 0$. Let S_ρ be the monodromy associated with a small circle around a_ρ. By $\mathcal{U}_\rho$ we will denote the universal covering of $E'_\rho = E_\rho - a_\rho$, with coordinate ρ defined by

$$\rho = \frac{1}{2\pi i} \log t. \quad (9.45)$$

The analog of (9.41) will be

$$\tau(\rho + 1) = S_\rho \tau(\rho). \quad (9.46)$$

If we go around the points a_ρ, k times, we should act with S_ρ^k; hence, we parametrize each path by the winding number k. The group of automorphisms (9.43), reduced to small closed paths around a_ρ, becomes

$$g(k, n_1, n_2)(\rho, \lambda) = (\rho + k, f_k(\rho)[\lambda + n_1\tau(\rho) + n_2]). \quad (9.47)$$

Denoting by $\mathcal{G}_\rho$ the group (9.47), we define the elliptic fibration around a_ρ as

$$(\mathcal{U}_\rho \times \mathbf{C})/\mathcal{G}_\rho. \tag{9.48}$$

Next, we will extend the elliptic fibration to the singular point a_ρ. We can consider two different cases, depending on the finite or infinite order of S_ρ.

9.3.1 Singularities of Type $\hat{D}_4$: $\mathbf{Z}_2$ Orbifolds

Let us assume S_ρ is of finite order,

$$(S_\rho)^m = \mathbf{1}_d. \tag{9.49}$$

In this case, we can extend (9.48) to the singular points, simply defining a new variable σ as

$$\sigma^m = t. \tag{9.50}$$

Let us denote D a local neigbourhood in the σ-plane of the point $\sigma = 0$, and define the group G_D of automorphisms

$$g(n_1, n_2) : (\sigma, \lambda) = (\sigma, \lambda + n_1\tau(\sigma) + n_2), \tag{9.51}$$

and the space

$$F = (D \times \mathbf{C})/G_D. \tag{9.52}$$

Obviously, F defines an elliptic fibration over D, with fiber F_σ at each point $\sigma \in D$ an elliptic curve of modulus $\tau(\sigma)$. From (9.49) and (9.42), it follows that

$$f_k(\sigma) = 1, \tag{9.53}$$

with $k = O(m)$. Thus, we can define a normal subgroup $\mathcal{N}$ of $\mathcal{G}_\rho$ as the set of transformations (9.47):

$$g(k, n_1, n_2) : (\rho, \lambda) \to (\rho + k, \lambda + n_1\tau(\rho) + n_2). \tag{9.54}$$

Comparing now (9.51) and (9.54), we get

$$(\mathcal{U}_\rho \times \mathbf{C})/\mathcal{N} = (D' \times \mathbf{C})/G_D \equiv F - F_0. \tag{9.55}$$

Using (9.54) and (9.47) we get

$$\mathcal{C} = \mathcal{G}/\mathcal{N}, \tag{9.56}$$

with $\mathcal{C}$ the cyclic group of order m, defined by

$$g_k : (\sigma, \lambda) \to (e^{2\pi i k/m}\sigma, f_k(\sigma)\lambda). \tag{9.57}$$

¿From (9.56) and (9.55), we get the desired extension to a_ρ, namely

$$F/\mathcal{C} = (\mathcal{U}_\rho \times \mathbf{C})/\mathcal{G}_\rho \cup F_0/\mathcal{C}. \tag{9.58}$$

Thus, the elliptic fibration extended to a_ρ, in case S_ρ is of finite order, is defined by $F/\mathcal{C}$. Now, $F/\mathcal{C}$ can have singular points that we can regularize. The simplest example corresponds to

$$S_\rho = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9.59}$$

i. e., a parity transformation. In this case, the order is $m = 2$, and we define σ by $\sigma^2 \equiv t$. The cyclic group (9.57) in this case simply becomes

$$(\sigma, \lambda) \to (-\sigma, -\lambda), \tag{9.60}$$

since from (9.59) and (9.42) we get $f_1 = -1$. At the point $\sigma = 0$ we have four fixed points, the standard $\mathbf{Z}_2$ orbifold points,

$$(0, \frac{a}{2}\tau(0) + \frac{b}{2}), \tag{9.61}$$

with $a, b = 0, 1$. The resolution of these four singular points will produce four irreducible components, $\Theta^1, \ldots, \Theta^4$. In addition, we have the irreducible component Θ_0, defined by the curve itself at $\sigma = 0$. Using the relation $\sigma^2 = t$, we get the $\hat{D}_4$ cycle,

$$\mathcal{C} = 2\Theta_0 + \Theta^1 + \Theta^2 + \Theta^3 + \Theta^4, \tag{9.62}$$

with $(\Theta_0, \Theta^1) = (\Theta_0, \Theta^2) = (\Theta_0, \Theta^3) = (\Theta_0, \Theta^4) = 1$. In general, the four external points of D-diagrams can be associated with the four $\mathbf{Z}_2$ orbifold points of the torus.

9.3.2 Singularities of Type $\hat{A}_{n-1}$

We will now consider the case

$$S_\rho = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \tag{9.63}$$

which is of infinite order. A local model for this monodromy can be defined by

$$\tau(t) = \frac{1}{2\pi i} n \log t. \tag{9.64}$$

Using the variable ρ defined in (9.45), we get, for the group $\mathcal{G}_\rho$ of automorphisms,

$$g(k, n_1, n_2) : (\rho, \lambda) \to (\rho + k, \lambda + n_1 n\rho + n_2), \tag{9.65}$$

and the local model for the elliptic fibration, out of the singular point,

$$(\mathcal{U}_\rho \times \mathbf{C})/\mathcal{G}_\rho, \tag{9.66}$$

i. e., fibers of the type of elliptic curves, with elliptic modulus $n\rho$. A simple way to think about these elliptic curves is in terms of cyclic unramified coverings [31]. Let us recall that a cyclic unramified covering, $\Pi : \hat{C} \to C$, of order n, of a curve C of genus g, is a curve $\hat{C}$ of genus

$$\hat{g} = ng + 1 - n. \tag{9.67}$$

Thus, for $g = 1$, we get $\hat{g} = 1$, for arbitrary n. Denoting by τ the elliptic modulus of C, in case $g = 1$, the elliptic modulus of $\hat{C}$ is given by

$$\hat{\tau} = n\tau. \tag{9.68}$$

Moreover, the generators $\hat{\alpha}$ and $\hat{\beta}$ of $H_1(\hat{C}; \mathbf{Z})$ are given in terms of the homology basis α, β of C as

$$\begin{aligned} \Pi\hat{\alpha} &= \alpha, \\ \Pi\hat{\beta} &= n\beta, \end{aligned} \tag{9.69}$$

with Π the projection $\Pi : \hat{C} \to C$. From (9.68) and (9.65), we can interpret the elliptic fibration (9.66) as one with elliptic fibers given by n-cyclic unramified coverings of a curve C with elliptic modulus ρ or, equivalently, $\frac{1}{2\pi i} \log t$. There exits a simple way to define a family of elliptic curves, with elliptic modulus given by $\frac{1}{2\pi i} \log t$, which is the plumbing fixture construction. Let D_0 be the unit disc around $t = 0$, and let C_0 be the Riemann sphere. Define two local coordinates, $z_a : \mathcal{U}_a \to D_0$, $z_b : \mathcal{U}_b \to D_0$, in disjoint neigborhoods $\mathcal{U}_a$, $\mathcal{U}_b$, of two points P_a and P_b of $\mathcal{C}_0$. Let us then define

$$\begin{aligned} W = \{(p,t) | t \in D_0, p \in C_0 - \mathcal{U}_a - \mathcal{U}_b, \text{ or } p \in \mathcal{U}_a, \\ \text{with } |z_a(p)| > |t|, \text{ or} \\ p \in \mathcal{U}_b, \text{ with } |z_b(p)| > |t|\}, \end{aligned} \tag{9.70}$$

and let S be the surface

$$S = \{xy = t; (x, y, t) \in D_0 \times O_0 \times D_0\}. \tag{9.71}$$

We define the family of curves through the following identifications

$$\begin{aligned}(p_a, t) \in W \cap \mathcal{U}_a \times D_0 &\simeq (z_a(p_a), \frac{t}{z_a(p_a)}, t) \in S, \\ (p_b, t) \in W \cap \mathcal{U}_b \times D_0 &\simeq (\frac{t}{z_b(p_b)}, z_b(p_b), t) \in S.\end{aligned} \tag{9.72}$$

For each t we get a genus one curve, and at $t = 0$ we get a nodal curve by pinching the non zero homology cycles. The pinching region is characterized by

$$xy = t, \tag{9.73}$$

which defines a singularity of type A_0. The elliptic modulus of the curves is given by

$$\tau(t) = \frac{1}{2\pi i} \log t + C_1 t + C_2, \tag{9.74}$$

for some constants C_1 and C_2. We can use an appropriate choice of coordinate t, such that $C_1 = C_2 = 0$. The singularity at $t = 0$ is a singularity of type $\hat{A}_0$, in Kodaira's classification, corresponding to

$$S_\rho = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{9.75}$$

Using now (9.68) and (9.74) we get, for the cyclic covering of order n, the result (9.64), and the group (9.65). The pinching region of the cyclic unramified covering is given by

$$xy = t^n, \tag{9.76}$$

instead of (9.73), i. e., for the surface defining the A_{n-1} singularity, $\mathbf{C}^2/\mathbf{Z}_n$. Now, we can proceed to the resolution of the singularity at $t=0$. The resolution of the singularity (9.76) requires $n-1$ exceptional divisors, $\Theta_1, \ldots, \Theta_{n-1}$. In addition, we have the rational curve Θ_0, defined by the complement of the node. Thus, we get, at $t=0$,

$$\mathcal{C} = \Theta_0 + \cdots + \Theta_{n-1}, \tag{9.77}$$

with $(\Theta_0, \Theta_1) = (\Theta_0, \Theta_{n-1}) = 1$, and $(\Theta_i, \Theta_{i+1}) = 1$, which is the $\hat{A}_{n-1}$ Dynkin diagram. The group of covering transformations of the n^{th} order cyclic unramified covering is $\mathbf{Z}_n$, and the action over the components (9.77) is given by

$$\begin{aligned} \Theta_i &\rightarrow \Theta_{i+1}, \\ \Theta_{n-1} &\rightarrow \Theta_0. \end{aligned} \tag{9.78}$$

9.3.3 Singularities of Type $\hat{D}_{n+4}$

This case is a combination of the two previous examples. Through the same reasoning as above, the group $\mathcal{G}_\rho$ is given, for

$$S_\rho = \begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}. \tag{9.79}$$

by

$$g(k, n_1, n_2) : (\rho, \lambda) \rightarrow (k+\rho, (-1)^k(\lambda + n_1 n \rho + n_2)). \tag{9.80}$$

Using a new variable $\sigma^2 = t$, what we get is a set of irreducible components $\Theta_0, \ldots \Theta_{2n}$, with the identifications $\Theta_i \rightarrow \Theta_{2n-i}$. In addition, we get the four fixed $\mathbf{Z}_2$ orbifold points described above. The singular fiber is then given by

$$\mathcal{C} = 2\Theta_0 + \cdots + 2\Theta_n + \Theta^1 + \Theta^2 + \Theta^3 + \Theta^4, \tag{9.81}$$

with the intersections of the $\hat{D}_{n+4}$ affine diagram. It is easy to see that in this case we also get

$$(\mathcal{C})^2 = 0. \tag{9.82}$$

Defining the genus of the singular fiber by $C^2 = 2g - 2$, we conclude that $g = 1$, for all singularities of Kodaira type. Notice that for rational singularities, characterized by non affine Dynkin diagrams of ADE type [32], we get self intersection $\mathcal{C}^2 = -2$, which corresponds to genus equal zero.

9.4 Kodaira's Classification

Next, we summarize the main results of Kodaira's classification. According to Kodaira's notation [33], we will define an elliptic fibration V onto Δ, where Δ will be chosen as a compact Riemann surface. In general, we take Δ to be of genus equal zero. The elliptic fibration,

$$\Phi : V \to \Delta, \tag{9.83}$$

will be singular at some discrete set of points, a_ρ. The singular fibers, $C_{a\rho}$, are given by

$$C_{a\rho} = \sum n_s \Theta_{\rho s}, \tag{9.84}$$

with $\Theta_{\rho s}$ irreducible curves. According to Kodaira's theorem, all possible types of singular curves are of the following types:

- $$I_{n+1} : \;\; C_{a\rho} = \Theta_0 + \Theta_1 + \cdots + \Theta_n, \quad n+1 \geq 3, \tag{9.85}$$

  where Θ_i are non singular rational curves with intersections $(\Theta_0, \Theta_1) = (\Theta_1, \Theta_2) = \cdots = (\Theta_n, \Theta_0) = 1$.

The A_n affine Dynkin diagram can be associated to I_{n+1}. Different cases are

i) I_0, with $C_\rho = \Theta_0$ and Θ_0 elliptic and non singular.

ii) I_1, with $C_\rho = \Theta_0$ and Θ_0 a rational curve, with one ordinary double point.

iii) I_2, with $C_\rho = \Theta_0 + \Theta_1$ and Θ_0 and Θ_1 non singular rational points, with intersection $(\Theta_0, \Theta_1) = p_1 + p_2$, i. e., two points.

Notice that I_1 and I_2 correspond to diagrams A_0 and A_1, respectively.

- Singularities of type I^*_{n-4} are characterized by

$$I^*_{n-4} : \; C_\rho = \Theta_0 + \Theta_1 + \Theta_2 + \Theta_3 + 2\Theta_4 + 2\Theta_5 + \cdots + 2\Theta_n, \tag{9.86}$$

with intersections $(\Theta_0, \Theta_4) = (\Theta_1, \Theta_4) = (\Theta_2, \Theta_4) = (\Theta_3, \Theta_4) = (\Theta_4, \Theta_5) = (\Theta_5, \Theta_6) = \cdots = 1$, these singularities correspond to the D_n Dynkin diagram.

- Singularities of type II^*, III^* and IV^* correspond to types E_6, E_7 and E_8.

In addition to these singularities, we have also the types

- $II : \; C_\rho = \Theta_0$, with Θ_0 a rational curve with a cusp.

- $III : \; C_\rho = \Theta_0 + \Theta_1$, with Θ_0 and Θ_1 non singular rational curves, with intersection $(\Theta_0, \Theta_1) = 2p$.

- $IV : \; C_\rho = \Theta_0 + \Theta_1 + \Theta_2$, with Θ_0, Θ_1 and Θ_2 non singular rational curves, with intersections $(\Theta_0, \Theta_1) = (\Theta_1, \Theta_2) = (\Theta_2, \Theta_0) = p$.

The monodromies at the singularities are given at the table below.

Matrix	Type of singularity
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	I_1
$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	I_b
$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$	I_b^*
$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	II
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	III
$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	IV

9.5 Elliptically Fibered $K3$

We are now going to consider singularities in the $K3$ manifold. Let C be a rational curve in the $K3$ manifold; then, by equation (9.31), $C \cdot C = -2$. If the curve C is holomorphically embedded it will be an element of the Picard lattice. Its volume is defined as

$$\mathrm{Vol}(C) = J \cdot C, \tag{9.87}$$

with J the Kähler class. A singularity will appear whenever the volume of C goes zero, i. e., whenever the Kähler class J is orthogonal to C. Notice that this implies that C should be orthogonal to the whole 3-plane defined by Ω and J, as C is in fact $(1,1)$, and therefore orthogonal to Ω.

Now, we can define the process of blowing up or down a curve C in X. In fact, a way to blow up is simply changing the moduli space of metrics J, until $J \cdot C$ becomes different from zero. The opposite process is the blow down of the curve. The other way to get rid off the singularity is simply changing the complex structure in such a way that the curve is not in $H^{1,1}$, i. e., the curve does not exist anymore.
We can have different types of singularities, according to how many rational curves C_i are orthogonal to J. The type of singularity will be given by the lattice generated by these C_i curves. Again, these lattices would be characterized by Dynkin diagrams.
Let us now consider an elliptically fibered $K3$ manifold,

$$E \rightarrow X \rightarrow B. \tag{9.88}$$

Elliptic singularities of Kodaira type are characterized by the set of irreducible components X_i of the corresponding singularities. The Picard lattice for these elliptic fibrations contains the $\Gamma^{1,1}$ lattice generated by the fiber and the base, and the contribution of each singularity as given by the Shioda-Tate formula [30]. Defining the Picard number $\rho(X)$ as $1+t$ for a Picard lattice of type $(1,t)$ we get

$$\rho(X) = 2 + \sum_{\nu} \sigma(F_\nu), \tag{9.89}$$

where the sum is over the set of singularities, and where σ is given by $\sigma(A_{n-1}) = n-1$, $\sigma(D_{n+4}) = n+4$, $\sigma(E_6) = 6$, $\sigma(E_7) = 7$, $\sigma(E_8) = 8$, $\sigma(IV) = 2$, $\sigma(III) = 1$, $\sigma(II) = 0$ (equation (9.89) is true provided the Mordell-Weyl group of sections is trivial).
As described in the previous section, the mirror map goes from a manifold X, with Picard lattice of type $(1,t)$, to X^*,

with Picard lattice $(1, 18 - t)$ or, equivalently,

$$\rho(X) + \rho(X^*) = 20. \tag{9.90}$$

Through mirror, we can then pass from an elliptically fibered $K3$ surface, with Picard number $\rho(X) = 2$, which should for instance have all its singularities of type A_0, to a $K3$ surface of Picard number $\rho(X^*) = 18$, which should have 16 singularities of A_1 type, or some other combination of singularities.

