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A new proof for the Euler theorem in the complex numbers theory

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Abstract

In the paper, a new proof for the Euler equation ($exp(ix) = \cos x + i \sin x$) has been presented. At first, a new and general formula has been proved from which the Euler equation has been derived.

key words. Euler theorem, Complex numbers, Analytic function

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1 Introduction

Euler equation in the theory of the complex numbers is usually proved by expansion of $\sin(x), \cos(x)$ and $exp(x)$ into power series. A general proof of this equation based on direct mathematical analysis does not exist. In this paper, at first a new formula has been proved from which the Euler equation has been derived as a special result.

2 Analysis

Let f be an analytic function with the following characteristics

$$f(z) = u(x, y) + iv(x, y),\ f(z) \neq \pm iz_0,\ z_0 = a + ib \neq 0,\ z = x + iy,\ i = \sqrt{-1} \tag{1}$$

Since f is an analytic function [1].

$$\frac{\partial u}{\partial v} = \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

U and V are defined as follows

$$U(\phi,\varphi)=\frac{\phi}{\phi^2+\varphi^2},\ V(\phi,\varphi)=\frac{\varphi}{\phi^2+\varphi^2},\ \phi=\phi(x,y),\ \varphi=\varphi(x,y)\Rightarrow$$

$$\frac{\partial U}{\partial x}=\frac{(\varphi^2-\phi^2)\dfrac{\partial \phi}{\partial x}-2\phi\varphi\dfrac{\partial \varphi}{\partial x}}{(\phi^2+\varphi^2)^2},\ \frac{\partial U}{\partial y}=\frac{(\varphi^2-\phi^2)\dfrac{\partial \phi}{\partial y}-2\phi\varphi\dfrac{\partial \varphi}{\partial y}}{(\phi^2+\varphi^2)^2} \tag{3}$$

$$\frac{\partial V}{\partial x} = \frac{(\varphi^2-\phi^2)\dfrac{\partial \phi}{\partial x}-2\phi\varphi\dfrac{\partial \varphi}{\partial x}}{(\phi^2+\varphi^2)^2},\ \frac{\partial V}{\partial y}=\frac{(\varphi^2-\phi^2)\dfrac{\partial \phi}{\partial y}-2\phi\varphi\dfrac{\partial \varphi}{\partial y}}{(\phi^2+\varphi^2)^2}$$

Let define g as

$g(z)=\dfrac{1}{f^2(z)+{z_0}^2}=\dfrac{1}{\phi_1+i\varphi_1}=U(\phi_1,\varphi_1)+iV(\phi_1,\varphi_1), \phi_1=u^2-v^2+a^2-b^2,\ \varphi_1=2(uv+ab)$

Using Eq. 2.

$$\frac{\partial \phi_1}{\partial x}=2u\frac{\partial u}{\partial x}-2v\frac{\partial v}{\partial x},\ \frac{\partial \phi_1}{\partial y}=-2u\frac{\partial v}{\partial x}-2v\frac{\partial u}{\partial x},\ \frac{\partial \phi_1}{\partial x}=2u\frac{\partial v}{\partial x}+2v\frac{\partial u}{\partial x},\ \frac{\partial \phi_1}{\partial y}=2u\frac{\partial u}{\partial x}-2v\frac{\partial v}{\partial x} \tag{4}$$

From Eqs. 3 and 4.

