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Matrix representation for involute curves of biharmonic curves in terms of exponential maps in the special three-dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

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Abstract

In this paper, we study involute curve of biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We obtain matrix representation for involute curve of biharmonic curve in terms of curvature and torsion of biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

key words. Involute curve, Biharmonic curve, Para-Sasakian manifold.

AMS subject classifications. 53C41, 53A10.

1 Introduction

There are several methods to render matrices into a more easily accessible form. They are generally referred to as matrix transformation or matrix decomposition techniques. The interest of all these decomposition techniques is that they preserve certain properties of the matrices in question, such as determinant, rank or inverse, so that these quantities can be calculated after applying the transformation, or that certain matrix operations are algorithmically easier to carry out for some types of matrices.

Matrices can be generalized in different ways. Abstract algebra uses matrices with entries in more general fields or even rings, while linear algebra codifies properties of matrices in the notion of linear maps. It is possible to consider matrices with infinitely many columns and rows. Another extension are tensors, which can be seen as higher-dimensional arrays of numbers, as opposed to vectors, which can often be realised as sequences of numbers, while matrices are rectangular or two-dimensional array of numbers. Matrices, subject to certain requirements tend to form groups known as matrix groups.

The aim of this paper is to study matrix representation of involute curves in terms of exponential maps in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

In this paper, we study involute curve of biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. We obtain matrix representation for involute curve of biharmonic curve in terms of curvature and torsion of biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

2 Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi = 0, \quad \eta(\xi) = 1, g(X,\xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y on M [1].

We consider the three-dimensional manifold

$$\mathbb{P} = \left\{ \left(x^1, x^2, x^3\right) \in \mathbb{R}^3 : \left(x^1, x^2, x^3\right) \neq (0,0,0) \right\},$$

where $\left(x^1, x^2, x^3\right)$ are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$\mathbf{e}_1 = e^{x^1} \frac{\partial}{\partial x^2}, \quad \mathbf{e}_2 = e^{x^1} \left(\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3} \right), \quad \mathbf{e}_3 = -\frac{\partial}{\partial x^1} \tag{2.4}$$

are linearly independent at each point of $\mathbb{P}$. Let g be the Riemannian metric defined by

$$\begin{aligned} g(\mathbf{e}_1, \mathbf{e}_1) &= g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1, \\ g(\mathbf{e}_1, \mathbf{e}_2) &= g(\mathbf{e}_2, \mathbf{e}_3) = g(\mathbf{e}_1, \mathbf{e}_3) = 0. \end{aligned} \tag{2.5}$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, \mathbf{e}_3) \text{ for any } Z \in \chi(\mathbb{P}).$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1) = \mathbf{e}_2, \ \phi(\mathbf{e}_2) = \mathbf{e}_1, \ \phi(\mathbf{e}_3) = 0. \tag{2.6}$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3) = 1, \tag{2.7}$$

$$\phi^2(Z) = Z - \ \eta(Z)\mathbf{e}_3, \tag{2.8}$$

$$g\left(\phi Z, \phi W\right) = g\left(Z, W\right) - \eta(Z)\eta(W), \tag{2.9}$$

for any $Z, W \in \chi(\mathbb{P})$. Thus for $\mathbf{e}_3 = \xi$, (ϕ, ξ, η, g) defines an almost para-contact metric structure on $\mathbb{P}$ [1].

Let ∇ be the Levi-Civita connection with respect to g. Then, we have

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_2.$$

Taking $\mathbf{e}_3 = \xi$ and using the Koszul's formula, we obtain

$$\begin{array}{lll} \nabla_{\mathbf{e}_1}\mathbf{e}_1 = -\mathbf{e}_3, & \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0, & \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_1 = 0, & \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3, & \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \\ \nabla_{\mathbf{e}_3}\mathbf{e}_1 = 0, & \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0, & \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{array} \tag{2.10}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3.

$$R_{122} = -\mathbf{e}_{1,}, \quad R_{133} = -\mathbf{e}_{1,}, \ R_{233} = -\mathbf{e}_2,$$

and

$$R_{1212} = \ R_{1313} = R_{2323} = 1. \tag{2.11}$$

3 Biharmonic Curves in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0, \tag{3.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (3.1).

