

Interior Controllability of a Strongly Damped Wave Equation

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Abstract

In this paper we prove the interior controllability of the strongly damped wave equation with Dirichlet boundary conditions

$$\begin{cases} w_{tt} + \eta(-\Delta)^{1/2}w_t + \gamma(-\Delta)w = 1_\omega u(t, x), & \text{in } (0, \tau) \times \Omega, \\ w = 0, & \text{in } (0, \tau) \times \partial\Omega, \\ w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x), & \text{in } \Omega, \end{cases}$$

in the space $Z_{1/2} = D((-\Delta)^{1/2}) \times L^2(\Omega)$, where Ω is a bounded domain in \mathbb{R}^n , ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control $u \in L^2(0, \tau; L^2(\Omega))$ and η, γ are positive numbers. We shall prove that for all $\tau > 0$ and any nonempty open subset ω of Ω the system is approximately controllable on $[0, \tau]$. Moreover, we exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time.

key words. Strongly damped wave equation, interior approximate controllability, strongly continuous semigroups

AMS(MOS) subject classifications. 93B05, 93C25.

1 Introduction

The interior approximate controllability is a well known, fascinating and important subject in systems theory; there are some important works in this area, including [17], [18], [19], [20] and [21]. In particular, Zuazua in [21], proves the interior approximate controllability of the heat equation

$$\begin{cases} z_t = \Delta z + 1_\omega u(t, x), & \text{in } (0, \tau) \times \Omega, \\ z = 0, & \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) = z_0(x), & \text{in } \Omega, \end{cases} \quad (1.1)$$

in two different ways. In the first one, he uses the Hahn-Banach theorem, integration by parts, the adjoint equation, the Carleman estimates, and the Holmgren uniqueness theorem([14]). The

second method is constructive and uses a variational technique: fix the control time $\tau > 0$, the initial and final state $z_0 = 0$, $z_1 \in L^2(\Omega)$ respectively and $\epsilon > 0$; the control steering the initial state z_0 to a ball of radius $\epsilon > 0$ and center z_1 is given by the point in which the following functional achieves its minimum value

$$J_\epsilon(\varphi_\tau) = \frac{1}{2} \int_0^\tau \int_\omega \varphi^2 dx dt + \epsilon \|\varphi_\tau\|_{L^2(\Omega)} - \int_\Omega z_1 \varphi_\tau,$$

where φ is the solution of the corresponding adjoint equation with initial data φ_τ .

In [11], S. Chen and R. Triggiani proved that the uncontrolled dynamics (corresponding to $u = 0$) generates an analytic semigroup in a very general setting, in [2] Avalos and Lasiecka studied the null controllability for the structurally damped abstract wave equation given by:

$$\begin{cases} w_{tt} + \rho A^\alpha w_t + Aw = u(t), & t \in (0, \tau) \\ w(0) = w_0, \quad w_t(0) = w_1, \end{cases}$$

where, $0 \leq \alpha < 1$, $[w_0, w_1] \in D(A^{\frac{1}{2}}) \times H$, H is a Hilbert space, $u \in L^2(0, \tau; H)$ (the control acts on the whole set Ω if $H = L^2(\Omega)$) and $A : D(A) \subset H \rightarrow H$ is a strictly positive, self-adjoint operator. Also, in [3], G. Avalos and P. Cokely studied the boundary local null controllability of structural damped elastic systems of the form:

$$\begin{cases} w_{tt} - \rho \Delta w_t + \Delta^2 w = 0, & \text{in } (0, \tau) \times \Omega, \\ w = u_1, & \text{on } (0, \tau) \times \partial\Omega, \\ \Delta w = u_2, & \text{on } (0, \tau) \times \partial\Omega, \\ w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x), & \text{in } \Omega, \end{cases}$$

Another works concerning with structurally damped elastic systems are the papers [6] and [7] by A.N. Carvalho and J.W. Cholewa, where the authors studied the existence of an attractor for the following strongly damped wave equation with critical nonlinearities

$$\begin{cases} w_{tt} + (-\Delta)^\alpha w_t = \Delta w + f(w), & \text{in } (0, \infty) \times \Omega, \\ w = 0, & \text{on } (0, \tau) \times \partial\Omega, \\ [w(0), w_t(0)] = [w_0, w_1] \in H_0^1(\Omega) \times L^2(\Omega), \end{cases}$$

where $\alpha \in [\frac{1}{2}, 1]$.