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}=2\frac{({\varphi_1}^2u-{\phi_1}^2u-2\phi_1\varphi_1v)\dfrac{\partial u}{\partial x}+(-{\varphi_1}^2v+{\phi_1}^2v-2\phi_1\varphi_1u)\dfrac{\partial v}{\partial x}}{({\phi_1}^2+{\varphi_1}^2)^2}$$

$$\frac{\partial U}{\partial y}=-\frac{\partial U}{\partial y}=2\frac{-({\varphi_1}^2u-{\phi_1}^2u-2\phi_1\varphi_1v)\dfrac{\partial v}{\partial x}+(-{\varphi_1}^2v+{\phi_1}^2v-2\phi_1\varphi_1u)\dfrac{\partial u}{\partial x}}{({\phi_1}^2+{\varphi_1}^2)^2}$$

Therefore, g is an analytic function. Let define h as follows

$h(z)=\dfrac{1}{f(z)+iz_0}=\dfrac{1}{\phi_2+i\varphi_2}=U(\phi_2,\varphi_2)+iV(\phi_2,\varphi_2), \phi_2=u-b,\ \varphi_2=u+a$

Using Eq. 2

$$\frac{\partial \phi_2}{\partial x}=\frac{\partial u}{\partial x},\ \frac{\partial \phi_2}{\partial y}=-\frac{\partial v}{\partial x},\ \frac{\partial \phi_2}{\partial x}=\frac{\partial v}{\partial x},\ \frac{\partial \phi_2}{\partial y}=\frac{\partial u}{\partial x} \tag{5}$$

From Eqs. 3 and 5

$$\frac{\partial U}{\partial x}, \frac{\partial V}{\partial y} = \frac{({\varphi_2}^2 - {\phi_2}^2)\frac{\partial u}{\partial x} - 2\phi_2\varphi_2\frac{\partial v}{\partial x}}{({\phi_2}^2 + {\varphi_2}^2)^2}, \ \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} = \frac{-({\varphi_2}^2 - {\phi_2}^2)\frac{\partial v}{\partial x} - 2\phi_2\varphi_2\frac{\partial u}{\partial x}}{({\phi_2}^2 + {\varphi_2}^2)^2}$$

Therefore, h is an analytic function. Let define s as

$$s(z) = \frac{1}{f(z) + iz_0} = \frac{1}{\phi_3 + i\varphi_3} = U(\phi_3, \varphi_3) + iV(\phi_3, \varphi_3), \phi_3 = u + b, \ \varphi_3 = v - a$$

Like the procedure was used for $h(z)$, it can be shown similarly that $s(z)$ is also an analytic function. Since $f(z)$ is an analytic function, for any continuous curve C from z_0 to z [1]

$$\int_C f(z)dz = \int_{z_0}^{z} f(z)dz = F(z) - F(z_0) = F(z) + c_0, \ F'(z) = f(z)$$

$$\begin{aligned} g(z)f'(z) = h(z)s(z)f'(z) &= \frac{1}{2iz_0}(s(z) - h(z))f'(z) \Rightarrow \int_C \frac{f'(z)dz}{f^2(z) + z_0^2} \\ &= \frac{1}{2iz_0}\int_C \left(\frac{f'(z)}{f(z) - iz_0} - \frac{f'(z)}{f(z) + iz_0}\right) dz + c_0 \\ \Rightarrow \frac{1}{z_0}\tan^{-1}\frac{f(z)}{z_0} + c_0 &= \frac{1}{2iz_0}\ln\frac{f(z) - iz_0}{f(z) + iz_0} = e^{2i\tan^{-1}\frac{f(z)}{z_0} + 2ic_0z_0}, \ \text{for} \ f(z) = 0 \Rightarrow -1 = e^{2ic_0z_0} \end{aligned}$$

$$\frac{f(z) - iz_0}{f(z) + iz_0} = -e^{2i\tan^{-1}\frac{f(z)}{z_0}}, \ f(z) \neq \pm iz_0 \tag{6}$$

The function $f(z)$ can be defined as

$$f(z) = z_0\tan(p(z)/2) \Rightarrow \frac{f(z) - iz_0}{f(z) + iz_0} = -\cos p(z) - i\sin p(z) \ \text{and} \ -e^{2i\tan^{-1}\frac{f(z)}{z_0}} = -e^{ip(z)} \Rightarrow$$

$$e^{ip(z)} = \cos p(z) + i\sin p(z) \tag{7}$$

References

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