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Weierstrass representation formula in the group of rigid motions $E(2)$

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Abstract

In this paper, we prove a Weierstrass representation formula for simply connected immersed maximal surfaces in $E(2)$. Using the Weierstrass representation we also give a simple proof of the fact that maximal immersions is harmonic maps on the domain.

key words. Weierstrass representation, rigid motions, harmonic map.

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1 Introduction

Analytic methods to study surfaces and their properties are of great interest both in mathematics and in physics. A classical example of such an approach is given by the Weierstrass representation for minimal surfaces [7]. This representation allows us to construct any minimal surface in the three-dimensional Euclidean space $\mathbb{R}^3$ via two holomorphic functions. It is the most powerful tool for the analysis of minimal surfaces.

Weierstrass representations are very useful and suitable tools for the systematic study of minimal surfaces immersed in n-dimensional spaces [12]. This subject has a long and rich history. It has been extensively investigated since the initial works of Weierstrass [19]. In the literature there exists a great number of applications of the Weierstrass representation to various domains of Mathematics, Physics, Chemistry and Biology. In particular in such areas as quantum field theory [8], statistical physics [14], chemical physics, fluid dynamics and membranes [16], minimal surfaces play an essential role. More recently it is worth mentioning that works by Kenmotsu [10], Hoffmann [9], Osserman [15], Budinich [5], Konopelchenko [6,11] and Bobenko [3, 4] have made very significant contributions to constructing minimal surfaces in a systematic way and to understanding their intrinsic geometric properties as well as their integrable dynamics. The type of extension of the Weierstrass representation which has been useful in three-dimensional applications to multidimensional spaces will continue to generate many additional applications to physics and mathematics. According to [13] integrable deformations of surfaces are generated by the Davey–Stewartson hierarchy of 2+1 dimensional soliton equations. These deformations of surfaces inherit all the remarkable properties of soliton equations. Geometrically such deformations are characterised by the invariance of an infinite set of functionals over surfaces, the simplest being the Willmore functional.

D. A. Berdinski and I. A. Taimanov gave a representation formula for minimal surfaces in 3-dimensional Lie groups in terms of spinors and Dirac operators [1].

In this paper, we prove a Weierstrass representation formula for simply connected immersed maximal surfaces in $E(2)$. Using the Weierstrass representation we also give a simple proof of the fact that maximal immersions is harmonic maps on the domain. Furthermore, we show that any harmonic map of a simply connected coordinate region into $E(2)$ can be represented a form.

2 The Group of Rigid Motions $E(2)$

Let $E(2)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cos x_1 & -\sin x^1 & x_2 \\ \sin x_1 & \cos x_1 & x_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $E(2)$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$ under the map

$$E(2) \longrightarrow \mathbb{S}^1 \times \mathbb{R}^2 : \begin{pmatrix} \cos[x_1] & -\sin[x_1] & x_2 \\ \sin[x_1] & \cos[x_1] & x_3 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow ([x_1], x_2, x_3),$$

where $[x_1]$ means x modulo $2\pi z$. It's Lie algebra has a basis consisting of

$$\mathbf{e}_1 = \frac{\partial}{\partial x_1},\ \mathbf{e}_2 = \cos x_1 \frac{\partial}{\partial x_2} + \sin x_1 \frac{\partial}{\partial x_3},\ \mathbf{e}_3 = -\sin x_1 \frac{\partial}{\partial x_2} + \cos x_1 \frac{\partial}{\partial x_3}, \tag{2.1}$$

and coframe

$$\theta^1 = dx_1,\ \theta^2 = \cos x_1 dx_2 + \sin x_1 dx_3,\ \theta^3 = -\sin x_1 dx_2 + \cos x_1 dx_3.$$

It is easy to check that the metric g is given by

$$g = \left(\theta^1\right)^2 + \left(\theta^2\right)^2 - \left(\theta^3\right)^2. \tag{2.2}$$