Capítulo

10

Superstring Theories

10.1 Worldsheet Supersymmetry

Superstrings correspond to the supersymmetric generalization of the σ-model (7.2). This is performed adding the fermionic term

$$S_F = \int d^2\sigma i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu, \tag{10.1}$$

where ψ^μ are spinors, relative to the worldsheet, and vectors with respect to the spacetime Lorentz group, $SO(1, D-1)$. Spinors in (10.1) are real Majorana spinors, and the Dirac matrices ρ^α, $\alpha = 0, 1$, are defined by

$$\begin{aligned} \rho^0 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \rho^1 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \end{aligned} \tag{10.2}$$

satisfying

$$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}. \tag{10.3}$$

The supersymmetry transformations are defined by

$$\begin{aligned} \delta x^\mu &= \bar{\epsilon}\psi^\mu, \\ \delta\psi^\mu &= -i\rho^\alpha \partial_\alpha x^\mu \epsilon, \end{aligned} \tag{10.4}$$

with ϵ a constant anticonmuting spinor. Defining the components

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}, \tag{10.5}$$

the fermionic lagrangian (10.1) can be rewritten as

$$S_F = \int d^2\sigma(\psi^\mu_-\partial_+\psi^\mu_- + \psi^\mu_+\partial_-\psi^\mu_+), \tag{10.6}$$

with $\partial_\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. As was the case for the bosonic string, we need now to specify the boundary conditions for the fermion fields, both in the open and closed string case. For open strings, there are two possibilities [34], [35]:

$$\begin{aligned} \text{Ramond} \quad &: \quad \psi^\mu_+(\pi,\tau) = \psi^\mu_-(\pi,\tau), \\ \text{Neveu-Schwarz} \quad &: \quad \psi^\mu_+(\pi,\tau) = -\psi^\mu_-(\pi,\tau), \end{aligned} \tag{10.7}$$

which produce the mode expansions

$$\begin{aligned} \text{Ramond} \quad &: \quad \psi^\mu_\mp = \frac{1}{\sqrt{2}}\sum_z d^\mu_n e^{-in(\tau\mp\sigma)}, \\ \text{Neveu-Schwarz} \quad &: \quad \psi^\mu_\mp = \frac{1}{\sqrt{2}}\sum_{z+\frac{1}{2}} b^\mu_n e^{-in(\tau\mp\sigma)}. \end{aligned} \tag{10.8}$$

In the case of closed strings, we can impose either periodic or antiperiodic boundary conditions for the fermions, obtaining Ramond (R) or Neveu-Schwarz (NS) for both $\psi_\pm$ fields.
After quantization we get, following similar steps to those in the bosonic case, that the critical dimension is 10, and that the mass formulas and normal orderings are given by

$$M^2 = \frac{1}{\alpha'}(N_L - \delta_L) = \frac{1}{\alpha'}(N_R - \delta_R), \tag{10.9}$$

with $\delta = \frac{1}{2}$ in the NS sector, and $\delta = 0$ in the R sector. Using this formula, and the GSO projection, we easily get the massless spectrum. For the closed string, the spectrum is

$$\begin{aligned} \text{NS-NS sector} &: \; b^{\mu}_{-1/2} b^{\nu}_{-1/2} |0>, \\ \text{NS-R sector} &: \; b^{\mu}_{-1/2} |S>, \\ \text{R-R sector} &: \; |S> \otimes |S> . \end{aligned} \tag{10.10}$$

The state $|S>$ corresponds to the Ramond vacua (recall $\delta = 0$ in the Ramond sector).
The d_0^{μ} oscillators in (10.8) define a Clifford algebra,

$$\{d_0^{\mu}, d_0^{\nu}\} = \eta^{\mu\nu}. \tag{10.11}$$

Introducing new operators,

$$\begin{aligned} \hat{d}_i^{\pm} &= \frac{1}{\sqrt{2}}(d_0^{2i} \pm d_0^{2i+1}), \;\; i = 1, \ldots, 4, \\ \hat{d}_0^{\pm} &= \frac{1}{\sqrt{2}}(d_0^{1} \mp d_0^{0}), \end{aligned} \tag{10.12}$$

we get

$$\{\hat{d}_i^{+}, \hat{d}_j^{-}\} = \delta_{ij}. \tag{10.13}$$

Using the $\hat{d}_i^{\pm}$, we can define the 32 Ramond vacua as

$$S^{\alpha=\{\pm 1/2, \ldots, \pm 1/2\}} = \hat{d}_0^{\pm} \ldots \hat{d}_4^{\pm}, \tag{10.14}$$

which transform as $SO(10)$ spinors, with the weight of the representation given by $\alpha = \{\pm 1/2, \ldots, \pm 1/2\}$. The representation **32** of $SO(10)$ can be decomposed into **16** and $\hat{\mathbf{16}}$, corresponding to weights with even and odd number of $-\frac{1}{2}$, respectively. Notice that the conformal weight of S^{α} is $\frac{5}{8}$. Physical states, as can be easily seen in the light cone gauge, are

representations of $SO(8)$, which again decomposes into different representations, $\mathbf{8}_S$ and $\mathbf{8}_{S'}$, depending on the even or odd character of the number of $-1/2$'s. In this case, $i = 0, \ldots, 3$, and the spinors have conformal weight equal $\frac{1}{2} = \frac{4}{8}$.
Depending on what is the spinorial representation chosen we get, from (10.10), two different superstring theories. In the chiral case, we choose the same chirality for the two fermionic states in the NS-R and R-NS sectors. This will lead to two gravitinos of equal chirality. Moreover, in the R-R sector we get, for equal chirality,

$$\mathbf{8}_S \otimes \mathbf{8}_S = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_S, \tag{10.15}$$

corresponding to a scalar field being identified with the axion, an antisymmetric field, and a 4-form field. We will call this superstring theory, with $N = 2$ supersymmetry, type IIB. In case we choose different chiralities for the spinor representations associated to the Ramond vacua, what we get is type IIA superstring theory, which is also an $N = 2$ theory, but with two gravitinos of different chirality; now, the R-R sector contains

$$\mathbf{8}_S \otimes \mathbf{8}_{S'} = \mathbf{8}_V \oplus \mathbf{56}_V, \tag{10.16}$$

i. e., a vector field and a 3-form. These are the first two types of superstring theories that we will consider.
In the open case, the massless modes in the NS-sector are given by the states $b^{\mu}_{-1}|k>$, defining a vector representation $\mathbf{8}_S$ of $SO(8)$. In the Ramond sector, the massless states are given by $|S>$ transforming in the spinorial representation of $SO(8)$. This set of massless states defines the supermultiplet of $N = 1$ supersymmetric ten dimensional Yang-Mills theory. The open superstring theory possesses one supersymmetry in

spacetime, and is known as type I theory. The two different theories associated to the two chiralities, $\mathbf{8}_S$ and $\mathbf{8}_{S'}$, of the Ramond vacua are equivalent, differing only in spacetime parity.

The definition of vertex operators in the superstring requires the use of picture changing representations. Introducing a worldsheet superfield,

$$X^\mu(z, \bar{z}, \theta, \bar{\theta}) = x^\mu(z) + x^\mu(\bar{z}) + \theta\psi^\mu(z) + \bar{\theta}\bar{\psi}^\mu(\bar{z}), \quad (10.17)$$

and superderivatives

$$\begin{aligned} D &= \partial_\theta + \theta\partial_z, \\ \hat{D} &= \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}, \end{aligned} \quad (10.18)$$

we can define the bosonic vertices as

$$V = \int d^2z d\theta d\bar{\theta} DX^\mu \bar{D}X^\nu e^{ikX}, \quad (10.19)$$

where we should employ Berezin's integration rules, $\int d\theta = 0$, and $\int d\theta\theta = 1$. Fermion emission vertices are defined using the fermion spinor fields S^α, defined by (10.14), and the spinor ghost $e^{-\phi/2}$, with ϕ the field obtained after bosonization of the ghost current [36],

$$V_F = \int e^{-\phi/2} e^{-\bar{\phi}/2} S_\alpha \tilde{S}_\beta e^{ikX}, \quad (10.20)$$

where S_α, $\tilde{S}_\beta$ are in the $SO(10)$ spinor representation, and will be of different or equal chirality, depending on the string theory (type IIA or type IIB) under consideration. The vertex (10.20) is in the $(-1/2, 1/2)$ picture, where the labels of the picture refer to the ghost number of the spinor fields, $e^{\phi/2}$ and $e^{-\phi/2}$. In type IIA (IIB) string theory, the product of spinors in (10.20) will contain n-forms, with n even (odd), corresponding to the odd (even) $SO(8)$ tensors. Notice that as we are not working in the light cone gauge, the spinors in (10.20) are in $SO(10)$ representations, and therefore the vertices are related to the corresponding R-R field strength tensors. The condition of physical states implies the equations of motion and Bianchi identity for these R-R field strengths.

10.2 Green-Schwarz Superstring: String Scan

A manifestly spacetime supersymmetric action for the string can be obtained by generalization of the Nambu-Goto action, involving supersymmetric invariant line elements. Introducing spacetime fermionic variables, θ^α, we define

$$\begin{aligned} \Pi_i^\mu &= \partial_i x^\mu - \bar{\theta}\gamma^\mu\partial_i\theta, \\ \Pi_i^\alpha &= \partial_i\theta^\alpha. \end{aligned} \tag{10.21}$$

Introducing coordinates in superspace,

$$z^M = (x^\mu, \theta^\alpha), \tag{10.22}$$

where $M = (\mu, \alpha)$, and the supervielbein E_M^A, (10.21) can be written, in terms of the pull back, as

$$\Pi_i^A = \partial_i z^M E_M^A. \tag{10.23}$$

The simplest generalization of the Nambu-Goto action in terms of the line elements (10.21) would be

$$S = T\int d\sigma d\tau \sqrt{\det\ \Pi_i^a \Pi_j^b \eta_{ab}}. \tag{10.24}$$

However, the action (10.24) is problematic since, for instance, in ten dimensions we have as bosonic degrees of freedom $10 - 2 = 8$, and as fermionic degrees of freedom, 16, which is the number of independent components of spinors in ten dimensions. This seems to indicate that we need some extra invariance reducing by one half the number of spinor degrees of freedom. This invariance is known as kappa symmetry [37], and appears once we add to (10.24) the extra Wess-Zumino term,

$$S = T \int d\sigma d\tau \sqrt{\det \, \Pi_i^a \Pi_j^b \eta_{ab}} + \epsilon^{ij} \partial_i z^M \partial_j z^N B_{MN}, \quad (10.25)$$

with B_{MN} a 2-form superfield. The action (10.25) is invariant under kappa-transformations,

$$\begin{array}{rcl} \delta z^M E_M^a & = & 0, \\ \delta z^M E_M^\alpha & = & k^\beta (1+\gamma)_\beta^\alpha, \end{array} \quad (10.26)$$

with κ^β a spacetime spinor, and with

$$\gamma_\beta^\alpha = \frac{1}{3} \epsilon^{i_1 i_2 i_3} E_{i_1}^{a_1} E_{i_2}^{a_2} E_{i_3}^{a_3} \gamma_{a_1 a_2 a_3}, \quad (10.27)$$

where γ are Dirac matrices, and $\gamma_{a_1 a_2 a_3} = \gamma_{[a_1 a_2 a_3]}$. The kappa-transformation (10.26) allows us to gauge away half of the fermionic degrees of freedom, which leads to the correct matching

$$10 - 2 = \frac{16}{2}. \quad (10.28)$$

Classically, the matching (10.28) works in three, four six and ten dimensions, that can be denoted as the real, complex, quaternionic and octonionic classical strings.

In chapter 6 this construction will be generalized to more general extended objects.

10.3 The $SO(32)$ Type I Superstring

In open string theory, the standard way to introduce a gauge group, is by adding Chan-Paton [38] factors at the end points of the string. These factors can be visualized as a quark-antiquark, transforming in the fundamental representations R, $\bar{R}$ of some gauge group G. Thus, the open string states will be defined as $|\Lambda(ij)>$, with the labels i and $\bar{j}$ in the basis of R and $\bar{R}$, respectively. Allowed gauge groups are restricted by the consistency of string theory. First of all, the massless modes $b^{\mu}_{-1}|k>$ are required to transform as gluons in the adjoint representation. This is automatically obtained for oriented strings, when working with the gauge group $U(N)$, as in this case $R\times\bar{R}$ is exactly the adjoint representation. For other gauge groups, $R\times\bar{R}$ will contain more than the adjoint representation, so that extra constraints must be imposed. This can be done for unoriented strings, and for orthogonal and simplectic groups. In this cases, R and $\bar{R}$ are equivalent, and invariance can be imposed with respect to the reflection $\sigma\rightarrow\pi-\sigma$.

As is well known, the oriented case with $U(N)$ gauge group is ruled out by simple supersymmetry arguments. In fact, as has already been discussed, open and closed strings are coupled, at one loop level. In the oriented case, closed superstrings produce $N=2$ supergravity, so that in order to have consistently defined oriented open strings, with gauge group $U(N)$, we should be able to couple $N=1$ supersymmetric Yang-Mills to $N=2$ supergravity, which is in fact impossible. Thus, the only possible candidate will be unoriented strings, with orthogonal or simplectic gauge group. To check the consistency of these theories, the computation of tadpoles must be worked out. In the unoriented case we should consider, at one loop, the contribution of two different topologies: the anulus and the Moëbius strip (this second topology can be interpreted as a cylinder with a cross cup). As we are interested in tadpoles, we should expect two contributions; one associated to the disc, and the other to the cross cup, $\mathbf{RP}^2$. These tadpoles will generically be denoted by Γ_i. Let us now introduce boundary states, $|B>$ and $|C>$, for the disc and the cross cup. These boundary states are simply closed string states, defined through the corresponding boundary conditions. We will describe these states more precisely in a moment, but before this we will work out the argument at a formal level.

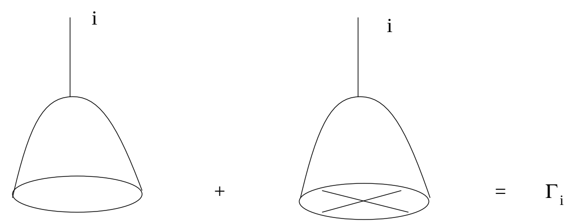

Figura 10.1: Dilaton contribution in the presence of a D-brane

Introducing a complete tadpole state in the NS sector, $|T>_{NS}$, Figure 10.3 should be interpreted as follows:

$$|T>_{NS}= Z_1|B> +Z_2|C>, \tag{10.29}$$

with the tadpoles related to

$$[<T|_{NS}(|B> +|C>)]^2 = (Z_1 + Z_2)^2. \tag{10.30}$$

Using (10.29), we can decompose the amplitude,

$$A = \int_0^\infty d\rho_{NS} < \theta T|e^{-\rho H}|T>_{NS}, \tag{10.31}$$

where H is the closed string hamiltonian, ρ represents an euclidean time on the worldsheet, and θ is the CPT worldsheet transformation of the sum of three different amplitudes with the topologies of the cylinder, the Moëbius strip and Klein bottle, respectively. The result is depicted in Figure 10.3.

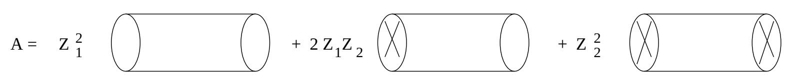

Figura 10.2: Dilaton contribution in the presence of a D-brane

The tadpoles can be approximated as the $\rho \to \infty$ limit is taken in (10.31). Then, (10.31) can be approximated through

$$A = \sum_i \left(\frac{\Gamma_i^2}{k^2 + m_i^2} \right)_{k=0}, \tag{10.32}$$

with Γ_i the different tadpoles at zero momentum. The amplitude A can be computed independently as a one loop string amplitude. For the cylinder and the Moëbius strip, this is an open string at one loop. The case of the Klein bottle is a closed string one loop effect. Thus, the total amplitude will be given by

$$A = \int \frac{dt}{t} \mathrm{tr}\, (e^{-t(p^2+m^2)}(-1)^{2J} P_{GSO} \frac{1}{2}(1+\Omega)), \tag{10.33}$$

where the trace is taken over both closed and open string states, with the GSO projector $\frac{1}{2}(1+(-1)^F)$ for open strings, and $\frac{1}{4}(1+(-1)^F)(1+(-1)^{\tilde{F}})$ for closed strings. The orientation operator Ω is the worldsheet parity, and $(-1)^{2J}$ is the spacetime fermion number. As is clear from the definition, the amplitude A contains the contributions of the three different topologies, cylinder, Moëbius strip and Klein bottle.

The amplitude (10.33), interpreted as a one loop open string process, includes the two open string sectors, R and NS. In addition, we can include boundary conditions in the time direction, corresponding to R-R and NS-NS sectors of the tree level closed string amplitude. Thus, depending on this boundary condition, we define two different amplitudes, $A^{\pm}$, one to one related to the NS-NS and R-R sectors of the closed string, satisfying

$$A^{+} = -A^{-} \tag{10.34}$$

as a consequence of supersymmetry.
In order to compare Figure 10.3 and (10.33), we should specify the boundary states $|B>$ and $|C>$. Instead of a detailed treatment, we will simply concentrate on the bosonic part of the $|B>$ states. Given a cylinder of length l, the boundary conditions on the bosonic field are

$$\partial_\sigma X^\mu(0,\tau) = \partial_\sigma X^\mu(l,\tau) = 0. \tag{10.35}$$

Hence, the state $|B>$, in its bosonic part, is determined by the conserved charge condition

$$P^\mu|B>= 0. \tag{10.36}$$

The complete solution for $|B>$ requires taking into account boundary conditions for fermionic fields, as well as for ghost fields. Using this boundary states, it was shown in [17] that

$$\begin{aligned} Z_1 &= N.2^{-2}(2\pi)^{-5/2}, \\ Z_2 &= -32.2^{-2}(2\pi)^{-5/2}, \end{aligned} \tag{10.37}$$

for $SO(N)$ gauge group. Using (10.32) and (10.37) we get, for the dilaton tadpole,

$$\Gamma^2 = \frac{(N-32)^2}{64\pi^5}, \tag{10.38}$$

that only vanishes for $SO(32)$. For $N \neq 32$, we get a non vanishing dilaton tadpole, so that we may wonder about the existence of some sort of Fishler-Susskind mechanism in order to consistently kill the dilaton tadpole through a change of background. The situation is different in the bosonic and fermionic cases. By supersymmetry, equation (10.34), we get a vanishing one loop amplitude; hence, if there exists a non vanishing tadpole in the NS-NS sector, some tadpole for a massless field must exist in the R-R sector, of equal value and different sign. The problem with the Fishler-Susskind mechanism is cancelling, through a change of background, this tadpole of the R-R sector.

In fact, if we assume the existence of a R-R tadpole equal and of opposite sign to the NS-NS dilaton tadpole, we will get

$$dH = d^*H = \Gamma, \tag{10.39}$$

with H a R-R field strength form. From (10.39), we observe that H should be a 1-form, or its dual a 11-form. This field strength corresponds to a field potential being a 10 or a 2-form. We can now try to imagine a source for the 10-form. It should be a 9 dimensional object. After our discussion on D-branes, it is natural to interpret ten dimensional spacetime as the worldvolume of a D-9brane, so that it is natural to interpret the R-R tadpole as a way to associate to the D-9brane, in the supersymmetric case, a R-R charge. Moreover, as we are introducing the R-R tadpole in order to cancel the NS-NS dilaton tadpole, we can interpret the D-9brane as a BPS object, since charge and mass are, by construction, equal. In fact, the structure of the dilaton tadpole, as coming from the disc and the cross cup, allows us to associate the disc part with D-9branes, and the cross cup contribution to 9-orientifolds. Due to the orientifold geometry, the N in (10.38) can be associated to $\frac{N}{2}$ D-9branes, so that we can normalize the R-R charge of the D-9brane to two. However, the orientifold contributes with -32, i. e., 16 times the R-R charge of the D-9brane. Thus, we get

$$\mu_{9-O}^{R-R} = -2^4 \mu_{D-9}^{R-R} \tag{10.40}$$

for the relation between the R-R charge of the 9orientifold and the R-R charge of the D-9brane. This is a particular case of a more general relation,

$$\mu_{p-O}^{R-R} = -2^{9-p} \mu_{D-p}^{R-R}. \tag{10.41}$$

If $N \neq 32$ we get, from (10.39), a coupling of the form

$$(N - 32) \int_{D-9brane} A_{10}, \tag{10.42}$$

and the equations of motion imply that N should equal $N = 32$ or, equivalently, that the Fishler-Susskind mechanism can not be extended to cancel the tadpoles in the R-R sector, concluding that only the unoriented $SO(32)$ open string is consistent.