Let us consider biharmonicity of curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} &\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}, \\ &\nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}, \\ &\nabla_{\mathbf{T}}\mathbf{B} = -\tau\mathbf{N}, \end{aligned} \tag{3.2}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1, \ g(\mathbf{N}, \mathbf{N}) = 1, \ g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} &\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ &\mathbf{N} = N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ &\mathbf{B} = \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1. *All of biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$ are helices.*

4 Involute Curves of Biharmonic Curves in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

Definition 4.1. *Let unit speed curve $\gamma : I \longrightarrow \mathbb{P}$ and the curve $\Theta : I \longrightarrow \mathbb{P}$ be given. For $\forall s \in I$, then the curve Θ is called the involute of the curve γ, if the tangent at the point $\gamma(s)$ to the curve*

γ passes through the tangent at the point $\Theta(s)$ to the curve Θ and

$$g\left(\mathbf{T}^{*}\left(s\right),\mathbf{T}\left(s\right)\right)=0. \tag{4.1}$$

Let the Frenet-Serret frames of the curves γ and Θ be $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and $\{\mathbf{T}^{*},\mathbf{N}^{*},\mathbf{B}^{*}\}$, respectively.

Theorem 4.2. *(see [12]) Let $\gamma : I \longrightarrow \mathbb{P}$ be a non-geodesic biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then,*

$$\begin{aligned}\mathcal{A}\gamma &= \sqrt{-\mathcal{TR}\left(\mathcal{A}^{2}\right)}(-\cos\varphi,\sin\varphi e^{-s\cos\varphi+C_{1}}\left(\sin\left[\Bbbk s+C\right]+\cos\left[\Bbbk s+C\right]\right),\\ &\quad \sin\varphi e^{-s\cos\varphi+C_{1}}\sin\left[\Bbbk s+C\right]),\end{aligned}$$

$$\begin{aligned}\mathcal{A}^{2}\gamma = \frac{\left(\sqrt{\mathcal{TR}(\mathcal{A}^{4})}\right)}{\kappa}&(-\frac{\sin^{2}\varphi}{2}s^{2}+\overline{C}_{1}s+\overline{C}_{2},\\ &e^{-\frac{\sin^{2}\varphi}{2}s^{2}+\overline{C}_{1}s+\overline{C}_{2}}\left(\Bbbk\sin\varphi\sin\left[\Bbbk s+C\right]+\cos\varphi\sin\varphi\cos\left[\Bbbk s+C\right]\right)\\ &+e^{-\frac{\sin^{2}\varphi}{2}s^{2}+\overline{C}_{1}s+\overline{C}_{2}}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s+C\right]+\cos\varphi\sin\varphi\sin\left[\Bbbk s+C\right]\right),\\ &-e^{-\frac{\sin^{2}\varphi}{2}s^{2}+\overline{C}_{1}s+\overline{C}_{2}}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s+C\right]+\cos\varphi\sin\varphi\sin\left[\Bbbk s+C\right]\right)),\end{aligned} \tag{4.2}$$

