New representations of focal curves in the special $\phi$–Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

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Abstract

In this paper, we study matrix representations of focal curves in terms of biharmonic curves in the special three-dimensional $\phi$–Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We construct new parametric equations of focal curves in terms of matrix representations in the special three-dimensional $\phi$–Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

key words. Focal curve, Biharmonic curve, Matrices, Para-Sasakian manifold.

1 Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |T(\phi)|^2 \, dv_h,$$

where $T(\phi) := \text{tr} \nabla \phi d\phi$ is the tension field of $\phi$.

The Euler–Lagrange equation of the bienergy is given by $T_2(\phi) = 0$. Here the section $T_2(\phi)$ is defined by

$$T_2(\phi) = -\Delta_\phi T(\phi) + \text{tr} R(T(\phi), d\phi) \, d\phi,$$

and called the bitension field of $\phi$. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study matrix representations of focal curves in terms of biharmonic curves in the special three-dimensional $\phi$–Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We construct new parametric equations of focal curves in terms of matrix representations in the special three-dimensional $\phi$–Ricci symmetric para-Sasakian manifold $\mathbb{P}$.
2 Special Three-Dimensional $\phi$–Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

An $n$-dimensional differentiable manifold $M$ is said to admit an almost para-contact Riemannian structure $(\phi, \xi, \eta, g)$, where $\phi$ is a $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a Riemannian metric on $M$ such that

$$\phi \xi = 0, \quad \eta (\xi) = 1, \quad g (X, \xi) = \eta (X), \quad (2.1)$$

$$\phi^2 (X) = X - \eta (X) \xi, \quad (2.2)$$

$$g (\phi X, \phi Y) = g (X, Y) - \eta (X) \eta (Y), \quad (2.3)$$

for any vector fields $X, Y$ on $M$, [2].

**Definition 2.1.** A para-Sasakian manifold $M$ is said to be locally $\phi$-symmetric if

$$\phi^2 (\nabla_W R) (X, Y) Z = 0,$$

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$ [2].

**Definition 2.2.** A para-Sasakian manifold $M$ is said to be $\phi$-symmetric if

$$\phi^2 (\nabla_W R) (X, Y) Z = 0,$$

for all vector fields $X, Y, Z, W$ on $M$.

**Definition 2.3.** A para-Sasakian manifold $M$ is said to be $\phi$-Ricci symmetric if the Ricci operator satisfies

$$\phi^2 (\nabla_X Q) (Y) = 0,$$

for all vector fields $X$ and $Y$ on $M$ and $S(X, Y) = g(QX, Y)$.

If $X, Y$ are orthogonal to $\xi$, then the manifold is said to be locally $\phi$-Ricci symmetric.

We consider the three-dimensional manifold

$$\mathbb{P} = \{ (x^1, x^2, x^3) \in \mathbb{R}^3 : (x^1, x^2, x^3) \neq (0, 0, 0) \}.$$
where \((x^1, x^2, x^3)\) are the standard coordinates in \(\mathbb{R}^3\). We choose the vector fields
\[
e_1 = e^{x^1} \frac{\partial}{\partial x^2}, \quad e_2 = e^{x^1} \left( \frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3} \right), \quad e_3 = -\frac{\partial}{\partial x^1}
\]
are linearly independent at each point of \(\mathbb{P}\). Let \(g\) be the Riemannian metric defined by
\[
\begin{align*}
g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = 1, \\
g(e_1, e_2) &= g(e_2, e_3) = g(e_1, e_3) = 0.
\end{align*}
\]

Let \(\eta\) be the 1-form defined by
\[
\eta(Z) = g(Z, e_3) \text{ for any } Z \in \chi(\mathbb{P}).
\]

Let be the (1,1) tensor field defined by
\[
\phi(e_1) = e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.
\]

Then using the linearity of and \(g\) we have
\[
\eta(e_3) = 1,
\]
\[
\phi^2(Z) = Z - \eta(Z)e_3,
\]
\[
g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),
\]
for any \(Z, W \in \chi(\mathbb{P})\). Thus for \(e_3 = \xi\), \((\phi, \xi, \eta, g)\) defines an almost para-contact metric structure on \(\mathbb{P}\).