In summary, the previous discussion already provides some glimpses on the dynamical relevance of D-branes in the supersymmetric case. In fact, by identifying spacetime with the D-9brane worldvolume, we have learn that the R-R tadpoles have the interpretation of R-R charges on the D-9branes. The natural extension, that we will soon work out, is considering D-branes of smaller dimension, and computing the corresponding tadpoles for the disc with boundary on the D-brane.

10.4 Toroidal Compactification of Type IIA and Type IIB Theories. U-duality

Before considering different compactifications of superstring theories, we will first review some general results on the maximum number of allowed supersymmetry, depending on the spacetime dimension.

Spinors should be considered as representations of $SO(1, d-1)$. Irreducible representations have dimension

$$2^{[\frac{d+1}{2}]-1}, \tag{10.43}$$

where [] stands for the integer part. Depending on the dimension, the larger spinor can be real, complex or quaternionic,

$$\begin{aligned} \mathbf{R}, \text{ if } d &= 1,2,3 \mod 8, \\ \mathbf{C}, \text{ if } d &= 0 \mod 4, \\ \mathbf{H}, \text{ if } d &= 5,6,7 \mod 8. \end{aligned} \tag{10.44}$$

Using (10.43) and (10.44), we get the number of supersymmetries listed in the table below[α].

[α]This table is constrained by the physical requirement that particles with spin > 2 do not appear.

Dimension	N	Irreducible Representation
11	1	$\mathbf{R}^{32}$
10	2	$\mathbf{R}^{16}$
9	2	$\mathbf{R}^{16}$
8	2	$\mathbf{C}^{8}$
7	2	$\mathbf{H}^{8}$
6	4	$\mathbf{H}^{4}$
5	4	$\mathbf{H}^{4}$
4	8	$\mathbf{C}^{2}$
3	16	$\mathbf{R}^{2}$

The maximum number of supersymmetries in three dimensions is then 16. From the table it is also clear that through standard Kaluza-Klein compactification, starting with six dimensional $N = 1$ supersymmetry leads to four dimensional $N = 2$, and three dimensional $N = 4$ supersymmetry. We can also notice that ten dimensional $N = 1$ leads to $N = 4$ supersymmetry in four dimensions.

It must be stressed that the counting of supersymmetries after dimensional reduction is slightly more subtle if we compactify on manifolds with non trivial topology. Here, the adequate concept is the holonomy of the internal manifold; let us therefore recall some facts on the concept of holonomy. Given a Riemannian manifold $\mathcal{M}$, the holonomy group $H_{\mathcal{M}}$ is defined as the set of transformations M_γ associated with paths γ in $\mathcal{M}$, defined by parallel transport of vectors in the tangent bundle. The connection used in this definition is the Levi-Civita connection. In general, for a vector bundle $E \to \mathcal{M}$, the holonomy group $H_{\mathcal{M}}$ is defined by the parallel transport of v in the fiber, with respect to the connection on E. The Ambrose-Singer theorem shows how the holonomy is generated by the curvature.

Manifolds can be classified according to its holonomy group. Therefore, we get [?]

- $H_{\mathcal{M}} = O(d)$, for real manifolds of dimension d.
- $H_{\mathcal{M}} = U(\frac{d}{2})$, for Kähler manifolds.
- $H_{\mathcal{M}} = SU(\frac{d}{2})$, for Ricci flat Kähler manifolds.
- $H_{\mathcal{M}} = Sp(\frac{d}{4})$, for hyperkähler manifolds[α].

The answer to the question of what the role of holonomy is in the counting of the number of supersymmetries surviving after compactification is quite simple: let us suppose we are in dimension d, so that the spinors are in $SO(1, d-1)$. Now, the theory is compactified on a manifold of dimension d_1, down to $d_2 = d - d_1$. Supersymmetries in d_2 are associated with representations of $SO(1, d_2 - 1)$, so we need to decompose an irreducible representation of $SO(1, d-1)$, into $SO(1, d_2 - 1) \times SO(d_1)$. Now, the holonomy group of the internal manifold $H_{\mathcal{M}_{d_1}}$ will be part of $SO(d_1)$. Good spinors in d_2 dimensions would be associated with singlets of the holonomy group of the internal manifold. Let us consider the simplest case, with $d_1 = 4$; then,

$$SO(4) = SU(2) \otimes SU(2) \tag{10.45}$$

and, if our manifold is Ricci flat and Kähler, the holonomy will be one of these $SU(2)$ factors. Therefore, we will need a singlet with respect to this $SU(2)$. As an example, let us consider the spinor in ten dimensions, with $N=1$; as we can see from the above table, it is a **16**, that we can decompose with respect to $SO(1,5) \times SU(2) \times SU(2)$ as

$$\mathbf{16} = (\mathbf{4}, \mathbf{2}, \mathbf{1}) \otimes (\mathbf{4}, \mathbf{1}, \mathbf{2}). \tag{10.46}$$

[α]Notice that any hyperkähler manifold is always Ricci flat.

Therefore, we only get one surviving supersymmetry in six dimensions. This is a general result: if we compactify a ten dimensional theory on a manifold of dimension four, with $SU(2)$ holonomy, we will get a six dimensional theory with only one supersymmetry. However, if the compactification is on a torus with trivial holonomy, two supersymmetries are obtained (the maximum number of allowed supersymmetries).
As the first contact with type IIA string theory we will then consider its compactification on a d-dimensional torus, T^d. To start with, let us work in the particular case $d = 4$. From the above table, we learn that the number of supersymmetries in six dimensions is 4, as the holonomy of T^4 is trivial. If we do not take into account the R-R fields, the moduli of the string σ-model is

$$O(4,4;\mathbf{Z})\backslash O(4,4)/O(4) \times O(4), \tag{10.47}$$

with the T-duality $O(4,4;\mathbf{Z})$ corresponding to changes of the type $R_i \rightarrow \frac{\alpha'}{R_i}$, for the four S^1 cycles composing the torus. The situation becomes different if we allow R-R background fields. In such a case, we should take into account the possibility of including Wilson lines for the A_μ field (the $\mathbf{8}_V$ in (10.16)), and also a background for the 3-form $A_{\mu\nu\rho}$ (the $\mathbf{56}_V$ of (10.16)). The number of Wilson lines is certainly 4, one for each non contractible loop in T^d, so we need to add 4 dimensions to the 16-dimensional space (10.47). Concerning an $A_{\mu\nu\rho}$ background, the corresponding moduli is determined by $H_3(T^4)$, which implies 4 extra parameters. Finally, the dimension equals

$$16 + 4 + 4 = 24. \tag{10.48}$$

Now, a new extra dimension coming form the dilaton field must be added. It is important here to stress this fact: in the approach in previous section to σ-model moduli space the dilaton moduli has not been considered. This corresponds to interpreting the dilaton as a string coupling constant, and allowing changes only in the string. Anyway, this differentiation is rather cumbersome. Including the dilaton moduli in (10.48), we get a moduli space of dimension (8.2), that can be written as

$$O(5,5;\mathbf{Z})\backslash O(5,5)/O(5)\times O(5). \qquad (10.49)$$

The proposal of moduli (10.49) for type IIA on T^4 already contains a lot of novelties. First of all, the modular group $O(5,5;\mathbf{Z})$ now acts on the dilaton and the resting Ramond fields. In fact, relative to the $O(4,4;\mathbf{Z})$ T-duality of toroidal compactifications, we have now an extra symmetry which is known as S-duality [39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51],

$$g\to\frac{1}{g},\quad R^i\to\frac{R^i}{\sqrt{g}} \qquad (10.50)$$

with g the string coupling constant. Transformations (10.50), together with T-duality transformations (8.5) and (8.10) combine into a new modular symmetry, which is called in the physics literature U-duality [43]. The phenomena found here resembles very much what arises from mirror symmetry in the analysis of $K3$. There, the "classical" modular group was $O(\Gamma^{3,19};\mathbf{Z})$, and quantum mirror symmetry creates the enhancement to $O(\Gamma^{4,20};\mathbf{Z})$ where, in addition to T-duality, we have mirror transformations. In the case of type IIA on T^4, it is because we include the R-R backgrounds and the dilaton that the modular symmetry $O(4,4;\mathbf{Z})$ is enhanced to the U-duality symmetry. However, in spite of the analogies, the physical meaning is different. To appreciate this, let us now consider type IIA on $K3$. The dilaton moduli can be added, but the R-R fields are not producing any new moduli. In fact, recall that $\Pi_1(K3) = 0$, and $H_3 = 0$, so that the moduli of type IIA on $K3$ is simply

$$O(4,20;\mathbf{Z})\backslash O(4,20)/O(4) \times O(20) \quad \times \mathbf{R}, \tag{10.51}$$

with $\mathbf{R}$ parametrizing the dilaton, and the modular group not acting on it.

The way to interpret the moduli (10.49) goes under the name of M-theory. Before entering a more precise definition of M-theory, the basic idea is thinking of (10.49) simply as the moduli of a toroidal compactification on T^5; however, in order to obtain a six dimensional $N = 4$ theory, we need to start with some theory living in 11 dimensions. The theory satisfying this is M-theory, a theory whose low energy supergravity description is well understood: it should be such that through standard Kaluza-Klein compactification it gives the field theory limit of type IIA strings; but this a theory known as eleven dimensional type IIA supergravity.

Once we have followed the construction of the type IIA string theory moduli on T^4, let us consider the general case of compactification on T^d. The dimension of the moduli is

$$\dim \; = d^2 + 1 + d + \frac{d(d-1)(d-2)}{3}, \tag{10.52}$$

where d^2 is the NS-NS contribution, the 1 summand comes form the dilaton, d from the Wilson lines, and $\frac{d(d-1)(d-2)}{3}$ from the 3-form $A_{\mu\nu\rho}$. The formula (10.52) has to be completed, for $d \geq 5$, by including dual scalars. For $d = 5$, the dual to the 3-form $A_{\mu\nu\rho}$ is a scalar. The result is

$$\frac{d(d-1)(d-2)(d-3)(d-4)}{5} \quad \text{duals} \quad \text{to } A_{\mu\nu\rho},$$

$$\frac{d(d-1)\dots(d-6)}{7} \quad \text{duals} \quad \text{to } A_\mu. \tag{10.53}$$

The moduli spaces, according to the value of the dimension of the compactification torus, are listed in the table below.

Dimension	**Moduli**
$d=4$	$O(5,5;\mathbf{Z})\backslash O(5,5)/O(5)\times O(5)$
$d=5$	$E_{6,(6)}(\mathbf{Z})\backslash E_{6,(6)}/Sp(4)$
$d=6$	$E_{7,(7)}(\mathbf{Z})\backslash E_{7,(7)}/SU(8)$
$d=3$	$Sl(5,\mathbf{Z})\backslash Sl(5)/SO(5)$
$d=2$	$Sl(3,\mathbf{Z})\times Sl(2,\mathbf{Z})\backslash Sl(3)/SO(3)\ \ Sl(2)/SO(2)$

For supergravity practitioners, the appearance of E_6 and E_7 in this table should not be a surprise.
Let us now see what happens in the type IIB case. The moduli on, for instance, T^4, is again the 16 dimensional piece coming from the NS-NS sector; now, the R-R sector is determined by the cohomology groups H^0, H^2 and H^4 (see equation (10.15)). From the Hodge diamond for T^4,

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array} \tag{10.54}$$

we get 8 extra modulis, exactly the same number as in the type IIA case. This is a general result for any T^d compactification. The reason for this is that type IIA and type IIB string theories are, after toroidal compactification, related by T-duality. However, on a manifold as $K3$, with $\Pi_1=0$, the moduli for IIA and IIB are drastically different, as can be derived from direct inspection of the $K3$ Hodge diamond (see equation (9.10)). Therefore, for type IIB we get, from the R-R sector, 1 coming from H^0, 22 from H^2, and 1 from H^4, which sums up a total of 24 extra modulis to be added to the $58 + 22$ of the NS-NS sector. Then, including the dilaton,

$$\dim \mathrm{IIB}(K3) = 22 + 58 + 24 + 1 = 105. \tag{10.55}$$

Therefore, the natural guess for the moduli is

$$O(5, 21; \mathbf{Z}) \backslash O(5, 21)/O(5) \times O(21). \tag{10.56}$$

Here, something quite surprising is taking place. As we can see from (10.51), when type IIA is compactified on $K3$, we do not find any appearance of U-duality or, in other words, S-duality. By contrast, in the type IIB case we find a modular group $O(5, 2; \mathbf{Z})$, that contains the dilaton and, therefore, the S-duality transformation. This is what can be called the S-duality of type IIB string theory [**?**], which can already be observed from equation (10.15). In fact, the R-R and NS-NS sectors both contain scalar fields and the antisymmetric tensor.

10.5 Heterotic String and $K3$ Surfaces

The idea of "heterosis", one of the most beautiful and productive ideas in the recent history of string theory [52] was motivated by two basic facts. First of all, the need to find a natural way to define non abelian gauge theories in string theory, without entering the use of Chan-Paton factors, and, secondly, the sharpness of the gap in string theory between left and right moving degrees of freedom. Here, we will concentrate on some of the ideas leading to the construction of heterosis. In the toroidal compactification of the bosonic string on T^d, we have found that the momenta live in a $\Gamma^{d,d}$ lattice. This is also true for the NS sector of the superstring. The lattice $\Gamma^{d,d}$, where the momenta live, is even and self dual. Taking into account the independence between left and right sectors, we can think on the possibility to compactify the left and right components on different tori, T^{d_L} and T^{d_R}, and consider as the corresponding moduli the manifold

$$O(d_L, d_R; \mathbf{Z})\backslash O(d_L, d_R)/O(d_L) \times O(d_R). \tag{10.57}$$

Before trying to find out the consistency of this picture, let us try to get a simple interpretation of moduli (10.57). The dimension of this moduli is $d_L \times d_R$, and we can separate it into $d_L \times d_L + d_L \times (d_R - d_L)$. Let us interpret the first part, $d_L \times d_L$, as the standard moduli for compactifications on a torus T^{d_L}; then, the second piece can be interpreted as the moduli of Wilson lines for a gauge group

$$U(1)^{d_R - d_L}. \tag{10.58}$$

With this simple interpretation, we already notice the interplay in heterosis when working with a gauge group that can be potentially non abelian, the gauge group (10.58), and differentiating left and right parts. When we were working with type II string theory, and considered toroidal compactifications, we were also adding, to the moduli space, the contribution of the Wilson lines for the RR gauge field, A_μ (in case we are in type IIA). However, in the case of type IIA on T^4, taking into account the Wilson lines did not introduce any heterosis asymmetry in the moduli of the kind (10.57). However, T^4 is not the only Ricci flat four dimensional manifold; we can also consider $K3$ surfaces. It looks like if T^4, $K3$, and its orbifold surface in between, $T^4/\mathbf{Z}_2$, saturate all compactification manifolds that can be thought in four dimensions. In the case of $K3$, the moduli of type IIA string (see equation (10.51)) really looks like the heterotic moduli, of the kind (10.57), we are looking for. Moreover, in this case, and based on the knowledge of the lattice of the second cohomology group of $K3$ (see equation (9.15)),

$$E_8 \perp E_8 \perp \mathcal{U} \perp \mathcal{U} \perp \mathcal{U}, \tag{10.59}$$

we can interpret the $16 = d_R - d_L$ units as corresponding precisely to Wilson lines of the $E_8 \times E_8$ gauge group appearing in (10.59). In other words, and following a very distant path form the historical one, what we are suggesting is interpreting moduli (10.51), of type IIA on $K3$, as some sort of heterosis, with $d_L = 4$ and $d_R = 20$. The magic of numbers is in fact playing in our team, as the numbers we get for d_L and d_R strongly suggest a left part, of critical dimension 10, and a right part, of precisely the critical dimension of the bosonic string, 26. This was, in fact, the original idea hidden under heterosis: working out a string theory looking, in its left components, as the standard superstring, and in its right components as the 26 dimensional bosonic string. However, we are still missing something in the "heterotic" interpretation of (10.51), which is the visualization, from $K3$ geometry, of the gauge group. In order to see this, some of the geometrical material introduced in subsection 9.1 will be needed; in terms of the concepts there introduced, we would claim that the (p_L, p_R) momentum is living in the lattice $\Gamma^{4,20}$. We can then think that p_L is in the space-like 4-plane where the holomorphic top form Ω, and the Kähler class J, are included. Recall that they define a space-like 3-plane. Now, momentum vectors, orthogonal to this 4-plane, can be considered; they are of the type

$$(0, p_R). \tag{10.60}$$

Now, whenever $p_R^2 = -2$, this vector will define a rational curve inside $K3$, with vanishing volume (in fact, the volume is given by $p_R \cdot J = 0$). The points $p_R^2 = -2$ will be at the root lattice of $E_8 \times E_8$. Now, from the mass formulas (8.3) we easily observe that $p_R^2 = -2$ is the condition for massless vector particles. In fact, if we separate, in the spirit of heterosis, the p_R of a 26 dimensional bosonic string into $(p_R^{(16)}, p_R^{(10)})$, we get, from (8.3),

$$M^2 = 4(p_R^{(16)})^2 + 8(N-1), \tag{10.61}$$

so that $M^2 = 0$, for $N=0$, if $(p_R^{(16)})^2 = 2$. The sign difference appears here because (recall subsection 9.1) in the $K3$ construction used for the second cohomology lattice, the E_8 lattice was defined by minus the Cartan algebra of E_8. Therefore, we observe that massless vector bosons in heterotic string are related to rational curves in $K3$ of vanishing volume, which allows to consider enhancement of symmetries when moving in moduli space [46, 53, 54]. Some of these rational curves can be blown up, which would be the geometrical analog of the Higgs mechanism, or either blown down, getting extra massless stuff. Moreover, for elliptically fibered $K3$ surfaces, the different Kodaira singularities reflect, in its Dynkin diagram, the kind of gauge symmetry to be found.

The above discussion summarizes what can be called the first quasi-theorem on string equivalence [43, 46],

Quasi-Theorem 1 Type IIA string on $K3$ is equivalent to $E_8 \times E_8$ heterotic string on T^4.