A new technique is used in [15] to prove the interior approximate controllability of the following broad class of reaction diffusion equations in the Hilbert space $Z = L^2(\Omega)$ given by

$$z' = -Az + 1_\omega u(t), \quad t \in [0, \tau],$$

where Ω is a domain in \mathbb{R}^n , ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control $u \in L^2(0, \tau; L^2(\Omega))$ and $A : D(A) \subset Z \rightarrow Z$ is an

unbounded linear operator with the spectral decomposition: $Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}$. The eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \dots \lambda_n \rightarrow \infty$ of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors of A . Then, the controllability of the n D heat equation (1.1) follows trivially from this result by putting $A = -\Delta$.

Following [15], in this paper we study the interior approximate controllability of the strongly damped wave equation

$$\begin{cases} w_{tt} + \eta(-\Delta)^{1/2}w_t + \gamma(-\Delta)w = 1_{\omega}u(t, x), & \text{in } (0, \tau) \times \Omega, \\ w = 0, & \text{in } (0, \tau) \times \partial\Omega, \\ w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x), & \text{in } \Omega, \end{cases} \quad (1.2)$$

in the space $Z_{1/2} = D((-\Delta)^{1/2}) \times L^2(\Omega)$, where Ω is a bounded domain in \mathbb{R}^n , ω is an open nonempty subset of Ω , 1_{ω} denotes the characteristic function of the set ω and the distributed control $u \in L^2(0, \tau; L^2(\Omega))$.

The controllability of similar systems, with the controls acting on the whole set Ω was studied in [1]. Here we first prove the approximate controllability of the following strongly damped wave equation with the controls acting on Ω

$$\begin{cases} w_{tt} + \eta(-\Delta)^{1/2}w_t + \gamma(-\Delta)w = u(t, x), & \text{in } (0, \tau) \times \Omega, \\ w = 0, & \text{in } (0, \tau) \times \partial\Omega, \\ w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x), & \text{in } \Omega. \end{cases} \quad (1.3)$$

Then we focus on proving the interior approximate controllability of the system (1.2). This is an important problem from both, theoretical and applications point of view, since the control is acting only on a subset of Ω .

Also, we note that, for finite dimensional linear systems, all the concepts of controllability are equivalent (exact controllability, approximate controllability and null controllability); however, for infinite dimensional systems these concepts are completely different. Actually, there exist a broad class of infinite dimensional systems that are approximately controllable, but never exactly controllable (see [8]); namely, those systems associated with compact semigroups, particularly, the heat equation and some diffusive processes.

In this paper we shall prove that, for all $\tau > 0$ and an open nonempty subset ω of Ω , the system (1.2) is approximately controllable on $[0, \tau]$. Moreover, we can exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time (see Theorem 4.1).

This technique is based in the following results:

Theorem 1.1 [9], [15] *The eigenfunctions of $-\Delta$ with Dirichlet boundary condition on Ω are real analytic functions.*

Theorem 1.2 [4] *Suppose $\Omega \subset \mathbb{R}^n$ is an open, non-empty and connected set, and f is a real analytic function in Ω with $f = 0$ on a non-empty open subset ω of Ω . Then, $f = 0$ in Ω .*

2 Formulation of the Problem

In this section we shall choose the space where this problem will be set up as an abstract control system in a Hilbert space. Let $X = L^2(\Omega) = L^2(\Omega, \mathbb{R})$ and consider the linear unbounded operator $A : D(A) \subset X \rightarrow X$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H^2(\Omega, \mathbb{R}) \cap H_0^1(\Omega, \mathbb{R}). \quad (2.4)$$

Then, the eigenvalues λ_j of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspaces and $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \rightarrow \infty$. Moreover,

a) there exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenfunctions of A

b) for all $x \in D(A)$ we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle x, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j x, \quad (2.5)$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in L^2 and

$$E_j x = \sum_{k=1}^{\gamma_j} \langle x, \phi_{j,k} \rangle \phi_{j,k}, \quad (2.6)$$

which means the set $\{E_j\}_{j=1}^{\infty}$ is a complete family of orthogonal projections in X and $x = \sum_{j=1}^{\infty} E_j x$, $x \in X$,

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At} x = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j x. \quad (2.7)$$

d) The fractional powered spaces X^r are given by:

$$X^r = D(A^r) = \{x \in X : \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 < \infty\}, \quad r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r, \quad \text{and}$$

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \quad (2.8)$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert space endowed with the norm given by:

$$\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|w\|_r^2 + \|v\|^2.$$

Proposition 2.1 *The operator $P_j : Z_r \rightarrow Z_r$, $j \geq 0$, defined by*

$$P_j = \begin{bmatrix} E_j & 0 \\ 0 & E_j \end{bmatrix}, \quad j \geq 1, \quad (2.9)$$

is a continuous(bounded) orthogonal projections in the Hilbert space Z_r .