The bracket relations are

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \ \ [\mathbf{e}_2, \mathbf{e}_3] = 0, \ \ [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{e}_3 & \mathbf{0} & -\mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & \mathbf{0} \end{pmatrix}, \tag{2.3}$$

where the (i,j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Then, we write the Kozul formula for the Levi–Civita connection is:

$$2g(\nabla_{\mathbf{e}_i} \mathbf{e}_j, \mathbf{e}_k) = L^k_{ij}.$$

From (2.3), we get

$$L^3_{21} = 2, \ L^1_{23} = -2, \ L^3_{12} = 2, \ L^2_{31} = 2, \ L^1_{32} = -2 \tag{2.4}$$

3 Weierstrass Representation Formula in $E(2)$

$\Sigma \subset E(2)$ be a spacelike surface and $\wp : \Sigma \longrightarrow E(2)$ a smooth map. The pull-back bundle $\wp^*(TE(2))$ has a metric and compatible connection, the pull-back connection, induced by he Riemannian metric and the Levi–Civita connection of $E(2)$. Consider the complexified bundle $\mathbb{E} = \wp^*(TE(2)) \otimes \mathbb{C}$.

Let (u, v) be local coordinates on Σ, and $z = u + iv$ the (local) complex parameter and set, as usual,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{3.1}$$

Let

$$\frac{\partial \wp}{\partial u} \mid_p = \wp_{*p} \left(\frac{\partial}{\partial u} \mid_p \right), \quad \frac{\partial \wp}{\partial v} \mid_p = \wp_{*p} \left(\frac{\partial}{\partial v} \mid_p \right), \tag{3.2}$$

and

$$\phi = \wp_z = \frac{\partial \wp}{\partial z} = \frac{1}{2} \left(\frac{\partial \wp}{\partial u} - i \frac{\partial \wp}{\partial v} \right). \tag{3.3}$$

Let now $\wp : \Sigma \longrightarrow E(2)$ be a conformal immersion and $z = u+iv$ a local conformal parameter. Then, the induced metric is

$$ds^2 = \lambda^2 (du^2 - dv^2) = \lambda^2 |dz|^2, \tag{3.4}$$

and the Beltrami–Laplace operator on $E(2)$, with respect to the induced metric, is given by

$$\Delta = \lambda^{-2} (\frac{\partial}{\partial u} \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \frac{\partial}{\partial v}). \tag{3.5}$$

We recall that a map $\wp : \Sigma \longrightarrow E(2)$ is harmonic if its tension field

$$\tau(\wp) = \texttt{trace} \nabla d\wp = 0. \tag{3.6}$$

Let $\{x_1, x_2, x_3\}$ be a system of local coordinates in a neighborhood U of M such that $U \cap \wp(\Sigma) \neq \emptyset$. Then, in an open set $G \subset \Sigma$

$$\phi = \sum_{j=1}^{3} \phi_j \frac{\partial}{\partial x_j}, \tag{3.7}$$

for some complex-valued functions ϕ_j defined on G. With respect to the local decomposition of ϕ, the tension field can be written as

$$\tau(\zeta) = \sum_i \left\{ \Delta \wp_i + 4\lambda^{-2} \sum_{j,k=1}^{n} \Gamma^i_{jk} \frac{\partial \wp_j}{\partial \overline{z}} \frac{\partial \wp_k}{\partial z} \right\} \frac{\partial}{\partial x_i}, \tag{3.8}$$

where Γ^i_{jk} are the Christoffel symbols of $E(2)$.