The meaning of the duality relation between type IIA string theory on $K3$, and heterotic string on T^4 established in the previous theorem can be clarified when working at the level of effective lagrangians. For the heterotic string compactified on T^4 to six dimensions, a lagrangian of the form

$$L = \int d^6x \sqrt{g} e^{-2\phi}[R + |\nabla\phi|^2 + |dB|^2 + |dA|^2] \qquad (10.62)$$

is expected (numerical factors, of no relevance for the general argument, have been omitted). As is clear form (10.62), all terms scale like $e^{-2\phi}$. The field A stands for any gauge field appearing upon the toroidal compactification.
Let us now consider the case of type IIA string theory. The field content is the same, with the important difference that now the gauge fields A are part of the RR sector. Naming g', ϕ', B' and A' the fields in the type IIA theory, the lagrangian one expects to find would be

$$L = \int d^6x \sqrt{g'} e^{-2\phi'}[R' + |\nabla\phi'|^2 + |dB'|^2 + |dA'|^2]. \qquad (10.63)$$

The expected duality between the two theories should correspond, at the level of the effective lagrangians (10.62) and (10.63), to a precise change of variables. In fact, the change we need [46] is $g' = g'' e^{2\phi}$:

$$L = \int d^6x \sqrt{g''}[e^{2\phi'}(R + |\nabla\phi|^2) + e^{-2\phi'}|dB'|^2 + e^{2\phi'}|dA|^2] \qquad (10.64)$$

To recover (10.62), the relations

$$\phi' = -\phi,$$

$$dB = e^{-2\phi'} dB', \\ A = A'. \tag{10.65}$$

The first equation contains the core of the S-duality relation between the two theories: to go from the heterotic to type IIA string theory, the string coupling must transform as $g \to \frac{1}{g}$. Previous arguments were so general that we can probably obtain extra equivalences by direct inspection of the different $K3$ moduli spaces that have been discussed in subsection 9.1. In particular, let us consider the moduli space of complex structures for an elliptically fibered $K3$ surface, a fact represented, in terms of the Picard lattice, clamming that it is of $\Gamma^{1,1}$ type, generated by a section, and with the fiber satisfying relations (8.6). This moduli is

$$O(2, 18; \mathbf{Z}) \backslash O(2, 18)/O(2) \times O(18), \tag{10.66}$$

where we have used equation (9.32), and the fact that the transcendental lattice is of type $(2, 18)$. From the heterosis point of view, it would be reasonable to interpret (10.66) as heterotic $E_8 \times E_8$ string, compactified on a 2-torus, T^2. In fact, we will have 4 real moduli, corresponding to the Kähler class and complex structure of T^2, and 16 extra complex moduli associated to the Wilson lines. However, now the type II interpretation of (10.66) is far from being clear, as (10.66) is just the part of the moduli space that is preserving the elliptic fibration. Now, in order to answer how (10.66) can be understood as a type II compactification a similar problem appears as we try to work out an heterotic interpretation of the type IIB moduli on $K3$, given in (10.56). A simple way to try to interpret (10.66), as some kind of type II compactification, is of course thinking of an elliptically fibered $K3$, where the volume of the fiber is fixed to be equal zero; generically,

$$J \cdot F = 0, \tag{10.67}$$

where F indicates the class of the fiber. Now, we can think that we are compactifying a type II string on the base space of the bundle. However, this does not lead to (10.66) for the type IIA case, as the RR fields are in H^1 and H^3, which will vanish. But what about type IIB? In this case, we have the NS field ϕ, and the R field χ, and we should fix the moduli of possible configurations of these fields on the base space of the elliptic fibration. Here, type IIB S-duality, already implicit in moduli (10.56), can help enormously, mainly because we are dealing with an elliptically fibered $K3$ manifold [**?**, 68, 58]. To proceed, let us organize the fields ϕ and χ into the complex

$$\tau = \chi + ie^{-\phi}, \tag{10.68}$$

and identify this τ with the moduli of the elliptic fiber. Then, the 18 complex moduli dimension of (10.66) parametrizes the moduli of complex structures of the elliptic fibration, and therefore the moduli of τ field configurations on the base space (provided τ and $\frac{a\tau+b}{c\tau+d}$ are equivalent from the type IIB point of view). These moduli parametrize then the type IIB compactification on the base space B (it is $\mathbb{P}^1$; recall that in deriving (10.66) we have used a base space B such that $B \cdot B = -2$). There is still one moduli missing: the size of the base space B, that we can identify with the heterotic string coupling constant. Thus, we arrive to the following quasi-theorem,

> **Quasi-Theorem** 2 Heterotic string on T^2 is equivalent to type IIB string theory on the base space of an elliptically fibered $K3$.

The previous discussion is known, in the physics literature, under the generic name of F-theory [55, 56, 57].

We have been considering, until now, type II strings on $K3$, and compared them to heterotic string on a torus. To find out what is the expected moduli for the heterotic string on $K3$, we can use the following trick: if heterotic string on T^2 is type IIB on the base space of an elliptically fibered $K3$, by quasi-theorem 2 heterotic string on an elliptically fibered $K3$ should correspond to type IIB on the base space of an elliptically fibered Calabi-Yau manifold. More precisely, type IIB string should be compactified on the basis of an elliptic fibration, which is now four dimensional, and that can be represented as a fibration of a $\mathbb{P}^1$ space over another $\mathbb{P}^1$. This type of fibrations are known in the literature as Hirzebruch spaces, $\mathbf{F}_n$. Hirzebruch spaces can simply be determined through heterotic data, given by the $E_8 \times E_8$ bundle on the $K3$ manifold. The moduli of these bundles on $K3$ will put us in contact with yet another interesting topic: small instantons.

10.6 Heterotic Compactifications to Four Dimensions

Before considering some definite examples, let us simply summarize the different supersymmetries we can get when compactifying to four dimensions, depending on the holonomy of the target manifold. In order to do that, we will need the results in subsection 10.4, on the maximum number of supersymmetries allowed for a given spacetime dimension.

Type of String	Target Manifold	Holonomy	Supersymmetry
II	$K3 \times T^2$	$SU(2)$	$N=4$
Heterotic	T^6	Trivial	$N=4$
II	Calabi-Yau	$SU(3)$	$N=2$
Heterotic	$K3 \times T^2$	$SU(2)$	$N=2$
II	$B_{SU(4)}$	$SU(4)$	$N=1$
Heterotic	Calabi-Yau	$SU(3)$	$N=1$

In the table above we have not differentiated between type IIA and type IIB[α]. The first two lines, corresponding to cases with $N = 4$ and $N = 2$ supersymmetry in four dimensional spacetime, will be the basic examples we will use to introduce the concept of dual pairs of string compactifications down to four dimensions.

Before entering a discussion on the ingredients of this table, we yet need to consider the holonomy of the moduli space. This holonomy will of course depend on the number of supersymmetries and the type (real, complex or quaternionic) of the representation. Hence, from subsection 10.4, we can complete the table below.

Spacetime Dimension	Supersymmetries	Type	Holonomy
$d = 6$	$N=2$	$\mathbf{H}^4$	$Sp(1) \oplus Sp(1)$
$d = 4$	$N=4$	$\mathbf{C}^2$	$U(4)$
$d = 4$	$N=2$	$\mathbf{C}^2$	$U(2)$

[α]This will be relevant when discussing the third line where, by $B_{SU(4)}$, we are thinking in the spirit of the discussion in the last part of previous section, where a Calabi-Yau fourfold of $SU(4)$ holonomy, elliptically fibered, and with a zero volume fiber, is used for compactification.

Using this results, we can now decompose the tangent vectors to the moduli according to its transformation rules with respect to the holonomy group. Let us concentrate in the $d = 4$ case. For $U(4)$, we get

$$U(4) \simeq U(1) \oplus SO(6). \tag{10.69}$$

The matter multiplets will contain 6 (real) scalars each, i. e., the number of dimensions we compactify. Then, if we have m of these matter multiplets, the part of the moduli on which the $SO(6)$ part of the holonomy group is acting should be

$$O(6,m)/O(6) \times O(m). \tag{10.70}$$

The $U(1)$ part of (10.69) will act on the supergravity multiplet so we expect, just from holonomy arguments, a moduli of type

$$O(6,m)/O(6) \times O(m) \quad \times Sl(2)/U(1). \tag{10.71}$$

Now, we need to compute m. For heterotic string, the answer is clear: $m = 22$, and the total dimension of (10.71) will be 134. Let us now consider the case of type IIA. From the table, we see that we should consider $K3 \times T^2$ as compactification manifold. Let us then first compute the dimension of the moduli space:

$$\begin{aligned} \text{Moduli of metrics and B fields on } K3 &= 80 \\ \text{Moduli of metrics and B fields on } T^2 &= 4 \\ b_1(K3 \times T^2) &= 2 \\ b_3(K3 \times T^2) &= 44 \\ \text{Axion-Dilaton} &= 2 \\ \text{Duals in } \mathbf{R}^4 \text{ to } 2-\text{forms} &= 2 \end{aligned} \tag{10.72}$$

which sums up to 134. Notice that the 44 in $b_3(K3 \times T^2)$ is coming from the 3-cycles obtained from one S^1 of T^2, and the 22 elements in $H^2(K3;\mathbf{Z})$. The 3-form of IIA can be compactified on the S^1 cycles of T^2 to give 2-forms in four dimensions. Now, the dual of a 2-form in $\mathbf{R}^4$ is scalar, so we get the last two extra moduli.

Now, we need to compare the two moduli spaces. If we expect S-duality in $N = 4$ for the heterotic compactification, the moduli, once we have taken into account the $O(6,22;\mathbf{Z})$ T-duality, will look like

$$O(6,22;\mathbf{Z})\backslash O(6,22)/O(6)\times O(22) \;\; Sl(2,\mathbf{Z})\backslash Sl(2)/U(1). \tag{10.73}$$

Now, we have a piece in IIA looking naturally as the second term in (10.73), namely the moduli of the σ-model on T^2, where $Sl(2,\mathbf{Z})$ will simply be part of the T-duality. Thus, it is natural to relate the moduli of IIA on the torus with the part of the moduli in (10.71) coming form the supergravity multiplet.

Let us now consider dual pairs in the second line of our table. There is a simple way to visualize under what general conditions on the Calabi-Yau manifold with $SU(3)$ holonomy such dual pairs can exist. In fact, imagine that $K3$ is elliptically fibered in $K3 \times T^2$; then, what we get is a fibration on $\mathbb{P}^1$ of the T^4 tori. Now, heterotic on T^4 is equivalent to type IIA on $K3$, so we expect that the Calabi-Yau manifold should be a $K3$ fibration on $\mathbb{P}^1$, and that duality works fiberwise. Therefore, from general arguments, we expect to get heterotic-type II dual pairs with $N = 2$ if we use Calabi-Yau manifolds which are $K3$ fibrations [59, 58]. In order to get a more precise picture, we need again to work out the holonomy, which is $U(2)$ in this case. In $N = 2$ we have two types of multiplets, vec-

tor and hypermultiplets. The vector multiplet contains two real scalars, and the hypermultiplet four real scalars. Then, we decompose $U(2)$ into $U(1) \oplus Sp(1)$, and the moduli into vector and hypermultiplet part.
Let us first consider type IIA string on the Calabi-Yau manifold. The moduli will contain $h^{1,1}$ deformations of B and J, $h^{2,1}$ complex deformations and b^3 RR deformations (b^1 does not contribute, as we are working with a Calabi-Yau manifold). The total number, in real dimension, is

$$2h^{1,1} + 4(h^{2,1} + 1), \tag{10.74}$$

where we have used that $b^3 = 2(h^{2,1} + 1)$, in real dimension. ¿From (10.74) we conclude that we have $h^{1,1}$ vector multiplets, and $h^{2,1}+1$ hypermultiplets. Notice that $4(h^{2,1}+1)$ is counting the 2 coming from the dilaton and the axion so, for type II we have combined dilaton and axion into an hypermutiplet.
Now, let us consider heterotic string on $K3 \times T^2$. The moduli we must now consider, of $E_8 \times E_8$ bundles on $K3$, is much more elaborated than that of T^4, or T^6, that we have worked out. Part of the difficulty comes from anomaly conditions. However, we know, according to Mukai's theorem, that the moduli of holomorphic bundles on $K3$ is quaternionic, i. e., hyperkähler, and that the moduli of the σ-model on $K3$ is of dimension 80. We have yet the moduli on T^2, that will be a manifold of $O(2,m)/O(2) \times O(m)$ type, and therefore a good candidate for representing the vector multiplet. Thus, we get

$$\begin{aligned} \text{Type IIA hypermultiplets} &\leftrightarrow K3 \text{ Heterotic}, \\ \text{Vector multiplets} &\leftrightarrow T^2. \end{aligned} \tag{10.75}$$

¿From our previous discussion we know that vector multiplets, in type IIA are related to $h^{1,1}$. Working fiberwise on a $K3$

fibered Calabi-Yau manifold we get, for $h^{1,1}$,

$$h^{1,1} = 1 + \rho, \tag{10.76}$$

with ρ the Picard number of the $K3$ manifold. Then, in order to get a dual pair in the sense of (10.75) we need m in the heterotic to satisfy

$$m = \rho. \tag{10.77}$$

In order to control the value of m, from the heterotic point of view, we need to watch out for possible Wilson lines that can be defined on T^2 after the gauge group has been fixed from the $K3$ piece. From (10.76) (and this was the logic for the identification (10.77)), the heterotic dilaton-axion is related to the 1 term contributing in (10.76), i. e., the 2-cycle defined by the base space of the $K3$-fibration.

As can be observed from (10.77), if we do not freeze either the Kähler class or the complex structure of T^2, the minimum value for ρ is 2. This is the contribution to the Picard lattice of a Dynkin diagram of type A_2, i. e., $SU(3)$. A possible line of work opens here, in order to identify the moduli spaces of vector multiplets for type IIA theories with the quantum moduli, defined according to Seiberg and Witten, for gauge theories, with

$$\text{rank } G = \rho. \tag{10.78}$$

10.7 Heterotic–Type I Duality

Another example of a relation through S-duality between two different string theories that can be easily obtained just working at the level of effective lagrangians is that between heterotic $SO(32)$ and type I string theory [46], [60], [61]. For the heterotic string we have the lagrangian

$$L = \int d^{10}x\sqrt{g}e^{-2\phi}[R + |\nabla\phi|^2 + F^2 + |dB|^2], \qquad (10.79)$$

while for the $SO(32)$ open string the effective low energy lagrangian is given by

$$L = \int d^{10}x\sqrt{g'}[e^{-2\phi'}(R' + |\nabla\phi'|^2) + e^{-\phi'}F^2 + |dB|^2], \quad (10.80)$$

where the different dilaton factor multiplying F^2 reflects the fact that gauge vector bosons are in the open string sector of the spectrum, and the factor in $|dB|^2$ reflects the fact that in type I string theory B is a RR field. The relation between both lagrangians is obtained through

$$\begin{aligned} \phi' &= -\phi, \\ g' &= e^{-\phi}g, \end{aligned} \qquad (10.81)$$

where again the first relation is the manifestation of S-duality.

10.8 The Quantum Fate of Moduli Singularities

In this section we will consider two types of moduli singularities in the context of heterotic and type II string theories, namely small instanton singularities for compactifications of heterotic string on $K3$ surfaces, and conifold singularities for type II theories compactified on Calabi-Yau threefolds. Both types of singularities will lead to unexpected non perturbative quantum effects.

10.8.1 Small Instantons for Heterotic String on $K3$

Let us start considering the effective lagrangian for the heterotic string in the string frame,

$$L = \frac{1}{\alpha'^4}\int d^{10}x\sqrt{g}e^{-2\phi}[R+4|\nabla\phi|^2-\frac{1}{3}H^2-\frac{\alpha'}{30}\mathrm{tr}\ F^2]+\mathcal{O}(\alpha'^2). \tag{10.82}$$

We will consider an "instanton" solution to (10.82), related to the solitonic fivebrane. Let us then label with indices $(0\dots5)$ the coordinates on which the fivebrane worldvolume lies, so that the four transversal coordinates will be $(6\dots9)$. A solution to (10.82) is obtained from the standard instanton solution on the four dimensional transversal space,

$$A^\mu = -\frac{2\Sigma_{\mu\nu}x^\nu}{(x^2+\rho^2)}, \quad \mu = 6,\dots,9, \tag{10.83}$$

where

$$\begin{aligned} e^\phi &= e^{\phi_0} + 8\alpha'\frac{x^2+2\rho^2}{(x^2+\rho^2)^2}, \\ H &= -\epsilon^\rho_{\mu\nu\lambda}\nabla_\rho\phi, \\ g_{ab} &= \eta_{ab}, \quad a,b = 0,\dots,5, \\ g_{\mu\nu} &= e^{2\phi}\delta_{\mu\nu}. \end{aligned} \tag{10.84}$$

Notice that up to the dilaton factor, the metric in the transversal four dimensional space is flat and euclidean,

$$ds^2 = d\bar{x}^2\left(e^{2\phi_0}+8\alpha'\frac{x^2+2\rho^2}{(x^2+\rho^2)^2}\right). \tag{10.85}$$

An interesting phenomena takes place as the $\rho \to 0$ limit for the instanton size is taken. In fact, as can be seen from

(10.85), the distance of any point to the origin $x = 0$ becomes infinite, that geometrically means that a long tube, on which the origin lies, arises, so that the origin is infinitely far away. Notice from (10.84) that the dilaton grows as we go further down the tube and, in this same way, also the effective coupling constant.

Once we have this geometrical picture in mind, we can work out as an example the compactification of the $SO(32)$ heterotic string on $K3$. Through this compactification a theory with $N = 1$ supersymmetry in six dimensions is obtained. From the Bianchi identity,

$$dH = \text{tr } (R \wedge R) - \text{tr } (F \wedge F), \tag{10.86}$$

and the fact that $\int \text{tr } (R \wedge R) = 24$ for $K3$ surfaces, we conclude that 24 $SO(32)$ Yang-Mills instantons are needed on $K3$. What we will work out now is the singularity appearing when we send the size of one of this instantons to zero. A natural way to address this question is considering the $\rho \to 0$ limit of the heterotic solitonic fivebrane given in (10.84). Using string weak coupling duality between heterotic $SO(32)$ and type I we can transform this question into a weak coupling problem in type I string theory. The effective lagrangian for type I strings is given by

$$L = \int d^{10}x\sqrt{g}[e^{-2\phi}(R + 4|\nabla\phi|^2) - \frac{1}{3}H^2 - \frac{1}{30}e^{-\phi}\text{tr } F^2], \tag{10.87}$$

which means that the instanton action behaves like $e^{-\phi} \sim \frac{1}{g_{string}}$. This is the behavior for a Dirichlet brane. This Dirichlet brane is a good candidate to what we are looking for because Dirichlet branes have, classically, zero thickness, and ρ is the magnitude for the thickness of the solitonic fivebrane.

Hence, we can expect that what is taking place at the singularity defined by a small instanton can be translated into the worldvolume dynamics on the D-fivebrane.