$$\begin{aligned}\mathcal{A}^3\gamma &= \frac{\left(\sqrt{-\mathcal{TR}\left(\mathcal{A}^2\right)}\right)^3}{\kappa}[\frac{\mathcal{TR}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathcal{TR}\left(\mathcal{A}^2\right)}\right)^6} - \frac{\left(\mathcal{TR}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathcal{TR}\left(\mathcal{A}^2\right)}\right)^5}]^{\frac{1}{2}}(-\sin\varphi e^{-s\cos\varphi+C_1}(\sin[\mathbb{k}s+C] \\ &+\cos[\mathbb{k}s+C])e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}.(-\mathbb{k}\sin\varphi\cos[\mathbb{k}s+C]+\cos\varphi\sin\varphi\sin[\mathbb{k}s+C]) \\ &-\sin\varphi e^{-s\cos\varphi+C_1}\sin[\mathbb{k}s+C]\,e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}((\mathbb{k}\sin\varphi\sin[\mathbb{k}s+C]+\cos\varphi\sin\varphi\cos[\mathbb{k}s+C]) \\ &+(-\mathbb{k}\sin\varphi\cos[\mathbb{k}s+C]+\cos\varphi\sin\varphi\sin[\mathbb{k}s+C])), \\ &(-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2)\sin\varphi e^{-s\cos\varphi+C_1}\sin[\mathbb{k}s+C] \\ &-\cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}\left(-\mathbb{k}\sin\varphi\cos[\mathbb{k}s+C]+\cos\varphi\sin\varphi\sin[\mathbb{k}s+C]\right), \\ &-\cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}((\mathbb{k}\sin\varphi\sin[\mathbb{k}s+C]+\cos\varphi\sin\varphi\cos[\mathbb{k}s+C]) \\ &+(-\mathbb{k}\sin\varphi\cos[\mathbb{k}s+C]+\cos\varphi\sin\varphi\sin[\mathbb{k}s+C])) \\ &-\sin\varphi e^{-s\cos\varphi+C_1}\left(\sin[\mathbb{k}s+C]+\cos[\mathbb{k}s+C]\right)(-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2) \\ &-\frac{\mathcal{TR}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathcal{TR}\left(\mathcal{A}^2\right)}\right)}(-\cos\varphi,\sin\varphi e^{x^1}\left(\sin[\mathbb{k}s+C]+\cos[\mathbb{k}s+C]\right),\sin\varphi e^{x^1}\sin[\mathbb{k}s+C]).\end{aligned}$$

where C, $\overline{C}_1, \overline{C}_2$ *are constants of integration and* $\mathbb{k} = \frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}$.

Now, let us define a special form of Definition 4.1.

Lemma 4.3. *Let* $\gamma : I \longrightarrow \mathbb{P}$ *be a regular curve in the special three-dimensional* $\phi-$*Ricci symmetric para-Sasakian manifold* $\mathbb{P}$. *Then,*

$$\Theta(s) = \gamma(s) + (\Lambda - s)\frac{\mathcal{A}\gamma}{\sqrt{-\mathcal{TR}\left(\mathcal{A}^2\right)}}. \tag{4.3}$$

Proof. Using Theorem 4.2, we immediately obtain

$$\mathbf{T} = \frac{\mathcal{A}\gamma}{\sqrt{-\mathcal{TR}\left(\mathcal{A}^2\right)}}.$$

According to the definition of Involute curve we have (4.3).

Theorem 4.4. *(see [12]) Let $\gamma : I \longrightarrow \mathbb{P}$ be a non-geodesic biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then the new curvatures of this curve are*

$$\kappa = -\frac{\sqrt{\mathcal{TR}\left(\mathcal{A}^4\right)}}{\mathcal{TR}\left(\mathcal{A}^2\right)}, \tag{4.4}$$

$$\tau = \Re[\frac{-\mathcal{TR}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathcal{TR}\left(\mathcal{A}^2\right)}\right)^4\sqrt{\mathcal{TR}\left(\mathcal{A}^4\right)}} - \frac{\left(\mathcal{TR}\left(\mathcal{A}^4\right)\right)^{\frac{3}{4}}}{\left(\sqrt{-\mathcal{TR}\left(\mathcal{A}^2\right)}\right)^6}],$$

where $\Re = [\frac{\mathcal{TR}(\mathcal{A}^6)}{\left(\sqrt{-\mathcal{TR}(\mathcal{A}^2)}\right)^6} - \frac{\left(\mathcal{TR}(\mathcal{A}^4)\right)^2}{\left(\sqrt{-\mathcal{TR}(\mathcal{A}^2)}\right)^5}]^{-\frac{1}{2}}$.