From (3.3), we have

$$\tau(\wp) = 4\lambda^{-2} \sum_i \left\{ \frac{\partial \phi_i}{\partial \overline{z}} + \sum_{j,k=1}^{n} \Gamma^i_{jk} \overline{\phi}_j \phi_k \right\} \frac{\partial}{\partial x_i}.$$

The section ϕ is holomorphic if and only if

$$\nabla_{\frac{\partial}{\partial \overline{z}}}\left(\sum_{i=1}^{3}\phi_i\frac{\partial}{\partial x_i}\right)=\sum_i\left\{\frac{\partial \phi_i}{\partial \overline{z}}\frac{\partial}{\partial x_i}+\phi_i\nabla_{\frac{\partial \zeta}{\partial \overline{z}}}\frac{\partial}{\partial x_i}\right\}.$$

Using (3.3), we get

$$\nabla_{\frac{\partial}{\partial \overline{z}}}\left(\sum_{i=1}^{3}\phi_i\frac{\partial}{\partial x_i}\right)=\sum_i\left\{\frac{\partial \phi_i}{\partial \overline{z}}\frac{\partial}{\partial x_i}+\phi_i\nabla_{\sum_j\overline{\phi}_j\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i}\right\}.$$

Making necessary calculations, we obtain

$$\nabla_{\frac{\partial}{\partial \overline{z}}}\left(\sum_{i=1}^{3}\phi_i\frac{\partial}{\partial x_i}\right)=\sum_i\left\{\frac{\partial \phi_i}{\partial \overline{z}}+\sum_{j,k}\Gamma^i_{jk}\overline{\phi}_j\phi_k\right\}\frac{\partial}{\partial x_i}=0.$$

Thus, ϕ is holomorphic if and only if

$$\frac{\partial \phi_i}{\partial \overline{z}}+\sum_{j,k}\Gamma^i_{jk}\overline{\phi}_j\phi_k=0,\quad i=1,2,3. \tag{3.9}$$

Theorem 3.1. *(Weierstrass representation) Let $E(2)$ be the group of rigid motions of Euclidean 2-space and $\{x_1,x_2,x_3\}$ local coordinates. Let ϕ_j, $j=1,2,3$ be complex-valued functions in an open simply connected domain $G\subset\mathbb{C}$ which are solutions of (3.9). Then, the map*

$$\wp_j(u,v)=2Re\left(\int_{z_0}^{z}\phi_j dz\right) \tag{3.10}$$

is well defined and defines a maximal conformal immersion if and only if the following conditions are satisfied :

$$\sum_{j,k=1}^{3}g_{ij}\phi_j\overline{\phi_k}\neq 0 \text{ and } \sum_{j,k=1}^{3}g_{ij}\phi_j\phi_k=0\ .$$

Let us expand Υ with respect to this basis to obtain

$$\Upsilon=\sum_{k=1}^{3}\psi_k\mathbf{e}_k\ . \tag{3.11}$$

Setting

$$\phi=\sum_i\phi_i\frac{\partial}{\partial x_i}=\sum_i\psi_i e_i, \tag{3.12}$$

for some complex functions $\phi_i, \psi_i : G \subset \mathbb{C}$. Moreover, there exists an invertible matrix $A = (A_{ij})$, with function entries $A_{ij} : \wp(G) \cap U \to \mathbb{R}$, $i, j = 1, 2, 3$, such that

$$\phi_i = \sum_j A_{ij}\psi_j. \tag{3.13}$$

Using the expression of ϕ, the section ϕ is holomorphic if and only if

$$\frac{\partial \psi_i}{\partial \overline{z}} + \frac{1}{2}\sum_{j,k} L^i_{jk}\overline{\psi}_j\psi_k = 0, \quad i = 1, 2, 3. \tag{3.14}$$

Theorem 3.2. *Let ψ_j, $j = 1, 2, 3$, be complex-valued functions defined in a open simply connected set $G \subset \mathbb{C}$, such that the following conditions are satisfied :*

i. $|\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 \neq 0$,

ii. $\psi_1^2 + \psi_2^2 - \psi_3^2 = 0$,

iii. ψ_j *are solutions of (3.21).*

Then, the map $\wp : G \to E(2)$ defined by

$$\wp_i(u, v) = 2Re\left(\int_{z_0}^{z} \sum_j A_{ij}\psi_j dz\right) \tag{3.15}$$

is a conformal maximal immersion.

Proof. By theorem 3.1 we see that $\wp$ is a harmonic map if and only if $\wp$ satisfy (3.15). Then, the map $\wp$ is a conformal maximal immersion.