At singular points in moduli space we expect new massless particles; thus, our candidate in the case of small instantons is the massless spectrum of the D-5brane. Here, there are two candidates to massless spectrum: open strings with Dirichlet conditions on both ends, and open strings with Dirichlet and Neuman conditions on each end. The spectrum of strings with Dirichlet conditions on both ends should in principle provide six dimensional massless vectors. However, as we are working in type I string theory, we should project out states which are not invariant under the orientation preserving operator, Ω. The vertex operator for vectors $A_\mu \partial_\tau x^\mu$ is odd under change of orientation; therefore, this piece of the spectrum is not symmetric under Ω. The way out from this is including Chan-Paton factors. As Chan-Paton factors for unoriented strings are only allowed for $SO(N)$ or $Sp(N)$ groups, if some massless spectrum appears at the small instanton singularity, it should correspond to one of these gauge symmetry groups. We will call this gauge group G.

The open string sector with boundary conditions of mixed type, Dirichlet and Neuman, will enjoy Chan-Paton factors of type G for the Dirichlet end, and the fundamental of $SO(32)$ for the Neuman end. From the point of view of the six dimensional worldvolume, these states will be associated to six dimensional hypermultiplets.

The next step in the characterization of the small instanton singularity will be discovering the gauge group G, and the number of massless hypermultiplets in six dimensions. In order to do that, the procedure will be comparing the moduli of hypermultiplets to the one instanton moduli. As the hypermultiplets transform under G, the moduli will have dimension

$$\dim \mathcal{M}_H = 4(k - d_G), \tag{10.88}$$

with k the number of hypermultiplets, and d_G the dimension of the gauge group, G. It must be recalled that for six dimensional $N = 1$ supersymmetry wehave, besides the gravitational multiplet containing 12 bosonic degrees of freedom, the tensor, the vector and the hypermultiplet, all of them with 4 bosonic degrees of freedom.

What we need to do now is comparing the moduli $\mathcal{M}_H$ to the moduli of the small instanton. If the size ρ of the instanton is small compared to the size of the $K3$ surface, we can certainly work in $\mathbf{R}^4$. We can embed the instanton configuration in $SO(N)$, which corresponds to the breaking of $SO(32)$ down to $SO(32-N)$. The dimension of the moduli of the instanton is

$$\dim \mathcal{M}_{ins} = 4N - 8. \tag{10.89}$$

Showing apart the 4 translations,

$$\mathcal{M}_{ins} = \mathcal{M}'_{ins} \times \mathbf{R}^4, \tag{10.90}$$

and hence

$$\dim \mathcal{M}'_{ins} = 4N - 12 = 4(N - 3). \tag{10.91}$$

Comparing now (10.91) and (10.88), we get $d_G = 3$ and $k = N$, i. e., gauge group $SU(2)$ and N-hypermultiplets. We conclude then that each instanton produces, as the size goes to zero, an extra $SU(2)$ gauge symmetry.

10.8.2 Conifold Singularities for Type II Strings

We will now consider conifold singularities in Calabi-Yau threefolds. Let X be a Calabi-Yau threefold, with third Betti number b_3, and A_I, B^J a homology basis,

$$A_I \cap B^J = \delta_I^J. \tag{10.92}$$

The complex structure of the manifold is described by the periods of the holomorphic top form, Ω,

$$\begin{aligned} \oint_{A_I} \Omega &\equiv F_I, \\ \oint_{B^J} \Omega &\equiv Z^J. \end{aligned} \tag{10.93}$$

Thus, on the moduli space of complex structures we are defining a bundle of dimension b_3, with $Sl(b_3;\mathbf{Z})$ group of modular transformations. Singularities of complex codimension one in the moduli space are characterized by their monodromy in $Sl(b_3;\mathbf{Z})$, while conifold singularities correspond to some vanishing 3-cycle. For the simplest Calabi-Yau threefold, the quintic, the monodromy at the conifold singularity is given by the transformation T in $Sl(2;\mathbf{Z})$,

$$\begin{pmatrix} F \\ Z \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F \\ Z \end{pmatrix} = \begin{pmatrix} F+Z \\ Z \end{pmatrix}, \tag{10.94}$$

which implies a dependence

$$F(z) \sim \frac{1}{2\pi i} Z \ln Z, \tag{10.95}$$

with the conifold singularity located at $z = 0$. Our task will now consist in understanding the quantum origin of the conifold singularity (10.95). In order to do this, let us compactify type IIB string theory on the Calabi-Yau threefold. What is left is a four dimensional $N = 2$ theory, with $b_{2,1}$ vector multiplets. The Z^I defined in (10.93) correspond to the scalar components. Using the well known Seiberg-Witten model for $SU(2)$ pure gauge theories, we should have a singularity at the point where the monopole becomes massless. In terms of Seiberg-Witten notation, this situation is characterized by a point u_0 in the moduli space of vacua such that $a(u_0)$, to be identified with the monopole mass, is zero. In the neigbourhood of this point, we can describe the infrared physics in terms of $N = 2$ magnetic QED, with one massless monopole hypermultiplet. The one loop effect of this hypermultiplet produces a correction to the magnetic gauge coupling constant, of the type

$$\frac{1}{g^2} \sim \ln(a) \tag{10.96}$$

or, in terms of the dual variable a_D, a dependence

$$a_D(a) \sim a \ln(a) \tag{10.97}$$

of the same type as (10.95), with a_D and a replaced by F and Z, respectively. In order to use a similar argument to understand relation (10.95), we should first discover the analog of the monopole in the type IIB case. It was conjectured by Strominger [62] that the appropriated object becoming massless at the conifold point is a black hole, that can be interpreted as a 3-brane wrapping the vanishing 3-cycle.

Let us consider the extremal Reissner-Nordstrom black hole,

$$ds^2 \;\; = \;\; -\left(1 + \frac{r_+^4}{r^4}\right)^{1/2} dt^2 + \frac{dr^2}{\left(1 - \frac{r_+^4}{r^4}\right)^2} + r^2 d\Omega_5^2$$

$$+\left(1-\frac{r_+^4}{r^4}\right)^{1/2} dx_i dx^i, \tag{10.98}$$

with $i = 0,1,2,3$. The black hole (10.98) is charged with respect to the RR 4-form,

$$\int_{\Sigma_5} F^{(5)} = Q, \tag{10.99}$$

and with constant dilaton. The charge Q is related to the radius r_+ through

$$Q = 2r_+^4. \tag{10.100}$$

A peculiarity of the Reissner-Nordstrom metric (10.98) is the existence of a long tube in the sense that along a static spatial slice the radial distance to the horizon at r_+ is infinity. The mass of this black hole is given by

$$M = Q.\text{Volume}. \tag{10.101}$$

Since it is BPS, i. e., extremal, and with RR charge, we get, in string units,

$$M \sim |Z|\frac{m_s}{g}, \tag{10.102}$$

with g, as usual, the string coupling constant, and Z as defined in (10.93). Thus, the relevant diagram associated to (10.95) will be that in Figure 10.3.

In order to understand Figure 10.3, where the double line in the loop represents a black hole, as the proper description of the conifold singularity (10.95), we need to fix the scale used in the computation of the loop diagram. Here comes an important issue we have not yet stressed: if we use as scale the string scale, m_s, then, from (10.102), we will get $\ln(|z|e^{-\phi})$, which implies a coupling of the dilaton to RR fields. This

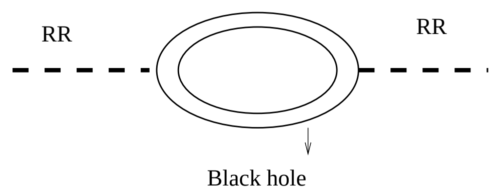

Figura 10.3: The black hole becoming massless generates the singularity at the conifold

coupling is ruled out by supersymmetry, and therefore we are forced to fix a different scale, $m_s e^{-\phi}$. The importance of this scale is that it becomes substringy at low string coupling constant. The conifold singularity is classical, in the sense that it is an effect of the classical moduli space of the Calabi-Yau threefold, which therefore survives even at $g_s = 0$ or, to put it more precisely, it should be independent of g_s. A possible way to see how the blackhole physics involved in Figure 10.3 can be a good description to the limit $g_s = 0$ is through a double scaling limit, $z \to 0$ and $g_s \to 0$, at the conifold point, and preserving constant the ratio $\frac{z}{g_s}$ that, from (10.102), is the black hole mass in string units.

Both the small instanton singularity and the conifold singularity are difficult to understand from the point of view of conformal field theory, where there is no available RR charged particle. A possible way to understand what is happening was suggested by Witten in reference [63], where it is argued that from the point of view of the conformal field theory, i. e., in the string frame at the conifold singularity, the target spacetime becomes effectively uncompactified, as a long tube as the one found in the solitonic fivebrane at $\rho = 0$ appears. Along this tube, the dilaton grows and perturbation theory becomes of no validity as we descend the tube. The process of descending along the tube might involve very complicated physics. What has been observed in the previous examples is a short cut to describe, through quantum mechanics, this conformal field theory physics, in terms of a finite number of degrees of freedom.

10.9 Point Particle Limit

A different approach to (**??**) and (**??**) is that based on geometric engineering [64]. In this case, the procedure is based on the following set of steps:

1. String theory is compactified on a Calabi-Yau threefold X, with the apropiate number of vector multiplets in four dimensions.

2. A point corresponding to classical enhancement of gauge symmetry in the moduli space of the Calabi-Yau threefold must be localized.

3. A rigid Calabi-Yau threefold is defined by performing a point particle limit.

4. The rigid Calabi-Yau manifold is used to define the Seiberg-Witten surface Σ.

5. Going form type IIB to type IIA string theory represents a brane configuration corresponding to an ALE space with singularity of some Dynkin type into a set of fivebranes that can be interpreted as a fivebrane with worldvolume $\Sigma \times \mathbf{R}^4$.

6. The BPS states are defined through the meromorphic one-form λ, derived from the Calabi-Yau holomorphic top form, in the rigid point particle limit.

As we can see from the previous set of steps, that we will explicitly show at work in one definite example, the main difference between both approaches is at the level of the meromorphic form in Seiberg-Witten theory. There is also an important difference in the underlying philosophy, related to the implicit use in the string approach, described in the above steps, of the heterotic-type II dual pairs, driving us to the choice of a particular Calabi-Yau manifold. The most elaborated geometric engineering approach uses, instead of a certain heterotic-type II dual pair, a set of local geometrical data, determined by the type of gauge symmetry we are interested on, and generalizes mirror maps to this set of local data. In all these cases, the four dimensional field theory we are going to obtain will not depend on extra parameters, as the string coupling constant. On the other hand, the M-theory approach, where field theories are obtained depending explicitly on the string coupling constant, might be dynamically rich enough as to provide a direct explanation of phenomena that can not be easily understood in the more restricted context of the point particle limit of string theory.

Next, we will follow steps 1 to 6 through an explicit example [65]. In order to obtain a field theory with gauge group $SU(n)$ we should start with a Calabi-Yau manifold with $h_{2,1} = n$, and admitting the structure of a $K3$-fibered threefold. We will use the language of weighted projective spaces. The weighted projective space $\mathbb{P}^{d+1}_{k_0,\ldots,k_{d+1}}$, with homogeneous coordinates $[z_0, \ldots, z_{d+1}]$ is defined by the equivalence relation

$$[X_0, \ldots, X_{d+1}] \sim [\lambda^{k_0} X_0, \ldots, \lambda^{k_{d+1}} X_{d+1}]. \qquad (10.103)$$

A Calabi-Yau manifold of complex dimension d can be defined as the vanishing locus of a homogeneous polynomial W, of degree $\sum_i k_i = 0$,

$$W = \sum a_{i_0} \ldots a_{i_{d+1}} X_0^{i_0} \ldots X_{d+1}^{i_{d+1}}. \qquad (10.104)$$

The values of $a_{i_0}, \ldots, a_{i_{d+1}}$ for which the defined manifold is not smooth define the discriminant locus of the Calabi-Yau manifold.
Let us the begin with an example with $h_{1,1} = 2$, defined by

$$\begin{aligned} W^* &= \frac{1}{12}x_1^{12} + \frac{1}{12}x_2^{12} + \frac{1}{6}x_3^6 + \frac{1}{6}x_4^6 + \frac{1}{2}x_5^2 \\ &\quad -\psi x_1 x_2 x_3 x_4 x_5 - \frac{1}{6}\phi (x_1 x_2)^6. \end{aligned} \qquad (10.105)$$

The moduli space of complex deformations of W^* is parameterized by (ψ, ϕ), subject to the global symmetry

$$\mathcal{A} : (\psi, \phi) \to (\beta\psi, -\phi) \ , \qquad \beta^{12} = 1. \qquad (10.106)$$

This symmetry forces to introduce invariant quantities; we will use $b = 1/\phi^2$ and $c = -\phi/\psi^6$. The $K3$-fibration structure of (10.105) becomes manifest by the change of variables $x_1/x_2 \equiv z^{1/6} b^{-1/12}$, $x_1^2 \equiv x_0 z^{1/6}$ [65]:

$$W^* = \frac{1}{12}(z + \frac{b}{z} + 2)x_0^6 + \frac{1}{6}x_3^6 + \frac{1}{6}x_4^6 + \frac{1}{2}x_5^2 + c^{-1/6} x_0 x_3 x_4 x_5, \qquad (10.107)$$

with the variable z acting as coordinate on the $\mathbb{P}^1$ base space. It is convenient to define

$$d(z; b) = \frac{1}{2}(z + \frac{b}{z} + 2) \ , \qquad \hat{c}(z; b, c) = c\, d(z; b). \qquad (10.108)$$

Substituting this into (10.107) and rescaling x_0, W^* acquires the explicit form of a $K3$-surface

$$W^* = \frac{1}{6}x_0^6 + \frac{1}{6}x_3^6 + \frac{1}{6}x_4^6 + \frac{1}{2}x_5^2 + \hat{c}(z;b,c)^{-1/6} x_0 x_3 x_4 x_5. \quad (10.109)$$

As we move in $\mathbb{P}^1$, the $K3$-fiber can become singular. From (10.109) it is easy to deduce that this occurs for the $K3$ modulus values $\hat{c}(z;b,c) = 0, 1$. These values of $\hat{c}$ are acquired at the following $\mathbb{P}^1$ points, $z = e_i^{\pm}$:

$$\begin{aligned} \hat{c} = 0 \quad &\to \quad e_0^{\pm} = -1 \pm \sqrt{1-b}, \\ \hat{c} = 1 \quad &\to \quad e_1^{\pm} = \frac{1 - c \pm \sqrt{(1-c)^2 - bc^2}}{c}. \end{aligned} \quad (10.110)$$

The discriminant of (10.109) is therefore given by $\Delta(z;b,c) = \prod_{i=0}^{1}(z - e_i^+(b,c))(z - e_i^-(b,c))$. There is an additional singularity at $\hat{c}(z;b,c) = \infty$, which is originated in the quotient by discrete reparameterizations of (10.109) inherited from the orbifold construction of W^*. It corresponds to the points

$$\hat{c} = \infty \quad \to \quad z = 0, \infty \qquad (b \neq \infty). \quad (10.111)$$

The Calabi-Yau manifold becomes singular when some of the points (10.110)-(10.111) coalesce. We will now analyze the regions in moduli space where this situation happens [66, 67] (we will follow notation in [66]). The loci

$$\begin{aligned} \mathcal{C}_1 &= \{b = 1\}, \\ \mathcal{C}_C &= \{(1-c)^2 - bc^2 = 0\}, \end{aligned} \quad (10.112)$$

are respectively obtained from the identifications $e_0^+ = e_0^-$ and $e_1^+ = e_1^-$. $\mathcal{C}_C$ is the conifold locus, where 3-cycles of the threefold degenerate to points, while $\mathcal{C}_1$ corresponds to the appearance of a genus two curve of A_1 singularities. We can also consider

$$\begin{aligned}\mathcal{C}_0 &= \{c = \infty\}, \\ \mathcal{C}_\infty &= \{b = 0\}, \end{aligned} \tag{10.113}$$

which are defined, respectively, by the identifications $e_1^{\pm} = e_0^{\pm}$ and $e_0^+ = e_1^+ = 0$. $\mathcal{C}_0$ is an orbifold locus, given by the fixed points under $\mathcal{A}^2$. $\mathcal{C}_\infty$ corresponds to the weak coupling limit locus, once we identify the heterotic dilaton with the size of the base space [68].

Other examples of $K3$ fibered Calabi-Yau manifolds, with $h_{1,1} = 2$, are $A : \mathbb{P}^4_{\{1,1,2,2,2\}}[8]$, $B : \mathbb{P}^5_{\{1,1,2,2,2,2\}}[4,6]$ and $C : \mathbb{P}^6_{\{1,1,2,2,2,2,2\}}[4,4,4]$ (B and C are defined as intersections). The $K3$ fibration structure is given by

$$\begin{aligned}
B: \quad & x_0^4/4 + x_3^4/4 + x_4^4/4 + x_5^4/4 + \hat{c}^{-1/4} x_0 x_3 x_4 x_5 = 0, \\
C: \quad & x_0^2/2 + x_3^2/2 + x_4^2/2 + \hat{c}^{-1/5} x_5 x_6 = 0 \\
& x_5^3/3 + x_6^3/3 + \hat{c}^{-1/5} x_0 x_3 x_4 = 0\,, \\
D: \quad & x_0^2/2 + x_3^2/2 + \hat{c}^{-1/6} x_4 x_5 = 0 \\
& x_4^2/2 + x_5^2/2 + \hat{c}^{-1/6} x_6 x_7 = 0 \\
& x_6^2/2 + x_7^2/2 + \hat{c}^{-1/6} x_0 x_3 = 0,
\end{aligned} \tag{10.114}$$

with $\hat{c}(z; b, c)$ defined in (10.108).