Proof. First we shall show that $P_j(Z_r) \subset Z_r$, which is equivalent to show that $E_j(X^r) \subset X^r$. In fact, let x be in X^r and consider $E_j x$. Then

$$\sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n E_j x\|^2 = \lambda_j^{2r} \|E_j x\|^2 < \infty$$

Therefore, $E_j x \in X^r, \forall x \in X^r$.

Now, we shall prove that this projection is bounded. In fact, from the continuous inclusion $X^r \subset X$, there exists a constant $k > 0$ such that

$$\|x\| \leq k \|x\|_r, \quad \forall x \in X^r.$$

Then, for all $x \in X^r$ we have the following estimate

$$\begin{aligned} \|E_j x\|_r^2 &= \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n E_j x\|^2 = \lambda_j^{2r} \|E_j x\|^2 \\ &\leq \lambda_j^{2r} \|x\|^2 \leq \lambda_j^{2r} k^2 \|x\|_r^2 \end{aligned}$$

Hence $\|E_j x\| \leq \lambda_j^r k \|x\|_r$, which implies the continuity of $E_j : X^r \rightarrow X^r$. So, P_j is a continuous projection on Z_r .

□

Hence, the equations (1.2) and (1.3) can be written respectively as an abstract second order ordinary differential equations in X as follows

$$w'' + \eta A^{1/2} w' + \gamma A w = 1_\omega u, \quad t \in (0, \tau]. \quad (2.10)$$

$$w'' + \eta A^{1/2} w' + \gamma A w = u, \quad t \in (0, \tau] \quad (2.11)$$

With the change of variables $w' = v$, we can write the second order equations (2.10) and (2.11) as a first order system of ordinary differential equations in the Hilbert space $Z = Z_{1/2} = X^{1/2} \times X$ as follows:

$$z' = \mathcal{A}z + B_\omega u \quad z \in Z_{1/2}, \quad (0, \tau], \quad (2.12)$$

$$z' = \mathcal{A}z + Bu \quad z \in Z_{1/2}, \quad (0, \tau], \quad (2.13)$$

where

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad B_\omega = \begin{bmatrix} 0 \\ 1_\omega I \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad (2.14)$$

and

$$\mathcal{A} = \begin{bmatrix} 0 & I_X \\ -\gamma A & -\eta A^{1/2} \end{bmatrix} \quad (2.15)$$

is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times D(A^{1/2})$.

As we had mentioned in the introduction this operator \mathcal{A} generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ in the space $Z = Z_{1/2} = X^{1/2} \times X$, which is also analytic (see [11], S. Chen and R. Triggiani). Now, using Lemma 2.1 from [16] or Lemma 3.1 from [10], one can get the following representation for this semigroup.

Proposition 2.2 *The semigroup $\{T(t)\}_{t \geq 0}$ generated by the operator \mathcal{A} has the following representation*

$$T(t)z = \sum_{nj=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z_{1/2}, \quad t \geq 0, \quad (2.16)$$

where $\{P_j\}_{j \geq 0}$ is a complete family of orthogonal projections in the Hilbert space $Z_{1/2}$ given by (2.9) and

$$A_j = R_j P_j, \quad R_j = \begin{bmatrix} 0 & 1 \\ -\gamma \lambda_j & -\eta \lambda_j^{1/2} \end{bmatrix}, \quad j \geq 1. \quad (2.17)$$

Moreover, $e^{A_j t} = e^{R_j t} P_j$, the eigenvalues of R_j are:

$$\lambda = -\lambda_j^{1/2} \left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2} \right), \quad j = 1, 2, \dots,$$

and

$$A_j^* = R_j^* P_j, \quad R_j^* = \begin{bmatrix} 0 & -1 \\ \gamma \lambda_j & -\eta \lambda_j^{1/2} \end{bmatrix}.$$