Since the parameter z is conformal, we have

$$\langle \Upsilon, \Upsilon \rangle = 0, \tag{3.16}$$

which is rewritten as

$$\psi_1^2 + \psi_2^2 - \psi_2^2 = 0. \tag{3.17}$$

Case I

From (3.17), we have

$$(\psi_1 - i\psi_2)(\psi_1 + i\psi_2) = \psi_3^2, \tag{3.18}$$

which suggests the definition of two new complex functions

$$\Omega := \sqrt{\frac{1}{2}(\psi_1 - i\psi_2)}, \quad \Phi := \sqrt{\frac{1}{2}(\psi_1 + i\psi_2)}. \tag{3.19}$$

The functions Ω and Φ are single-valued complex functions which, for suitably chosen square roots, satisfy

$$\begin{aligned} \psi_1 &= \Omega^2 + \Phi^2, \\ \psi_2 &= i\left(\Omega^2 - \Phi^2\right), \\ \psi_3 &= 2\Omega\Phi. \end{aligned} \tag{3.20}$$

Lemma 3.3. *If Υ satisfies the equation (3.14), then*

$$\Omega\Omega_{\bar{z}} - \Phi\Phi_{\bar{z}} = -i\left(\bar{\Omega}\Phi - \Omega\bar{\Phi}\right)(|\Omega| - |\Phi|), \tag{3.21}$$

$$\Omega\Omega_{\bar{z}} + \Phi\Phi_{\bar{z}} = |\Phi|\,\bar{\Omega}\Phi - |\Omega|\,\Omega\bar{\Phi}, \tag{3.22}$$

$$\Omega_{\bar{z}}\Phi + \Omega\Phi_{\bar{z}} = \frac{i}{2}\left(|\Omega|^2 + \bar{\Omega}^2\Phi^2 - \Omega^2\bar{\Phi}^2 - |\Phi|^2\right). \tag{3.23}$$

Proof. Using (2.4) and (3.14), we have

$$\begin{aligned} \frac{\partial\psi_1}{\partial\bar{z}} &= \left(\overline{\psi}_2\psi_3 + \overline{\psi}_3\psi_2\right), \\ \frac{\partial\psi_2}{\partial\bar{z}} &= -\overline{\psi}_3\psi_2, \\ \frac{\partial\psi_3}{\partial\bar{z}} &= -\overline{\psi}_2\psi_1. \end{aligned} \tag{3.24}$$

Substituting (3.20) into (3.24), we have (3.21)-(3.23).

Corollary 3.4.

$$\Omega\Omega_{\bar{z}} = -\frac{i}{2}\left(\bar{\Omega}\Phi - \Omega\bar{\Phi}\right)\left(|\Omega| - |\Phi|\right) + \frac{1}{2}\left(|\Phi|\,\bar{\Omega}\Phi - |\Omega|\,\Omega\bar{\Phi}\right) \tag{3.25}$$

Corollary 3.5.

$$\Phi\Phi_{\bar{z}} = \frac{i}{2}\left(\bar{\Omega}\Phi - \Omega\bar{\Phi}\right)\left(|\Omega| - |\Phi|\right) + \frac{1}{2}\left(|\Phi|\,\bar{\Omega}\Phi - |\Omega|\,\Omega\bar{\Phi}\right). \tag{3.26}$$

Theorem 3.6. *Let Ω and Φ be complex-valued functions defined in a simply connected domain $G \subset \mathbb{C}$. Then the map $\wp : G \to E(2)$, defined by*

$$\wp_1(u,v) = Re\left(\int_{z_0}^{z}\left[\Omega^2 + \Phi^2\right]dz\right),$$

$$\wp_2(u,v) = Re\left(\int_{z_0}^{z}\left(i\left(\Omega^2 - \Phi^2\right)\cos x_1 - 2\Omega\Phi\sin x_1\right)dz\right), \tag{3.27}$$

$$\wp_3(u,v) = Re\left(\int_{z_0}^{z}(i\left(\Omega^2 - \Phi^2\right)\sin x_1 + 2\Omega\Phi\cos x_1)dz\right),$$

is a conformal maximal immersion.