We will now consider an $SU(3)$ case, corresponding to a Calabi-Yau manifold whose mirror is the weighted projective space $\mathbb{P}^{24}_{1,1,2,8,12}$,

$$\begin{aligned}
&\frac{1}{24}(x_1^{24} + x_2^{24}) + \frac{1}{12} x_3^{12} + \frac{1}{2} x_5^2 \\
&-\psi_0 x_1 x_2 x_3 x_4 x_5 - \frac{1}{6}(x_1 x_2 x_3)^6 - \frac{1}{12}(x_1 x_2)^{12} \quad = \quad 0.
\end{aligned} \tag{10.115}$$

In order to clearly visualize (10.115) as a $K3$-fibration we will perform the change of variables

$$x_1/x_2 \equiv \hat{z}^{1/12} b^{-1/24}, \qquad x_1^2 \equiv x_0 \hat{z}^{1/12}, \tag{10.116}$$

so that (10.115) can be rewritten in the form

$$\begin{aligned} \frac{1}{24}(\hat{z} + \frac{b}{\hat{z}} + 2)x_0^{12} + \frac{1}{12}x_3^{12} + \frac{1}{3}x_4^3 + \frac{1}{2}x_5^2 & \\ + \frac{1}{6\sqrt{c}}(x_0x_3)^6 + \left(\frac{a}{\sqrt{c}}\right)^{1/6} x_0x_3x_4x_5 &= 0, \end{aligned} \tag{10.117}$$

which represents a $K3$ surface, fibered over a $\mathbb{P}^1$ space parametrized by the coordinate z. Parameters in (10.117) are related to those in (10.115) through

$$a = -\psi_0^6/\psi_1, \quad b = \psi_2^{-2}, \quad c = \psi_2/\psi_1^2. \tag{10.118}$$

The parameter b can be interpreted as the volume of $\mathbb{P}^1$:

$$-\log b = \text{Vol }(\mathbb{P}^1). \tag{10.119}$$

Next, we should look for the points $\hat{z}$ in $\mathbb{P}^1$ over which the $K3$ surface is singular. The discriminant can be written as

$$\Delta_{K3} = \prod_{i=0}^{2}(\hat{z} - e_i^+(a,b,c))(\hat{z} - e_i^-(a,b,c)), \tag{10.120}$$

where

$$\begin{aligned} e_0^\pm &= -1 \pm \sqrt{1-b}, \\ e_1^\pm &= \frac{1 - c \pm \sqrt{(1-c)^2 - bc^2}}{c}, \\ e_2^\pm &= \frac{(1-a)^2 - c \pm \sqrt{((1-a)^2 - c)^2 - bc^2}}{c} \end{aligned} \tag{10.121}$$

The Calabi-Yau manifold will be singular whenever two roots e_i coalesce, as

$$\Delta_{\text{Calabi-Yau}} = \prod_{i<j}(e_i - e_j)^2. \tag{10.122}$$

We will consider the singular point in the moduli space corresponding to $SU(3)$ symmetry. Around this point we will introduce new coordinates, through

$$\begin{aligned} a &= -2(\alpha' u)^{3/2}, \\ b &= \alpha' \Lambda^6, \\ c &= 1 - \alpha'^{3/2}(-2u^{3/2} + 3\sqrt{3}v). \end{aligned} \tag{10.123}$$

Going now to the $\alpha' \to 0$ limit in (10.122), we get a set of roots $e_i(u, v; \Lambda^6)$ on a z-plane, with z defined in $\alpha'^{3/2} z \equiv \hat{z}$:

$$\begin{aligned} e_0 &= 0, \; e_\infty = \infty, \\ e_1^\pm &= 2u^{3/2} + 3\sqrt{3}v \pm \sqrt{(2u^{3/2} + 3\sqrt{3}v)^2 - \Lambda^6}, \\ e_2^\pm &= -2u^{3/2} + 3\sqrt{3}v \pm \sqrt{(2u^{3/2} - 3\sqrt{3}v)^2 - \Lambda^6} \end{aligned} \tag{10.124}$$

Now, we can use (10.124) as the definition of a Riemann surface Σ, defined by the Calabi-Yau data at the singular $SU(3)$ point, and in the point particle limit. There exits a natural geometrical picture for understanding the parameters u and v in (10.123), which is the definition of the blow up, in the moduli space of complex structures of $\mathbb{P}^{24}_{1,1,2,8,12}$, of the $SU(3)$ singular point. From this point of view, the parameters u and v in (10.123) will be related to the volume of the set of vanishing two-cycles associated with a *rational* singularity, i. e., an orbifold singularity of type A_{n-1} (in the case we are considering, $n = 3$). These vanishing cycles, as is the case with rational singularities, are associated with Dynkin diagrams of non affine type. The branch points (10.124) on the z-plane define the curve

$$y^2 = \prod_i (x - e_i(u, v; \Lambda^6)), \tag{10.125}$$

which can also be represented as the vanishing locus of a polynomial $F(x, z) = 0$, with F given by [69, 70]

$$F(x, z) = z + \frac{\Lambda^6}{z} + B(x), \tag{10.126}$$

where $B(x)$ is a polynomial in x of degree three; in the general case of $SU(n)$ theories, the polynomial will be of degree n.
This has exactly the same look as what we have obtained using brane configurations, with the space $\mathcal{Q}$ replaced by the (x, z) space. The difference is that now we are not considering the (x, z) space as a part of spacetime, and Σ as embedded in it, but we use Σ as defined in (10.126) to define a Calabi-Yau space in a rigid limit by the equation

$$F(x, z) + y^2 + w^2 = 0, \tag{10.127}$$

which defines a threefold in the (x, y, z, w) space. And, in addition, we think of (10.127) as a Calabi-Yau representation of the point particle limit. In order to get the meromorphic one-form λ, and the BPS states, we need to define a map from the third homology group, $H_3(CY)$, of the Calabi-Yau manifold, into $H_1(\Sigma)$. This can be done as follows. The three-cycles in $H_3(CY)$ of the general type $S^2 \times S^1$, with S^2 a vanishing cycle of $K3$, correspond to S^1 circles in the z-plane. The three-cycles with the topology of S^3 can be interpreted as a path from the north to the south pole of S^3, starting with a vanishing two-cycle, and ending at another vanishing two-cycle of $K3$. This corresponds, in the z-plane, to paths going from e_i^+ to e_i^-. Once we have defined this map,

$$f : H_3(CY) \longrightarrow H_1(\Sigma), \tag{10.128}$$

we define

$$\lambda(f(C)) = \Omega(C), \tag{10.129}$$

with Ω the holomorphic top form.

A similar analysis can be done for computing the mass of BPS states, and the meromorphic one-form λ in the brane framework. In fact, we can consider a two-cycle C in $\mathcal{Q}$ such that

$$\partial C \subset \Sigma, \tag{10.130}$$

or, in other words, $C \in H_2(\mathcal{Q}/\Sigma; \mathbf{Z})$. The holomorphic top form on $\mathcal{Q}$ is given by

$$\Omega = R\frac{dt}{t} \wedge dv, \tag{10.131}$$

and thus the BPS mass will be given by

$$M \sim R \int_C \frac{dt}{t} \wedge dv = R \int_{\partial C} \frac{dt}{t} v(t), \tag{10.132}$$

with $v(t)$ given by

$$F(t, v) = 0, \tag{10.133}$$

for the corresponding Seiberg-Witten curve, Σ. Notice that the same analysis, using (10.129) and the holomorphic top form for (10.127) will give, by contrast to the brane case, a BPS mass formula independent of R.

Next, we will compare the brane construction and geometric engineering in the more complicated case of $N = 1$ [71, 72].

Capítulo

11

Dirichlet-Branes

11.1 Supersymmetric D-Branes

In chapter 2 we have introduced the concept of D-branes as a necessary ingredient to understand T-duality in the framework of open bosonic strings. The same argument can be extended without changes to the case of type I unoriented superstrings. The main novelty in this case is that supersymmetric D-branes appear as natural sources of R-R fields [19]. This is an extremely important result, as if we simply consider fundamental strings it is not possible to define, at the worldsheet level, any coupling to Ramond-Ramond backgrounds. The importance of non trivial R-R backgrounds was clear in the discussion of U-dualities and, therefore, appears as a necessary ingredient in the proof of duality theorems relating string theories on different backgrounds.
Repeating the analysis of chapter 2, we can consider the interaction of D-branes in the supersymmetric case, now arising from the exchange of closed superstrings. The computation is identical to the bosonic case, except for the fact that both the contributions of the NS-NS and R-R sectors should be taken into account; these contributions correspond to the two types of closed string boundary conditions. The relevant amplitude for the interaction of two D-branes through the exchange of a

closed string is the cylinder amplitude,

$$\mathcal{A} = \int \frac{dt}{2t} \text{tr } [e^{-t(p^2+m^2)} P_{GSO}], \tag{11.1}$$

where $P_{GSO} = \frac{1}{2}(1 + (-1)^F)$. The other two topologies obtained when we include the orientation projector correspond to the orientifold D-brane interaction for the Moëbius strip, and orientifold-orientifold interaction for the Klein bottle (recall that orientifolds induce crosscups). In (11.1) we have, from the open string point of view, four contributions, coming from the NS sectors with $\frac{1}{2}$ and $\frac{1}{2}(-1)^F$, and the R with $\frac{1}{2}$ and $\frac{1}{2}(-1)^F$. From the tree level closed string point of view, these four contributions come from the NS-NS closed string sector, and the R-R sector, differing by the insertion of a factor $(-1)^F$ [17]. In order to see this more clearly, let us introduce explicitly the boundary conditions. Representing the cylinder as in Figure (((((, there are two labels characterizing the boundary conditions on fermions, namely $\hat{\eta}$ and η, with values ± 1. The value $\hat{\eta} = +1$ corresponds to the open string R sector, and $\hat{\eta} = -1$ to the NS sector. The boundary condition $\eta = +1$ is from the closed string R-R sector, while $\eta = -1$ is from the NS-NS sector. From the open string point of view, $\eta = +1$ is equivalent to insertion in the trace of the factor $(-1)^F$, and $\eta = -1$ to taking the trace without performing the insertion. In summary, supersymmetry implies

$$\mathcal{A} = \mathcal{A}_{NS-NS} + \mathcal{A}_{R-R} = 0, \tag{11.2}$$

with

$$\mathcal{A}_{NS-NS} = \int \frac{dt}{2t} \text{tr } [e^{-t(p^2+m^2)}]. \tag{11.3}$$

Taking into account a factor of 2 for the exchange of the two ends of the open string stretching along the D-branes we get, in the $t \to 0$ limit, for D-$p$branes (including a volume factor, as so far only a density amplitude has been calculated),

$$A =_{NS-NS}= V_{p+1} 2\pi (4\pi^2\alpha')^{3-p} \Gamma((7-p)/2) |b|^{p-7}, \quad (11.4)$$

which leads to a tension for the D-$p$brane

$$T_p = \frac{1}{g}\sqrt{2\pi}(4\pi\alpha')^{(3-p)/2}. \quad (11.5)$$

Using (11.2), we can get an equivalent result for the R-R sector. The interpretation of this contribution is as coming from a R-R tadpole, with external line a R-R fermionic vertex operator. The non vanishing tadpole for the R-R fermionic vertex at zero momentum is the proof that the D-brane is the desired R-R source. In order to interpret the non vanishing $\mathcal{A}_{RR}$ piece in (11.2), as reflecting the nature as R-R source of the D-brane, we must find the R-R vertex as associated to a $p+1$-form. This looks slightly difficult, as already discussed in chapter 2, because the fermion vertex operators in the $(-1/2, 1/2)$ picture,

$$V_F = e^{-\phi/2} e^{-\tilde{\phi}/2} S^\alpha \tilde{S}^\alpha e^{ikx}, \quad (11.6)$$

decompose into strength field forms. The solution to this puzzle [73] comes from the picture changing formalism. In fact, we are working an amplitude with the topology of a disc, so that the net ghost charge is -2; thus, if we want a non vanishing one point amplitude, we must employ either the $(-1/2, -3/2)$ or the $(-3/2, -1/2)$ pictures. It is this change of picture the one reducing the field strength into a field form showing the consistency of the claim on the R-R source nature of D-branes.

If we are interested in orientifolds, we can compute the contribution to the amplitude with the Klein bottle topology.. The relation between the orientifold on D-brane charges and masses is

$$\text{Orientifold}_p = \mp 2^{p-5}\text{D-brane}. \tag{11.7}$$

To end up this brief introduction to generalities on supersymmetric D-branes, let us now discuss supersymmetry. Relation (11.2) has become the identity relation between mass and charge of D-branes, a relation characterizing BPS solitons (solitons saturating the Bogomolny bound).
Typically, BPS states are characterized by being anhilated by one or more supersymmetry transformations,

$$Q|\text{BPS}>= 0. \tag{11.8}$$

If the supersymmetry algebra contains central extensions, then condition (11.8) implies the bound

$$M \geq |Z|, \tag{11.9}$$

with Z the central charge per unit volume. Thus, in order to check the BPS nature of D-branes, we should analyze equation (11.8) for the D-brane. In superstring theory, the supersymmetric charge is defined in terms of the fermionic vertex operator in the $-1/2$ picture,

$$\begin{aligned} Q^\alpha &= \oint V^F_{-1/2}, \\ \tilde{Q}^\alpha &= \oint \tilde{V}^F_{-1/2}, \end{aligned} \tag{11.10}$$

with $V \equiv e^{-\phi/2} S^\alpha e^{ikx}$, and $\tilde{V} \equiv e^{-\tilde{\phi}/2} \tilde{S}^\alpha e^{ikx}$. The open string sector is only invariant under the total supersymmetry $Q^\alpha +$

$\tilde{Q}^\alpha$; thus, in the presence of D-branes, i. e., for a D-brane background, only one supersymmetry is preserved. At this point, we can summarize the spectrum of different D-branes for type IIA and type IIB string theories. The field content of these theories in the Ramond-Ramond sector is

$$\begin{aligned} \text{IIA} &\rightarrow A^1, \quad A^3, \\ \text{IIB} &\rightarrow A^0, \quad A^2, \quad A^4, \end{aligned} \tag{11.11}$$

and the corresponding spectrum of D-branes is the one presented in Figure 11.1, where the arrows represent the relation through Hodge duality.

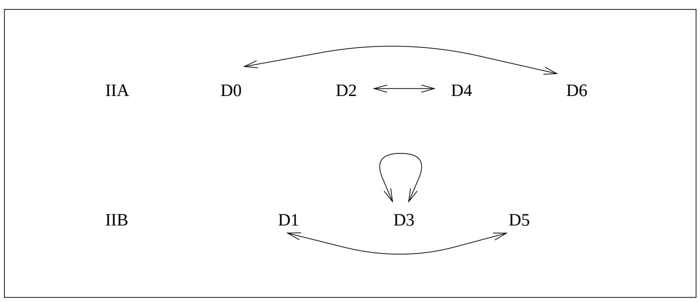

Figura 11.1: Dirichlet-brane spectrum in ten dimensional string theory

The whole set of connections between D-branes under T-duality is given in Figure 11.2.

11.2 D-Brane Scattering

11.2.1 Field Theory Effective Potentials

The effective potential arising in the scattering of Dirichlet pbranes is described in terms of the dimensional reduction,

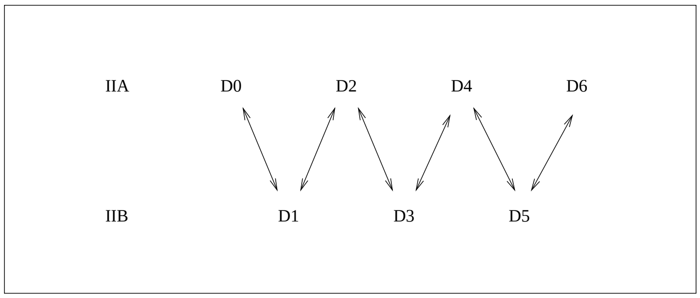

Figura 11.2: T-duality relates type IIA and type IIB D-branes

down to $p+1$ dimensions, of ten dimensional Yang-Mills with sixteen supercharges.

A system of N D-zerobranes is described through a set of 9 $N \times N$ dimensional matrices, X^i_{ab} ($i = 1, \ldots, 9$; $a, b = 1, \ldots, N$) locating the position of the zerobrane worldvolumes in transverse space, and a set of 16 fermionic superpartners, ψ^α ($\alpha = 1, \ldots, 16$), transforming as spinors under the $SO(9)$ group of transverse rotations. In order to perform field theoretic calculations with the dimensional reduction of ten dimensional Yang-Mills, and describe the zerobrane dynamics, the background field method is quite convenient, as it avoids the loss of gauge invariance when quantum effects are taken into account. In what follows, the effective potential governing the interaction between zerobranes will be shown to agree with the eleven dimensional result for graviton scattering in supergravity. The first approach will be the explicit evaluation of the determinants arising from a one loop calculation, through Riemann's zeta function techniques.

In units where $2\pi\alpha' = 1$, ten dimensional $N = 1$ supersymmetric Yang-Mills, including a gauge fixing term, is

$$S = \int dt \left(\frac{1}{2g} \mathrm{tr}\ F_{\mu\nu} F^{\mu\nu} - i \mathrm{tr}\ \bar{\psi} D_\mu \Gamma^\mu \psi + \frac{1}{g} \mathrm{tr}\ (\bar{D}^\mu A_\mu)^2) + \right) S_{\mathrm{ghost}}, \tag{11.12}$$

where the background field gauge fixing condition being used is

$$\bar{D}^\mu A_\mu = \partial^\mu A_\mu + [B^\mu, A_\mu], \tag{11.13}$$

with B^μ the background field. Dimensional reduction down to $0+1$ dimensions simply implies

$$\begin{aligned} F_{0i} &= \partial_t X_i + [A, X_i], \\ F_{ij} &= [X_i, X_j], \\ D_t \psi &= \partial_t \psi + [A, \psi], \\ D_i \psi &= [X_i, \psi], \end{aligned} \tag{11.14}$$

with $i, j = 1, \ldots, 9$ labelling the transverse directions. The background field can introduced through $X_i = B_i + \sqrt{g} Y_i$, (with Y_i the quantum fluctuations) which amounts, as the effective action is expanded in powers of g, to an expansion in the number of loops.