3 Controllability of System (2.13)

In this section we prove approximate controllability of the system (2.13). First we give the definition of approximate controllability for this system. To this end, for all $z_0 \in Z = Z_{1/2}$ and $u \in L^2(0, \tau; U)$ the initial value problem

$$\begin{cases} z' = Az + Bu(t), & z \in Z \\ z(0) = z_0, \end{cases} \quad (3.18)$$

where $U = L^2(\Omega)$ and the control function u belongs to $L^2(0, \tau; U)$, admits only one mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds, \quad t \in [0, \tau]. \quad (3.19)$$

Definition 3.1 (Approximate Controllability) *The system (2.13) is said to be approximately controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z$ and $\varepsilon > 0$ there exists $u \in L^2(0, \tau; U)$ such that the solution $z(t)$ of (3.18) corresponding to u verifies:*

$$z(0) = z_0, \quad \text{and} \quad \|z(\tau) - z_1\| < \varepsilon.$$

Definition 3.2 *For the system (2.13) we define the following concepts:*

a) *The controllability map (for $\tau > 0$) $\mathcal{B}^\tau : L^2(0, \tau; U) \longrightarrow Z$ is given by*

$$\mathcal{B}^\tau u = \int_0^\tau T(s)Bu(s)ds. \quad (3.20)$$

b) The grammian map $L_{\mathcal{B}^\tau} : Z \rightarrow Z$ is defined by $L_{\mathcal{B}^\tau} = \mathcal{B}^\tau \mathcal{B}^{\tau*}$, i.e.,

$$L_{\mathcal{B}^\tau} z = \mathcal{B}^\tau \mathcal{B}^{\tau*} u = \int_0^\tau T(s) B B^* T^*(s) z ds.$$

The following lemma is trivial:

Lemma 3.1 *The equation (2.13) is approximately controllable on $[0, \tau]$ if and only if $\overline{\text{Rang}(\mathcal{B}^\tau)} = Z$.*

The following theorem holds in general for evolution equations and follows from the above lemma and a characterization of range dense linear operator.

Theorem 3.1 [13] *The equation (2.13) is approximately controllable on $[0, \tau]$ if, and only if, one of the following statements holds:*

(i) $\text{Ker}(\mathcal{B}^{\tau*}) = \{0\}$.

(ii) $\langle L_{\mathcal{B}^\tau} z, z \rangle > 0, z \neq 0$ in Z .

(iii) $B^* T^*(\cdot) z = 0 \Rightarrow z = 0$.

The following corollary is easy to prove and allows us to find a formula for a sequence of controls steering the systems from initial state z_0 to an ϵ -neighborhood of the final state z_1 at time $\tau > 0$.

Corollary 3.1 *The equation (2.13) is approximately controllable on $[0, \tau]$ if, and only if,*

$$\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + L_{\mathcal{B}^\tau})^{-1} z = 0, \quad z \in Z. \quad (3.21)$$

Now, using part ii) from the foregoing theorem and Corollary 3.1, it is not hard to prove the following theorem:

Theorem 3.2 *The system (2.13) is approximately controllable on $[0, \tau]$. Moreover, a sequence of controls steering the system (2.13) from initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$ is given by*

$$u_\alpha(t) = B^* T(\tau - t)(\alpha I + L_{\mathcal{B}^\tau})^{-1}(z_1 - T(\tau)z_0),$$

and the error of this approximation E_α is given by

$$E_\alpha = \alpha(\alpha I + L_{\mathcal{B}^\tau})^{-1}(z_1 - T(\tau)z_0).$$

4 Proof of the Main Theorem

In this section we shall prove the main result of this paper on the interior approximate controllability of the linear system (2.12). To this end, we observe that the definition of approximate controllability for system (2.12) is similar to the one given for system (2.13) and, for all $z_0 \in Z$ and $u \in L^2(0, \tau; U)$ the initial value problem

$$\begin{cases} z' = \mathcal{A}z + B_\omega u(t), z \in Z \\ z(0) = z_0, \end{cases} \quad (4.22)$$

where the control function u belongs to $L^2(0, \tau; U)$, admits only one mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)B_\omega u(s)ds, \quad t \in [0, \tau]. \quad (4.23)$$

To prove the main result, we need the following two lemmas:

Lemma 4.1 [5]