Proof. Using (3.12), we get

$$\phi_1 = \psi_1,\ \phi_2 = \psi_2\cos x_1 - \psi_3\sin x_1,\ \phi_3 = \psi_2\sin x_1 + \psi_3\cos x_1. \tag{3.28}$$

From (3.10) we have the system (3.27). Using Theorem 3.2 $\wp : G \to E(2)$ is a conformal maximal immersion.

Case II

From (3.17), we have

$$\psi_1 = \Re\cos\Im,\ \psi_2 = \Re\sin\Im,\ \psi_3 = \Re, \tag{3.29}$$

which suggests the definition of two new complex functions

$$\Im = \arctan\frac{\psi_2}{\psi_1} \text{ and } \Re = \psi_1^2 + \psi_2^2. \tag{3.30}$$

Lemma 3.7. *If Υ satisfies the equation (3.14), then*

$$\Re_{\bar{z}} \cos \Im - \Im_{\bar{\bar{z}}} \Re \sin \Im = |\Re| \, \overline{\sin \Im} + |\Re| \sin \Im, \tag{3.31}$$

$$\Re_{\bar{z}} \sin \Im + \Im_{\bar{z}} \Re \cos \Im = - |\Re| \sin \Im, \tag{3.32}$$

$$\Re_{\bar{z}} = - |\Re| \, \overline{\sin \Im} \cos \Im. \tag{3.33}$$

Proof. Using (2.4) and (3.14), we have

$$\begin{aligned} \frac{\partial \psi_1}{\partial \bar{z}} &= \left(\overline{\psi}_2 \psi_3 + \overline{\psi}_3 \psi_2 \right), \\ \frac{\partial \psi_2}{\partial \bar{z}} &= -\overline{\psi}_3 \psi_2, \\ \frac{\partial \psi_3}{\partial \bar{z}} &= -\overline{\psi}_2 \psi_1. \end{aligned} \tag{3.34}$$

Substituting (3.29) into (3.34), we have (3.23)-(3.25).

Corollary 3.8.

$$- |\Re| \cos^2 \Im \overline{\sin \Im} - \Im_{\bar{z}} \Re \sin \Im = |\Re| \left(\overline{\sin \Im} + \sin \Im \right), \tag{3.35}$$

$$- |\Re| \, |\sin \Im| \cos \Im + \Im_{\bar{z}} \Re \cos \Im = - |\Re| \sin \Im. \tag{3.36}$$

Corollary 3.9.

$$\cos^3 \Im \overline{\sin \Im} + |\sin \Im| \cos \Im \sin \Im = - \left(\left(\overline{\sin \Im} + \sin \Im \right) \cos \Im - \sin^2 \Im \right). \tag{3.37}$$

Theorem 3.10. *Let $\Re$ and $\Im$ be complex-valued functions defined in a simply connected domain $G \subset \mathbb{C}$. Then the map $\wp : G \to E(2)$, defined by*

$$\begin{aligned} \wp_1(u,v) &= Re \left(\left(\Re \cos \Im \right) dz \right), \\ \wp_2(u,v) &= Re \left(\int_{z_0}^{z} \left(\Re \sin \Im \cos x_1 - \Re \sin x_1 \right) dz \right), \end{aligned} \tag{3.38}$$

$$\wp_3(u,v) = Re\left(\int_{z_0}^{z} (\Re \sin \Im \sin x_1 + \Re \cos x_1) dz\right),$$

is a conformal minimal immersion.

Proof. Using (3.10) and (3.32), we get

$$\phi_1 = \Re \cos \Im,\ \phi_2 = \Re \sin \Im \cos x_1 - \Re \sin x_1,\ \phi_3 = \Re \sin \Im \sin x_1 + \Re \cos x_1.$$

Using Theorem 3.2 $\wp : G \to E(2)$ is a conformal maximal immersion.

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