If the $U(2) \simeq U(1) \times SU(2)$, a center of mass–relative motion decomposition is used,

$$\begin{aligned} X^i &= \frac{i}{2}(X_0^i \mathbf{I} + X_a^i \sigma^a), \\ A &= \frac{i}{2}(A_0^i \mathbf{I} + A_a \sigma^a), \\ \psi &= \frac{i}{2}(\psi_0^i \mathbf{I} + \psi_a \sigma^a), \end{aligned} \tag{11.15}$$

the background field, which can be chosen as corresponding to motion on a straight line,

$$B^1 = \frac{i}{2}\begin{pmatrix} vt & 0 \\ 0 & -vt \end{pmatrix}, \quad B^2 = \frac{i}{2}\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \tag{11.16}$$

simply becomes

$$B_3^1 = vt, \quad B_3^2 = b. \tag{11.17}$$

Dropping the piece of the action describing the free motion of the center of mass, the action becomes

$$\begin{aligned}
\mathcal{S} &= i\int d\tau[\frac{1}{2}Y_1^i(\partial_\tau^2 - r^2)Y_1^i + \frac{1}{2}Y_2^i(\partial_\tau^2 - r^2)Y_2^i \\
&+ \frac{1}{2}Y_3^i(\partial_\tau^2)Y_3^i - \sqrt{g}\epsilon^{a3x}\epsilon^{cbx}B_3^iY_a^jY_b^iY_c^j - \frac{g}{4}\epsilon^{abx}\epsilon^{cdx}Y_a^iY_b^jY_c^iY_d^j] \\
&+ i\int d\tau[\frac{1}{2}A_1(\partial_\tau^2 - r^2)A_1 + \frac{1}{2}A_2(\partial_\tau^2 - r^2)A_2 + \frac{1}{2}A_3(\partial_\tau^2)A_3 \\
&+ 2\epsilon^{ab3}\partial_\tau B_3^iA_aY_b^i + \sqrt{g}\epsilon^{abc}\partial_\tau Y_a^iA_bY_c^i \\
&- \sqrt{g}\epsilon^{a3x}\epsilon^{bcx}B_3^iA_aA_bY_c^i - \frac{g}{2}\epsilon^{abx}\epsilon^{cdx}A_aY_b^iA_cY_d^i] \\
&+ i\int d\tau[\psi_+^T(\partial_\tau - v\tau\gamma_1 - b\gamma_2)\psi_- + \frac{1}{2}\psi_3^T\partial_\tau\psi_3 \\
&+ \sqrt{\frac{g}{2}}(Y_1^i - iY_2^i)\psi_+^T\gamma^i\psi_3 + \sqrt{\frac{g}{2}}(Y_1^i + iY_2^i)\psi_3^T\gamma^i\psi_- \\
&- i\sqrt{\frac{g}{2}}(A_1 - iA_2)\psi_+^T\psi_3 + i\sqrt{\frac{g}{2}}(A_1 + iA_2)\psi_-^T\psi_3 \\
&- \sqrt{g}Y_3^i\psi_+^T\gamma^i\psi_- + i\sqrt{g}A_3\psi_+^T\psi_-] \\
&+ i\int d\tau[C_1^*(-\partial_\tau^2 + r^2)C_1 + C_2^*(-\partial_\tau^2 + r^2)C_2 - C_3^*\partial_\tau^2C_3 \\
&+ \sqrt{g}\epsilon^{abc}\partial_\tau C_a^*C_bA_c - \sqrt{g}\epsilon^{a3x}\epsilon^{cbx}B_3^iC_a^*C_bY_c^i],
\end{aligned} \tag{11.18}$$

where the euclidean rotation $t \to i\tau$, $A \to -iA$ has been performed, the obvious distance between branes $r^2 = b^2 + (v\tau)^2$ is being used, and the decomposition of gamma matrices given in [1],

$$\begin{aligned}\Gamma^0 &= \sigma^3 \otimes \mathbf{I}_{16\times 16},\\ \Gamma^i &= i\sigma^1 \otimes \gamma^i,\end{aligned} \tag{11.19}$$

has been employed in order to simplify the action, and introduce the fermionic fields

$$\begin{aligned}\psi_+ &= \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2),\\ \psi_- &= \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2).\end{aligned} \tag{11.20}$$

The effective action will include contributions from the gaussian integration of the bosonic fields, and contributions from the integration over the Grassman variables describing the fermionic fields, which leads to a determinant product that can be evaluated through Riemann's zeta function techniques. Diagonalizing the bosonic mass matrix in the action (11.18), we obtain 10 massless bosons, 2 bosons of mass $m^2 = r^2 - 2v$, 2 bosons of mass $m^2 = r^2 + 2v$, 16 bosons of mass $m^2 = r^2$ and 10 massless bosons, all of them real while, from the ghost piece in (11.18) there are two complex bosons of mass r, and one massless complex boson; the fermionic action contains 8 fermions of mass $m^2 = r^2 - v$, and 8 fermions of mass $m^2 = r^2 + v$. Hence, the global determinant becomes the product

$$\det{}^{-6}(-\partial_\tau^2 + r^2)\det{}^{-1}(-\partial_\tau^2 + r^2 + 2v)\det{}^{-1}(-\partial_\tau^2 + r^2 - 2v)$$
$$\det{}^{4}(-\partial_\tau^2 + r^2 + v)\det{}^{4}(-\partial_\tau^2 + r^2 - v). \tag{11.21}$$

This determinant for the operators $A = -\partial_\tau^2 + \mu^2 + v^2\tau^2$, with $\mu^2 = b^2, b^2 \pm 2v, b^2 \pm v$, can evaluated through Riemann's zeta function,

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\sigma \sigma^{s-1} \int d^n \bar{x} G(\bar{x}, \bar{x}; \tau), \tag{11.22}$$

where $G(\bar{x}, \bar{y}; \sigma)$ is the propagator, solution to the heat equation $A_{\bar{x}} G(\bar{x}, \bar{y}; \tau) = -\partial_\sigma G$, and n is the spacetime dimension or, in brane terms, the dimension of the worldvolume of the scattering branes. Hence, for zerobranes,

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\sigma \sigma^{s-1} \int d\tau G(\tau, \tau; \sigma) =$$

$$\frac{1}{2\Gamma(s)} \int_0^\infty d\sigma \sigma^{s-1} e^{-\mu^2 \sigma} \frac{1}{\sinh(\sigma v)}. \tag{11.23}$$

However, the integration over the "proper time" variable σ is only defined for small values of $\frac{1}{b^2}$, as the semiclassical interaction between zerobranes can only be computed for large impact parameter. Finally, after power series expansions, the global product determinant becomes

$$\delta = \frac{v^3}{b^6}. \tag{11.24}$$

As from the eikonal approximation $\psi \sim e^{i\psi}$ is the wavefunction in a semiclassical analysis for the scattering of two zerobranes, with δ the solution to the stationary Hamilton-Jacobi equation,

$$|\vec{\nabla}\delta| = \sqrt{2\mu(E - V(r))}, \tag{11.25}$$

the phase shift (11.24) can be employed to obtain the potential between the scattering zerobranes. Imposing the wavefunction to become the free one as $\tau \to -\infty$,

$$\delta = \mu v \tau + \int_{\tau=-\infty}^{\tau} \{[\mu^2 v^2 - U(r)]^{1/2} - \mu v\} d\tau', \tag{11.26}$$

where μ is the zerobrane mass. In the $\mu^2 v^2 >> U(r)$ limit,

$$\delta \sim \mu v \tau - \frac{1}{2\mu v} \int_{-\infty}^{\tau} U(\sqrt{b^2 + v^2 \tau^2}) d\tau', \tag{11.27}$$

so that the wave function, after scattering, reads

$$\psi(\vec{b} + \mu \vec{v} \tau) \sim \phi(\vec{b} + \mu \vec{v} \tau). \exp\left[\frac{-i}{2\mu v} \int_{-\infty}^{\tau} U(\sqrt{b^2 + v^2 \tau^2}) d\tau'\right]. \tag{11.28}$$

Up to constants, choosing $V(r) = \frac{15}{8}\frac{v^4}{r^7}$, with $r^2 = b^2 + v^2\tau^2$ one gets $\delta = \frac{v^3}{b^6}$ or, equivalently, $\delta = \frac{v^3}{b^6}$ implies

$$V(r) = \frac{15}{8}\frac{v^4}{r^7}, \tag{11.29}$$

which is the potential governing the scattering of two zero-branes.

11.2.2 Brane-Antibrane Scattering

11.3 Elliptic Fibrations and D-7branes

Let us consider an elliptically fibered $K3$ manifold X. Recall from chapter 3 the relations

$$\begin{aligned} 24 &= \sum_i e(F - i), \\ \rho(X) &= 2 + \sum_i \sigma(F_i), \end{aligned} \tag{11.30}$$

where F_i represents the singular fibers, and where $e(F_i)$ are the Euler characteristics of the singularities. An interesting example is that of a model with four D_4 singularities. In this case, $e(D_4) = 6$ and $\sigma(D_4) = 4$, so that we get $\rho(X) = 18$.

The interest on D_4 singularities comes from the monodromy matrices

$$M_{D_4} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11.31}$$

Thus, in this case the elliptic modulus of the fiber $\tau(z)$ is constant. Let us consider a Calabi-Yau fourfold, which is $K3$ fibered, and with the $K3$ surface elliptically fibered in the way above described. A D-brane model of this spacetime can be obtained locating D-7branes and 7-orientifolds at the points in the base space of the elliptically fibered $K3$, where the elliptic fiber is singular. The question we want to address is how to represent different singularities in terms of D-branes and orientifold planes. The way to proceed is interpreting the Calabi-Yau fourfold as an F-theory compactification. As discussed in chapter 4, this is equivalent to considering type IIB string compactifications, with the elliptic modulus of the elliptic fiberbeing identified with the $Sl(2, \mathbf{Z})$ multiplet defined by the dilaton and axion fields of type IIB string theory,

$$\tau = \chi + ie^{-\phi}, \tag{11.32}$$

with χ the axion field. Using this identification, we can associate with the D-7brane R-R charge a monodromy transformation of the form

$$\chi \rightarrow \chi + 1, \tag{11.33}$$

i. e., a T-transformation, $\tau \rightarrow \tau + 1$. In fact, a 7brane is associated to an 8-form field, or a 9-form field strength. Its Hodge dual is a -1 form, namely $d\chi$. Thus, for a 7-brane of topological charge equal one we get

$$\oint_C d\chi = 1, \tag{11.34}$$

with the loop C in $\mathbb{P}^1$ around the point where the D-7brane is inserted. This is the typical monodromy of A_1 singularities. But, what about orientifolds? The charge of the orientifold, for a 7orientifold is

$$-2^{7-5} = -4, \tag{11.35}$$

corresponding to a monodromy transformation

$$\chi \to \chi - 4. \tag{11.36}$$

Therefore, in order to get a D_4 singularity, with monodromy matrix given by (11.31), we can take a 7orientifold and 4 D-7branes.

An interesting implication of these elliptic fibrations, as shown in [?], allows to get a D-brane model of Seiberg-Witten soltions for $N = 2$ supersymmetric Yang-Mills theories. The idea comes under D-brane probes. We can consider a D-3brane probe, parallel to the D-7branes and orientifolds. The supersymmetry on the D-brane is $N = 2$, as the D-7branes break half of the supersymmetries. We can now interpret the elliptically fibered $K3$ as a model of the Seiberg-Witten quantum moduli for the gauge theory defined on the worldvolume of the D-3branes. Under this interpretation, the elliptic modulus τ is identified with the complexified coupling constant for this four dimensional gauge theory. The example with singularities of type D_4 corresponds to constant τ, and therefore vanishing beta function, which is the case of $N = 2$ $SU(2)$ gauge theory, with four hypermultiplets.

A geometrical interpretation of the Seiberg-Witten phenomena of splitting the classical singularity of enhancement of gauge symmetry into a monopole and dyon singularities in the quantum moduli is now available. In fact, if we move away from the orientifold the D-7branes, we will get

$$\tau(z) = \tau_0 + \frac{1}{2\pi i}(\sum_{j=1}^{4} \ln(z - z_j) - 4 \ln z), \tag{11.37}$$

where z_i are the positions of the D-7branes, and $-4 \ln z$ is the orientifold contribution. The value τ_0 is the asymptotic value of the elliptic modulus. It is obvious, from (11.8), that this solution is not defining a good elliptic fibration, as Im τ is not always positive. The solution to this puzzle is breaking, through quantum effects, the orientifold into two pieces, corresponding to the monopole and dyon singularities in the Seiberg-Witten solution. From a mathematical point of view, what is going on is that the orientifold is not representing a good elliptic Kodaira singularity. If we split it into good singularities, we should do it consistently with the Euler number constraint, (11.30). A possible suggestion is relating the Seiberg-Witten solution with some $K3$ manifold, as it is clear from the Sioda-Tate formula (11.30), the solution with the splitted orientifold is having a completely different Picard number.

11.4 D-Brane Classical Supergravity Solutions

As we have discussed in chapter 4, the NS-NS sector of type IIA and type IIB strings is the same. Setting all R-R backgrounds equal to zero, the low energy effective action for the NS-NS sector of type II strings is given by

$$\mathcal{L}_{NS} = -\frac{1}{2\kappa^2} \int d^{10}x \sqrt{g} e^{-2\Phi} (R - 4 D_\mu \Phi D^\mu \Phi + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}). \tag{11.38}$$

The gravitational coupling, κ, can be absorbed in a shift of the dilaton field. The lagrangian (11.38) is given by the so called string frame. We can easily pass to the Einstein frame through a redefinition of the metric,

$$g_{\mu\nu} \to e^{\phi/2} g_{\mu\nu}. \tag{11.39}$$

We are interested in classical solutions to (11.38) that are sources of the 2-form field, B, i. e., solutions which are extended objects of dimension equal one. The magnetic duals should be sources of a 6-form field, and will therefore be five dimensional. We will call these solutions NS-$p$branes, with $p = 1$ and $p = 5$. In the string frame (11.38), these solutions are given by [74]

$$\begin{aligned} ds^2 &= H^{-1} dx^2_{\|_{(2)}} - dx^2_{I_{(8)}}, \\ e^{2\phi} &= H^{-1}, \\ H_{0,I} &= \partial_I H^{-1}, \end{aligned} \tag{11.40}$$

for the NS-1brane, where $x_{\|} = (x_0, x_1)$ are the worldvolume coordinates, and x_I the 8 transversal coordinates.

In order to check supersymmetry, we should consider variations on the metric (11.40) for the gravitino and dilatino fields,

$$\begin{aligned} \delta\lambda &= [\partial_\mu \phi \gamma^\mu \Gamma_{11} + \frac{1}{6} H_{\mu\nu\rho} \gamma^{\mu\nu\rho}]\eta = 0, \\ \delta\psi_\mu &= [\partial_\mu + \frac{1}{4}(\omega_\mu^{ab} + H_\mu^{ab} \Gamma 11)\Gamma_{ab})\eta = 0, \end{aligned} \tag{11.41}$$

with the gamma matrices Γ in flat space satisfying $\{\Gamma_\mu, \Gamma\nu\} = 2\eta_{\mu\nu}$, and $\gamma^\mu = e_a^\mu \Gamma^a$ for a and s target space indices. For

$\eta = \epsilon_R + \epsilon_L$, we get

$$\begin{aligned} \epsilon_{R,L} &= H^{-1/4}\epsilon^0_{R,L}, \\ \Gamma^0\Gamma^1\epsilon^0_R &= \epsilon^0_R, \\ \Gamma^0\Gamma^1\epsilon^0_L &= -\epsilon^0_R, \end{aligned} \tag{11.42}$$

with ϵ^0 a constant spinor. Introducing the ansatz (11.40) in the equations of motion for (11.38), we get

$$H = \left(1 + \frac{c}{r^6}\right), \tag{11.43}$$

for some arbitrary constant, c. This charge should be related to the B-charge of the fundamental string. The way to discover this charge will be using the BPS property of this relation, which is due to the fact that the solution preserves half supersymmetries. As it is BPS, we can identify this charge with the mass. The mass can be determined from (11.40) by the usual ADM procedure wjen going to the Einstein frame. The Hodge dual NS-5brane solution is [75]

$$\begin{aligned} ds^2 &= dx^2_{(6)} - H dx^2_{(4)}, \\ e^{2\phi} &= H, \\ {}^*H_{01\ldots 5I} &= \partial_I H^{-1}. \end{aligned} \tag{11.44}$$

The function H is the harmonic function

$$H = 1 + \frac{c}{(x_1^2 + \cdots + x_4^2)}, \tag{11.45}$$

where the constant should be determined imposing Dirac's duality condition on the electric magnetic pair defined by the solitonic fivebrane and onebrane described in (11.40). As derived in [75], the constant for the solitonic fivebrane is α'.

Once we have obtained these solitonic solutions for the string effective action (11.38), we can try to repeat the same argument for branes with R-R charge, i. e., for D-branes. The first thing we need is to complete (11.38) with the contributions from the R-R sectors for type IIA and type IIB. For type IIA,

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{NS} + \int d^{10}x\sqrt{g}\left[\frac{1}{4}(F^{(2)})^2 + \frac{1}{2.4}(F^{(4)} + A \wedge H)^2\right] \\ &\quad + \int F^4 \wedge F^4 \wedge B, \qquad (11.46)\end{aligned}$$

while for type IIB string theory

$$\mathcal{L} = \mathcal{L}_{NS} + \int d^{10}x\sqrt{g}\left[\frac{1}{2}(\nabla\chi)^2 + \frac{1}{2.3}(\chi H - (F^{(3)})^2\right], \quad (11.47)$$

where $F^{(4)} = dA^{(3)}$, $F^{(2)} = dA^{(1)}$ in type IIA strings, and $F^{(3)} = dA^{(2)}$ in type IIB. The field χ is the R-R axion of type IIB string theory. In (11.47), extra terms containing $F^{(5)} = dA^{(4)}$ should be added, but unfortunately can not be written in covariant form, due to the fact that $F^(5)$ is self dual in ten dimensions. For lagrangians (11.46) and (11.47), we can try to get solutions sources of R-R fields. For generic D-$p$branes, the solution is given by

$$\begin{aligned}ds^2 &= H^{-1/2}dx_{\|}^2 - H^{1/2}dx_{\perp}^2, \\ e^{2\phi} &= H^{-1/2(p-3)}, \\ H_{0\ldots pI} &= \partial_I H^{-1}, \qquad (11.48)\end{aligned}$$

from which is clear that the self dual case, with $p = 3$, is not dilatonic. The harmonic function H in (11.48) is given by

$$H = \left(1 + \frac{\tilde{c}}{r^{(7-p)}}\right), \qquad (11.49)$$

where the constant will be an integer times the minimum charge of the D-brane. For our future analysis of the D-brane representation of black holes it would be important to consider also D-brane solutions carrying momentum a parallel direction to the D-brane [76]. Macroscopically, we can interpret these oscillating solutions as resulting from a coherent superposition of open string excitations on the D-brane world-volume. The recipe to get these oscillating $p$brane solutions with momentum in one particular direction $\vec{\imath}$ is to replace in (11.44) or (11.48) $dt^2+dx_i^2$ by $-dt^2+dx_i^2+k(dt-dx_9)^2$, where k is a harmonic function,

$$k = \frac{\text{constant } .N}{r^{7-p}}, \tag{11.50}$$

with N measured again in momentum measured units, defined by the minimum momenta that will fix the constant in (11.50).

11.5 D-Branes and Black Holes

As an illustrative example, we will simply review the main aspects of the D-brane microscopic description of the Bekenstein-Hawking [77] entropy for a five dimensional black hole in type IIB supergravity, compactified on a five torus, T^5. The brane metric we will start with corresponds to wrapping on T^5 Q_5 D-5branes, and Q_1 D-1branes, along one particular direction of the torus. We will also include some Kaluza-Klein momenta, N along that direction, i. e., we will consider oscillating D-strings. This solution will be characterized by three harmonic functions, that we will call H_5, H_1 and K, describing, respectively, the D-5brane, the D-1brane and the Kaluza-Klein momentum. The supergravity solution for this configuration of branes is given by [78], [79], [80]

$$\begin{aligned} ds^2 &= H_1^{-1/2} H_5^{-1/2}(-dt^2 + dx_9^2 + k(dt - dx_9)^2) \\ &\quad + H_1^{1/2} H_5^{1/2}(dx_1^2 + \cdots + dx_4^2) \\ &\quad + H_1^{1/2} H_5^{-1/2}(dx_5^2 + \cdots + dx_8^2). \end{aligned} \tag{11.51}$$

The coordinates for the uncompactified directions are

$$(t, x_1, \ldots, x_4).$$

(11.51) is a solution where we have already imposed to branes being completely wrapped on the five dimensional internal spacetime. The harmonic functions in (11.51), at large distances, will describe, in the uncompactified five dimensional space, point like sources with different charges, so that all of them will depend on the radial distance as $\frac{1}{r^2}$, which is the spherically symmetric solution to the Laplace equation in $\mathbf{R}^4$. Thus, notice that all the harmonic functions in (11.51) depend in the same way on r, $\frac{c_1 Q_1}{r^2}$, $\frac{c_5 Q_5}{r^2}$, $\frac{c_N N}{r^2}$.