Let $\{\alpha_{1j}\}_{j \geq 1}$, $\{\beta_{1,j}\}_{j \geq 1}$ and $\{\alpha_{2j}\}_{j \geq 1}$, $\{\beta_{2,j}\}_{j \geq 1}$ be sequences of real numbers such that: $\alpha_{11} > \alpha_{12} > \alpha_{13} \cdots$; $\alpha_{21} > \alpha_{22} > \alpha_{23} \cdots$ and $\alpha_{1j} > \alpha_{2j}$, for all $j = 1, 2, 3, \dots$. Then, for any $\tau > 0$ we have

$$\sum_{j=1}^{\infty} \{e^{\alpha_{1j}t} \beta_{1,j} + e^{\alpha_{2j}t} \beta_{2,j}\} = 0, \quad \forall t \in [0, \tau],$$

iff

$$\beta_{1,j} = \beta_{2,j} = 0, \quad j = 1, 2, \dots, \infty.$$

Lemma 4.2 Let $\{\alpha_j\}_{j \geq 1}$, $\{\gamma_j\}_{j \geq 1}$, and $\{\beta_{1j}\}_{j \geq 1}$, $\{\beta_{2,j}\}_{j \geq 1}$ be sequences of real numbers such that: $\alpha_1 > \alpha_2 > \alpha_3 \cdots$. Then, for any $\tau > 0$ we have

$$\sum_{j=1}^{\infty} \{e^{(\alpha_j+i\gamma)t} \beta_{1,j} + e^{(\alpha_j-i\gamma)t} \beta_{2,j}\} = 0, \quad \forall t \in [0, \tau], \quad i = \sqrt{-1}. \quad (4.24)$$

iff

$$\beta_{1,j} = \beta_{2,j} = 0, \quad j = 1, 2, \dots, \infty.$$

Proof.

We use the same idea for the proof of Lemma 3.14 from [12] and Theorem 4.2.1 from [13]. In fact, because of the analyticity we may extend (4.24) for $t \geq 0$. Then, for all $t \geq 0$ we have

$$\sum_{j=1}^{\infty} \{e^{(\alpha_j+i\gamma)t} \beta_{1,j} + e^{(\alpha_j-i\gamma)t} \beta_{2,j}\} = \sum_{j=1}^{\infty} e^{\alpha_j t} \{e^{i\gamma t} \beta_{1,j} + e^{-i\gamma t} \beta_{2,j}\} = 0. \quad (4.25)$$

Then, putting $\alpha_j = \alpha_1 + \sigma_j$, with $\sigma_j < 0$, and dividing (4.25) by α_1 we get

$$\cos \gamma t(\beta_{1,1} + \beta_{2,1}) + i \sin \gamma t(\beta_{1,1} - \beta_{2,1}) + \sum_{j=2}^{\infty} e^{\sigma_j t} \{ \cos \gamma t(\beta_{1,j} + \beta_{2,j}) + i \sin \gamma t(\beta_{1,j} - \beta_{2,j}) \} = 0;$$

and passing to the limit as $t \rightarrow \infty$ we obtain that

$$\lim_{t \rightarrow \infty} \{ \cos \gamma t(\beta_{1,1} + \beta_{2,1}) + i \sin \gamma t(\beta_{1,1} - \beta_{2,1}) \} = 0.$$

i.e.,

$$\lim_{t \rightarrow \infty} \cos \gamma t(\beta_{1,1} + \beta_{2,1}) = \lim_{t \rightarrow \infty} \sin \gamma t(\beta_{1,1} - \beta_{2,1}) = 0.$$

Hence,

$$\begin{cases} \beta_{1,1} + \beta_{2,1} = 0 \\ \beta_{1,1} - \beta_{2,1} = 0 \end{cases} \iff \beta_{1,1} = \beta_{2,1} = 0.$$

Therefore,

$$\sum_{j=2}^{\infty} e^{\sigma_j t} \{ \cos \gamma t(\beta_{1,j} + \beta_{2,j}) + i \sin \gamma t(\beta_{1,j} - \beta_{2,j}) \} = 0, \quad \forall t \geq 0,$$

with $\sigma_2 > \sigma_3 > \sigma_4 \dots$.

So, in a similar manner we can prove that $\beta_{1,j} = \beta_{2,j} = 0$.

□

Now, we are ready to formulate and prove the main result of this work.