Let \(\nabla\) be the Levi-Civita connection with respect to \(g\). Then, we have
\[
[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.
\]
3 Matrix Representation of Focal Curves in the Special Three-Dimensional $\phi$–Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

For a unit speed curve $\gamma$, the curve consisting of the centers of the osculating spheres of $\gamma$ is called the parametrized focal curve of $\gamma$. The hyperplanes normal to $\gamma$ at a point consist of the set of centers of all spheres tangent to $\gamma$ at that point. Hence the center of the osculating spheres at that point lies in such a normal plane. Therefore, denoting the focal curve by $C_\gamma$, we can write

$$C_\gamma(s) = (\gamma + c_1 N + c_2 B)(s),$$

where the coefficients $c_1, c_2$ are smooth functions of the parameter of the curve $\gamma$, called the first and second focal curvatures of $\gamma$, respectively. Further, the focal curvatures $c_1, c_2$ are defined by

$$c_1 = \frac{1}{\kappa}, \quad c_2 = \frac{c_1'}{\tau}, \quad \kappa \neq 0, \quad \tau \neq 0.$$  \hspace{1cm} (3.2)

**Lemma 3.1.** Let $\gamma : I \rightarrow \mathbb{P}$ be a unit speed biharmonic curve and $C_\gamma$ its focal curve on $\mathbb{P}$. Then,

$$c_1 = \frac{1}{\kappa} = \text{constant and } c_2 = 0.$$  \hspace{1cm} (3.3)

**Proof.** Using (2.3) and (3.2), we get (3.3).

**Lemma 3.2.** Let $\gamma : I \rightarrow \mathbb{P}$ be a unit speed biharmonic curve and $C_\gamma$ its focal curve on $\mathbb{P}$. Then,

$$C_\gamma(s) = (\gamma + c_1 N)(s).$$  \hspace{1cm} (3.4)

**Theorem 3.3.** (see [13]) Let $\gamma : I \rightarrow \mathbb{P}$ be a non-geodesic biharmonic curve in the special three-dimensional $\phi$–Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then,

$$A_\gamma = \sqrt{-\text{trace} (A^2)} (-\cos \varphi, \sin \varphi e^{-s \cos \varphi + C_1} (\sin [ks + C] + \cos [ks + C]),$$

$$\sin \varphi e^{-s \cos \varphi + C_1} \sin [ks + C]),$$

$$A^2_\gamma = \left(\frac{\sqrt{-\text{trace}(A^4)}}{2} \cdot \begin{align*}
\sin^2 \varphi s^2 + C_1 s + C_2, \\
e^{-\sin^2 \varphi s^2 + C_1 s + C_2} (k \sin \varphi \sin [ks + C] + \cos \varphi \sin \varphi \cos [ks + C]), \\
+ e^{-\sin^2 \varphi s^2 + C_1 s + C_2} (k \sin \varphi \cos [ks + C] + \cos \varphi \sin \varphi \sin [ks + C]), \\
- e^{-\sin^2 \varphi s^2 + C_1 s + C_2} (-k \sin \varphi \cos [ks + C] + \cos \varphi \sin \varphi \sin [ks + C])).
\end{align*}\right)$$  \hspace{1cm} (3.5)
where $C, C_1, C_2$ are constants of integration and $k = \frac{\sqrt{\kappa^2 - \sin^2 \varphi}}{\sin \varphi}$.

Now, let us define a special form of focal curve.