In order to see the type fo metric associated to (11.51) in five dimensions, we will perform the usual Kaluza-Klein dimensional reduction [81], [82]. In order to compare later on with the Bekenstein-Hawking formula for the entropy, we should work in the Einstein frame; the metric thus obtained is

$$ds^2 = -\frac{1}{(H_1 H_5(1+k))^{2/3}} dt^2 + (H_1 H_5(1+k))^{1/3}(dx_1^2 + \cdots + dx_4^2). \tag{11.52}$$

In order to fix constants in the different harmonic functions, we should proceed as follows (see [83], for a complete reference). First, we will compute the mass of the different D-branes wrapped on the internal torus, in the Einstein frame. Next, we compare these masses with the ADM mass formula for the dimensionally reduced metric, (11.52). These ADM masses will depend on the coefficients of the harmonic functions. Identifying both quantities, we get the coefficients of the harmonic functions appearing in (11.52). These coefficients, and Newton's constant are given by

$$c_1 = \frac{4G_N^5 R_9}{\pi \alpha' g}, \quad c_5 = g\alpha', \quad c_p = \frac{4G_N^5}{\pi R_9}, \tag{11.53}$$

where $G_N^d = \frac{G_N^{10}}{V_{10-d}}$ ($G_N^{10} = 8\pi^6 g^2 \alpha'^4$). Using these constants we get, for the entropy of the extremal black hole (11.52),

$$S = 2\pi\sqrt{NQ_1Q_5}, \tag{11.54}$$

which is a surprisingly simple result, completely independent of the size of the internal five dimensional torus. Our task will now be getting a microscopic description of (11.54) (we will however proceed very schematically, just to give the reader the flavor of the procedure). We will then choose the D-string wrapped in the x_9-direction; it carries RR charge for the RR $B^{\mu\nu}$ 2-form. On the worldvolume of the D-5brane, we have a Chern-Simpons coupling of the type [84]

$$\int d^{5+1}x B \wedge F \wedge F. \tag{11.55}$$

As we have a superposition of Q_5 D-5branes, the effective gauge theory on the D-5brane worldvolume is $U(Q_5)$. Thus, from (11.55), we can interpret, for each value of x_9, that the D-string is represented by a gauge field in $U(Q_5)$, defining a four dimensional instanton living in the four dimensional section $\mathbf{R}^4$, and with topological charge $\int d^4xF \wedge F$ equal one. The number of instantons we have is equal to the number of D-strings, Q_1. Now, we can consider the moduli space of these instantons. We have four translational zero modes, no dilatations, and the orientations in $U(Q_5)$; thus, there are $4Q_1Q_5$ moduli. Next, we will put some momentum N along the ninth direction, corresponding to the oscillation of one string. This will correspond to small fluctuations of the instantons. The gas of instantons can be described in terms of a two dimensional conformal field theory defined on the (x^9, t) spacetime, and with target the moduli space of instantons. Most of the work is already done. The supersymmetry preserved on the D-5brane worldvolume will give rise to a fermionic partner for each of the $4Q_1Q_5$ bosonic collective coordinates. Hence, the conformal field theory will have, as cantral extension, $6Q_1Q_5$. To compute the entropy, we only need the multiplicity of states for given values of the momentum, $N = L_0 - \bar{L}_0$ ($L_0 = 0$):

$$d(N) \sim e^{2\pi\sqrt{N\,c/6}}, \tag{11.56}$$

with c the central extension. The entropy is

$$S = \ln d(N) = 2\pi\sqrt{NQ_1Q_5}, \tag{11.57}$$

in agreement with the Bekenstein-Hawking formula, (11.54).

Bibliografía

[1] M. B. Green, J. H. Schwarz and E. Witten, "*Superstring Theory*", Cambridge University Press, Cambridge, 1987.

[2] J. Polchinski, "What is String Theory?", **hep-th/9411028**.

[3] C. Vafa, "Lectures on Strings and Dualities", **hep-th/9702201**.

[4] E. Kiritsis, "Introduction to Superstring Theory", **hep-th/9709062**.

[5] Y. Nambu, "Quark Model and the Factorization of the Veneziano Amplitude", in *Symmetries and Quark Models*, Ed. R. Chand, Gordon and Breach (1970).

T. Goto, "Relativistic Quantum Mechanics of One-Dimensional Mechanical Continuum and Subsidiary Condition of Dual Resonance Model", Prog.Theor.Phys. **46** (1971), 1560.

[6] A. M. Polyakov, "Quantum Geometry of Bosonic Strings", Phys. Lett. **B103** (1981), 207.

[7] P. S. Howe and R. W. Tucker, “A Locally Supersymmetric and Reparametrization Invariant Action For a Spinnig Membrane”, J. Phys. **A10** (1977), L155.

[8] C. Rebbi, “Dual Models and Relativistic Quantum Strings”, Phys. Rept. **12** (1974), 1.

[9] S. Mandelstam, “Dual-Resonance Models”, Phys. Rept. **13** (1974), 259.

[10] L. D. Faddeev and V. N. Popov, “Feynman Diagrams for the Yang-Mills Field”, Phys. Lett. **B25** (1967), 29.

[11] S. Fubini and G. Veneziano, “Duality in Operator Formalism”, Nuovo Cim. **A67** (1970), 29.

[12] C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, “Strings in Background Fields”, Nucl. Phys. **B262** (1985), 593.

[13] W. Fischler and L. Susskind, “Dilaton Tadpoles, String Condensates and Scale Invariance”, Phys. Lett. **B171** (1986), 383.

W. Fischler and L. Susskind, “Dilaton Tadpoles, String Condensates and Scale Invariance. 2”, Phys. Lett. **B173** (1986), 262.

[14] L. Brink, M. B. Green and J. H. Schwarz, “$N = 4$ Yang-Mills and $N = 8$ Supergravity as Limits of String Theories”, Nucl. Phys. **B198** (1982), 474.

K. Kikkawa and M. Yamasaki, “Casimir Effects in Superstring Theories”, Phys. Lett. **B149** (1984), 357.

N. Sakai and I. Senda, "Vacuum Energies of String Compactified on Torus", Prog. Theor. Phys. **75** (1986), 692. Erratum-ibid. **77** (1987), 773.

[15] K. S. Narain, "New Heterotic String Theories in Uncompactified Dimensions < 10", Phys. Lett. **B169** (1986), 41; K. S. Narain, M. H. Samadi and E. Witten, "A Note on the Toroidal Compactification of Heterotic String Theory", Nucl. Phys. **B279** (1987), 369.

[16] For reviews see A. Giveon, M. Porrati and E. Rabinovici, "Target Space Duality in String Theory", Phys. Rep. **244** (1994), 77.

E. Álvarez, L. Álvarez-Gaumé and Y. Lozano, "An Introduction to T-Duality in String Theory", **hep-th/9410237**.

[17] J. Polchinski and Y. Cai, "Consistency of Open Superstring Theories", Nucl. Phys. **B296** (1988), 91.

[18] M. Green, "Space-Time Duality and Dirichlet String Theory", Phys. Lett. **B266** (1991), 325.

[19] J. Polchinski, "Dirichlet Branes and Ramond-Ramond Charges", Phys. Rev. Lett. **75** (1995), 4724.

[20] J. Polchinski, S. Chaudhuri and C. V. Johnson, "Notes on D-Branes", **hep-th/9602052**, and "TASI Lectures on D-Branes", **hep-th/9611050**.

[21] T. H. Buscher, "A Symmetry of the String Background Field Equations", Phys. Lett. **B194** (1987), 59.

T. H. Buscher, "Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models", Phys. Lett. **B201** (1988), 466.

[22] E. S. Fradkin and A. A. Tseylin, "Effective Field Theory from Quantized Strings", Phys. Lett. **B158** (1985), 316.

E. S. Fradkin and A. A. Tseylin, "Effective Action Approach to Superstring Theory", Phys. Lett. **B160** (1985), 69.

E. S. Fradkin and A. A. Tseylin, "Quantum String Theory Effective Action", Nucl. Phys. **B261** (1985), 1.

[23] C.G. Callan, C. Lovelace, C.R. Nappi and S.A. Yost, "Loop Corrections to Superstring Equations of Motion", Nucl. Phys. **B308** (1988), 221.

[24] R. G. Leigh, "Dirac-Born-Infeld Action From Dirichlet Sigma Model", Mod. Phys. Lett. **A4** (1989), 2767.

[25] P. Griffiths and J. Harris, "*Principles of Algebraic Geometry*", Wiley-Interscience, 1978.

[26] P. A. Aspinwall, "$K3$ Surfaces and String Duality", **hep-th/9611137**.

[27] U. Persson, Lecture Notes in Mathematics, 1124, Springer-Verlag.

[28] N. Seiberg, "Observations on the Moduli Space of Superconformal Field Theories", Nucl. Phys. **B303** (1988), 286.

[29] B. Greene and M. R Plesser, "Duality in Calabi-Yau Moduli Space", Nucl Phys. **B338** (1990), 15.

P. Candelas, M. Lynker and R. S. Schimmrigk, "Calabi-Yau Manifolds in Weighted P(4)", Nucl. Phys. **B341** (1990), 383.

P. Candelas, X. de la Ossa, P. Green and L. Parkes, "A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory", Nucl. Phys. **B359** (1991), 21.

V. Batyrev, "Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties", **alg-geom/9410003**.

For a series of references, see S. -T. Yau ed., "Essays on Mirror Manifolds", International Press, Hong-Kong, 1992.

[30] I. V. Dolgachev, "Mirror Symmetry for Lattice Polarized $K3$-Surfaces", **alg-geom/9502005**.

[31] J. D. Fay, "Theta Functions on Riemann Surfaces", Lecture Notes in Mathematics, **352**, Springer-Verlag, 1973.

[32] M. Artin, "On Isolated Rational Singularities of Surfaces", Amer. J. Math. **88** (1966), 129.

[33] K. Kodaira, Ann. Math. **77**, 3 (1963), 563.

[34] P. Ramond, "Dual Theory for Free Fermions", Phys. Rev. **D3** (1971), 2415.

[35] A. Neveu and J. H. Schwarz, "Factorizable Dual Model of Pions", Nucl. Phys. **B31** (1971), 86.

[36] D. Friedan, E. Martinec and S. Shenker, "Conformal Invariance, Supersymmetry and String theory", Nucl. Phys. **B271** (1986), 93.

[37] M. B. Green and J. H. Schwarz, "Supersymmetrical Dual String Theory", Nucl. Phys. **B181** (1981), 502.

M. B. Green and J. H. Schwarz, "Supersymmetric Dual String Theory (II). Vertices and Trees", Nucl. Phys. **B198** (1982), 252.

M. B. Green and J. H. Schwarz, "Supersymmetrical String Theories", Phys. Lett. **B109** (1982), 444.

[38] J. C. Paton and H. M. Chan, "Generalized Veneziano Model with Isospin", Nucl. Phys. **B10** (1969), 516.

[39] C. Montonen and D. Olive, "Magnetic Monopoles as Gauge Particles ?", Phys. Lett. **B72** (1977), 177.

[40] J. Cardy and E. Rabinovici, "Phase Structure of $Z(P)$ Models in the Presence of a Theta Parameter", Nucl. Phys. **B205** (1982), 1.

[41] A. Font, L. Ibañez, D. Lüst and F. Quevedo, "Strong-Weak Coupling Duality and Non-Perturbative Effects in String theory", Phys. Lett. **B249** (1990), 35.

[42] A. Sen, "Strong-Weak Coupling Duality in Four Dimensional Field theory", Int. J. Mod. Phys. **A9** (1994), 3707.

[43] C. Hull and P. Townsend, "Unity of Superstring Dualities", Nucl. Phys. **B438** (1995), 109.

[44] M. Duff, “Strong/Weak Coupling Duality from the Dual String”, Nucl. Phys. **B442** (1995), 47.

[45] P. Townsend, “The Eleven-Dimensional Supermembrane Revisited”, Phys. Lett. **B350** (1995), 184.

[46] E. Witten, “String Theory Dynamics in Various Dimensions”, Nucl. Phys. **B443** (1995), 85.

[47] A. Sen, “String-String Duality Conjecture in Six Dimensions and Charged Solitonic Strings”, Nucl. Phys. **B450** (1995), 103.

[48] J. Harvey and A. Strominger, “The Heterotic String is a Soliton”, Nucl. Phys. **B449** (1995), 535.

[49] B. Greene, D. Morrison and A. Strominger, “Black Hole Condensation and the Unification of String Vacua”, Nucl. Phys. **B451** (1995), 109.

[50] C. Vafa and E. Witten, “A One-Loop Test Of String Duality”, Nucl. Phys. **B447** (1995), 261.

[51] C. Hull and P. Townsend, “Enhanced Gauge Symmetries in Superstring Theories”, Nucl. Phys. **B451** (1995), 525.

[52] D. J. Gross, J. A. Harvey E. Martinec and R. Rohm, “Heterotc String”, Phys. Rev. Lett. **54** (1985), 502; “Heterotic String Theory (I). The Free Heterotic String.”, Nucl. Phys. **B256** (1985), 253; “Heterotic String Theory (II). The Interacting Heterotic String.”, Nucl. Phys. **B267** (1986), 75.

[53] S. Katz, D. R. Morrison and M. R. Plesser, "Enhanced Gauge Symmetry in Type II String Theory", Nucl. Phys. **477** (1996), 105.

[54] A. Klemm and P. Mayr, "Strong Coupling Singularities and Non-abelian Gauge Symmetries in $N = 2$ String Theory", Nucl. Phys. **B469**, 37. (1996),

[55] C. Vafa, "Evidence for F-Theory", Nucl. Phys. **B469** (1996), 403.

[56] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi-Yau Threefolds. I.", Nucl. Phys. **B473** (1996), 74.

[57] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi-Yau Threefolds. II.", Nucl. Phys. **B476** (1996), 437.

[58] A. Klemm, W. Lerche and P. Mayr,"$K3$-Fibrations and Heterotic-Type II String Duality", Phys. Lett. **B357** (1995), 313.

[59] S. Kachru and C. Vafa, "Exact Results for $N = 2$ Compactifications of Heterotic Strings", Nucl. Phys. **450**, (1995), 69.

[60] A. Dabholkar, "Ten Dimensional Heterotic String as a Soliton", Phys. Lett. **B357** (1995), 307.

[61] C. Hull, "String-String Duality in Ten-Dimensions", Phys. Lett. **B357** (1995), 545.

[62] A. Strominger, "Massless Black Holes and Conifolds in String Theory", Nucl. Phys. **B451** (1995), 96.

[63] E. Witten, "Small Instantons in String Theory", Nucl. Phys. **B460** (1996), 541.

[64] S. Katz, A. Klemm, C. Vafa, "Geometric Engineering of Quantum Field Theories", Nucl. Phys. **B497** (1997), 173.

[65] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, "Self-Dual Strings and $N = 2$ Supersymmetric Field Theory", Nucl. Phys. **B477**, (1996), 746.

[66] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. Morrison, "Mirror Symmetry for Two Parameter Models (I)", Nucl. Phys. **B416** (1994) 481.

[67] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, "Mirror Symmetry, Mirror Map, and Applications to Calabi-Yau Hypersurfaces", Comm. Math. Phys. **167** (1995) 301.

[68] P. S. Aspinwall and J. Louis, "On the Ubiquity of $K3$ Fibrations in String Duality", Phys. Lett. **B369** (196), 233..

[69] A. Gorskii, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, "Integrability and Exect Seiberg-Witten Solution", Phys. Lett. **B355** (1995), 466.

[70] E. Martinec and N. P. Warner, "Integrable Systems and Supersymmetric Gauge Theories", Nucl. Phys. **B459** (1996), 97.

[71] S. Katz and C. Vafa, "Geometric Engineering of Quantum Field Theories", Nucl. Phys. **B497** (1997),

173, and "Geometric Engineering of $N = 1$ Quantum Field Theories", Nucl. Phys. **B497** (1997), 196.

[72] M. Bershadsky, A. Johansen, T. Pantev, V. Sadov and C. Vafa, "F-Theory, Geometric Engineering and $N=1$ Dualities", **hep-th/9612052**.

[73] S. H. Shenker, "Another Length Scale in String Theory?", **hep-th/9509132**.

[74] A. Dabholkar and J. A. Harvey, "Nonrenormalization of the Superstring Tension", Phys. Rev. Lett. **63** (1989), 478.

A. Dabholkar, G. Gibbons, J. A. Harvey and F. Ruiz Ruiz, "Superstrings and Solitons", Nucl. Phys. **B340** (1990), 33.

[75] C. G. Callan, J. A. Harvey and A. Strominger, "World Sheet Approach to Heterotic Instantons and Solitons", Nucl. Phys. **B359** (1991), 611.

[76] D. Garfinkle and T. Vachaspati, "Cosmic String Traveling Waves", Phys. Rev. **D42** (1990), 1960.

D. Garfinkle, "Black String Traveling Waves", Phys. Rev. **D46**, (1992), 4286.

[77] J. D. Bekenstein, "Black Holes and Entropy", Phys. Rev. **D7** (1973), 2333.

J. D. Bekenstein, "Generalized Second Law of Thermodynamics in Black Hole Physcis", Phys. Rev. **D9** (1974), 3292.

S. W. Hawking, "Black Holes and Thermodynamics", Phys. Rev. **D13** (1976), 191.

[78] M. Cvetic and D. Youm, "Dyonic BPS Saturated Black Holes of Heterotic String on a Six Torus", Phys. Rev. **D53** (1996), 584.

[79] A. A. Tseytlin, "Extreme Dyonic Black Holes in String Theory", Mod. Phys. Lett. **A11** (1996), 689.

[80] C. G. Callan and J. M. Maldacena, "D-Brane Approach to Black Hole Quantum Mechanics", Nucl. Phys. **B472** (1996), 591.

[81] J. Maharana and J. H. Schwarz, "Noncompact Symmetries in String Theory", Nucl. Phys. **B390** (1993), 3.

[82] A. Sen, "Electric Magnetic Duality in String Theory", Nucl. Phys. **B404** (1993), 109.

[83] J. M. Maldacena, "Black Holes in String Theory", Ph. D. Thesis, **hep-th/9607235** .

[84] M. R. Douglas, "Branes within Branes", **hep-th/9512077**.