Using above Lemma and Theorem we have following result:

Theorem 4.5. *Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed biharmonic curve and Θ its involute curve on $\mathbb{P}$. Then,*

$$\begin{aligned} x^1_\Theta(s) &= -\Lambda\cos\varphi + C_1, \\ x^2_\Theta(s) &= C_2 - \tfrac{\sin^3\varphi\left(\mathcal{TR}\left(\mathcal{A}^2\right)\right)^2}{\mathcal{TR}(\mathcal{A}^4)-\left(\mathcal{TR}(\mathcal{A}^2)\right)^2\sin^4\varphi} e^{-s\cos\varphi + C_1}([\Omega+\cos\varphi]\cos[\Omega s + C] \\ &\quad + [-\Omega+\cos\varphi]\sin[\Omega s + C]) + (\Lambda - s)\sin\varphi e^{x^1}\left(\sin[\Omega s + C] + \cos[\Omega s + C]\right), \\ x^3_\Theta(s) &= C_3 - \tfrac{\sin^3\varphi\left(\mathcal{TR}\left(\mathcal{A}^2\right)\right)^2}{\mathcal{TR}(\mathcal{A}^4)-\left(\mathcal{TR}(\mathcal{A}^2)\right)^2\sin^4\varphi} e^{-s\cos\varphi + C_1}(-\cos\varphi\cos[\Omega s + C] \\ &\quad + [\Omega s + C]\sin[\Omega s + C]) + (\Lambda - s)\sin\varphi e^{x^1}\sin[\Omega s + C]), \end{aligned} \tag{4.5}$$

where Λ, C, C_1, C_2, C_3 are constants of integration and $\Omega = \left[\frac{\sqrt{\mathcal{TR}(\mathcal{A}^4)-\left(\mathcal{TR}(\mathcal{A}^2)\right)^2\sin^2\varphi}}{(\mathcal{TR}(\mathcal{A}^2))\sin\varphi}\right]$

Proof. If we substitute (4.2) into (4.3), we have above system, which completes the proof.

References

[1] D. E. Blair: *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Springer-Verlag 509, Berlin-New York, 1976.

[2] R. Caddeo and S. Montaldo: *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math. 12(8) (2001), 867–876.

[3] R. Caddeo, S. Montaldo and C. Oniciuc: *Biharmonic submanifolds of* $\mathbb{S}^n$, Israel J. Math., to appear.

[4] M.P. Carmo: *Differential Geometry of Curves and Surfaces*, Pearson Education, 1976.

[5] B. Y. Chen: *Some open problems and conjectures on submanifolds of finite type,* Soochow J. Math. 17 (1991), 169–188.

[6] I. Dimitric: *Submanifolds of* $\mathbb{E}^m$ *with harmonic mean curvature vector,* Bull. Inst. Math. Acad. Sinica 20 (1992), 53–65.

[7] J. Eells and L. Lemaire: *A report on harmonic maps,* Bull. London Math. Soc. 10 (1978), 1–68.

[8] J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109–160.

[9] T. Hasanis and T. Vlachos: *Hypersurfaces in* $\mathbb{E}^4$ *with harmonic mean curvature vector field,* Math. Nachr. 172 (1995), 145–169.

[10] G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130–144.

[11] G. Y. Jiang: *2-harmonic maps and their first and second variational formulas,* Chinese Ann. Math. Ser. A 7(4) (1986), 389–402.

[12] T. Körpınar, E. Turhan and V. Asil: *Frenet equations of Biharmonic curves in terms of exponential maps in the special three-dimensional* ϕ*- Ricci symmetric para-Sasakian manifold* P, (submitted).

[13] E. Loubeau and C. Oniciuc: *On the biharmonic and harmonic indices of the Hopf map*, Transactions of the American Mathematical Society 359 (11) (2007), 5239–5256.

[14] H. Matsuda, S. Yorozu: *Notes on Bertrand curves*, Yokohama Math. J. 50 (1-2), (2003), 41-58.

[15] D. J. Struik: *Differential geometry*, Second ed., Addison-Wesley, Reading, Massachusetts, 1961.

[16] E. Turhan and T. Körpınar: *Characterize on the Heisenberg Group with left invariant Lorentzian metric*, Demonstratio Mathematica 42 (2) (2009), 423-428.

[17] E. Turhan, T. Körpınar, *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis*^{3}, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

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