**Lemma 3.4.** Let $\gamma : I \rightarrow \mathbb{P}$ be a regular curve in the special three-dimensional $\phi$–Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then,

$$C_{\gamma}(s) = \gamma(s) + c_1 \frac{A^2 \gamma}{\sqrt{\text{trace}(A^4)}}. \quad (3.6)$$

**Proof.** Using Theorem 3.3, we immediately obtain

$$N = \left( \frac{\sqrt{\text{trace}(A^4)}}{\text{trace}(A^4)} \right)^{-1} A^2 \gamma.$$

According to the definition of focal curve we have (3.6).

**Theorem 3.5.** (see [13]) Let $\gamma : I \rightarrow \mathbb{P}$ be a non-geodesic biharmonic curve in the special three-dimensional $\phi$–Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then the new curvatures of this curve are

$$\kappa = -\frac{\sqrt{\text{trace}(A^4)}}{\text{trace}(A^2)}, \quad (3.7)$$

$$\tau = \Re\left[ \frac{-\text{trace}(A^6)}{\left(\sqrt{-\text{trace}(A^4)}\right)^4 \text{trace}(A^4)} - \frac{(\text{trace}(A^4))^{\frac{3}{2}}}{\left(\sqrt{-\text{trace}(A^4)}\right)^6} \right],$$

where $\Re = \left[ \frac{\text{trace}(A^6)}{\left(\sqrt{-\text{trace}(A^4)}\right)^4 \sqrt{-\text{trace}(A^4)}} - \frac{(\text{trace}(A^4))^{\frac{3}{2}}}{\left(\sqrt{-\text{trace}(A^4)}\right)^6} \right]^{-\frac{1}{2}}$.

Using above Lemma and Theorem we have following result:

**Theorem 3.6.** Let $\gamma : I \rightarrow \mathbb{P}$ be a biharmonic curve parametrized by arc length. If $C_\gamma$ is a focal curve of $\gamma$, then the parametric equations of $C_\gamma$ are
\[
\begin{align*}
\tilde{x}^1(s) &= -\left(\frac{\text{trace}(A^2)}{\text{trace}(A^4)}\right)^2 \left(\frac{-\sin^2 \varphi}{2} s^2 + \overline{C}_1 s + \overline{C}_2\right) - \cos \varphi s + C_1, \\
\tilde{x}^2(s) &= -\left(\frac{\text{trace}(A^2)}{\text{trace}(A^4)}\right)^2 \frac{\sin^3 \varphi}{e^{-s \cos \varphi + C_1}} \left(\Pi \cos [\Pi s + C] + [-\Pi + \cos \varphi] \sin [\Pi s + C]\right) \\
&\quad - \left(\frac{\text{trace}(A^2)}{\text{trace}(A^4)}\right)^2 \frac{\sin^2 \varphi}{e^{s \cos \varphi} + \overline{C}_1 s + \overline{C}_2} \left(\Pi \sin \varphi \sin [\Pi s + C] + \cos \varphi \sin \varphi \cos [\Pi s + C]\right) + C_2, \\
\tilde{x}^3(s) &= -\left(\frac{\text{trace}(A^2)}{\text{trace}(A^4)}\right)^2 \frac{\sin^3 \varphi}{e^{-s \cos \varphi + C_1}} \left(\cos \varphi \cos [\Pi s + C] + \sin [\Pi s + C]\right) + C_3,
\end{align*}
\]

where \( C, \overline{C}_1, \overline{C}_2, C_1, C_2, C_3 \) are constants of integration and \( \Pi = \left[\frac{\sqrt{\text{trace}(A^4) - (\text{trace}(A^2))^2 \sin^2 \varphi}}{\text{trace}(A^2) \sin \varphi}\right] \).

**Proof.** Assume that \( \gamma \) be a spacelike biharmonic curve and \( C_\gamma \) its focal curve on \( \mathbb{P} \). Substituting equation (3.5) into Lemma 3.4, and by using the Mathematica program we have above system. This completes the proof of the theorem.

**References**


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