Theorem 4.1 *If $\eta^2 \neq 4\gamma$, then for all $\tau > 0$ and all open nonempty subset ω of Ω the system (2.12) is approximately controllable on $[0, \tau]$.*

Moreover, a sequence of controls steering the system (2.12) from initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$ is given by

$$u_\alpha(t) = B_\omega^* T(\tau - t)(\alpha I + L_{\mathcal{B}^\tau})^{-1}(z_1 - T(\tau)z_0),$$

and the error of this approximation E_α is given by

$$E_\alpha = \alpha(\alpha I + L_{\mathcal{B}^\tau})^{-1}(z_1 - T(\tau)z_0),$$

where

$$\begin{aligned} \mathcal{B}^\tau u &= \int_0^\tau T(s) B_\omega u(s) ds, \\ L_{\mathcal{B}^\tau} z &= \mathcal{B}^\tau \mathcal{B}^{\tau*} u = \int_0^\tau T(s) B_\omega B_\omega^* T^*(s) z ds. \end{aligned}$$

Proof. We apply Theorem 3.1 to prove the controllability of system (2.12). To this end, we observe that

$$T^*(t)z = \sum_{j=1}^{\infty} e^{A_j^* t} P_j^* z, \quad z \in Z, \quad t \geq 0, \quad B_{\omega}^* = \begin{bmatrix} 0 & 1_{\omega} I \end{bmatrix}.$$

Case1. $\eta^2 > 4\gamma$. In this case the eigenvalues of the matrix R_j are real and simple. Then, there exists a family of complete complementary projections $\{q_1(j), q_2(j)\}$ on \mathbb{R}^2 such that

$$e^{R_j^* t} = e^{-\lambda_j^{\frac{1}{2}} \rho_1 t} q_1^*(j) + e^{-\lambda_j^{\frac{1}{2}} \rho_2 t} q_2^*(j), \quad \rho_1 = \frac{\eta - \sqrt{\eta^2 - 4\gamma}}{2}, \quad \rho_2 = \frac{\eta + \sqrt{\eta^2 - 4\gamma}}{2}.$$

Therefore,

$$B_{\omega}^* T^*(t)z = \sum_{j=1}^{\infty} B_{\omega}^* e^{A_j^* t} P_j^* z = \sum_{j=1}^{\infty} \sum_{s=1}^2 e^{-\lambda_j^{\frac{1}{2}} \rho_s t} B_{\omega}^* P_{s,j}^* z,$$

where $P_{s,j} = q_s(j)P_j = P_j q_s(j)$.

Suppose now that $B_{\omega}^* T^*(t)z = 0, \quad \forall t \in [0, \tau]$. Then,

$$\begin{aligned} B_{\omega}^* T^*(t)z &= \sum_{j=1}^{\infty} B_{\omega}^* e^{A_j^* t} P_j^* z = \sum_{j=1}^{\infty} \sum_{s=1}^2 e^{-\lambda_j^{\frac{1}{2}} \rho_s t} B_{\omega}^* P_{s,j}^* z = 0. \\ \iff \sum_{j=1}^{\infty} \{e^{-\lambda_j^{\frac{1}{2}} \rho_1 t} (B_{\omega}^* P_{1,j}^* z)(x) + e^{-\lambda_j^{\frac{1}{2}} \rho_2 t} (B_{\omega}^* P_{2,j}^* z)(x)\} &= 0, \quad \forall x \in \Omega. \end{aligned}$$

Since $0 < \rho_1 < \rho_2$, the sequences $\{\alpha_{1,j} = -\lambda_j^{\frac{1}{2}} \rho_1 : j = 1, 2, \dots\}$ and $\{\alpha_{2,j} = -\lambda_j^{\frac{1}{2}} \rho_2 : j = 1, 2, \dots\}$ satisfy Lemma 4.1. Then

$$(B_{\omega}^* P_{1,j}^* z)(x) = (B_{\omega}^* P_{2,j}^* z)(x) = 0, \quad \forall x \in \Omega, \quad j = 1, 2, \dots$$

Because

$$q_i^*(j) = \begin{bmatrix} a_{11}^{ij} & a_{12}^{ij} \\ a_{21}^{ij} & a_{22}^{ij} \end{bmatrix}, \quad i = 1, 2; \quad j = 1, 2, \dots,$$

we get that

$$(B_{\omega}^* P_{i,j}^* z)(x) = 1_{\omega} [a_{21}^{ij} E_j z_1(x) + a_{22}^{ij} E_j z_2(x)] = 0, \quad \forall x \in \Omega.$$

That is,

$$(B_{\omega}^* P_{i,j}^* z)(x) = a_{21}^{ij} E_j z_1(x) + a_{22}^{ij} E_j z_2(x) = 0, \quad \forall x \in \omega.$$

Also, we know from Theorem 1.1 that $\phi_{j,k}$ are analytic functions, which implies the analyticity of $E_j z_i$. Then, from Theorem 1.2 we get that

$$(B_{\omega}^* P_{s,j}^* z)(x) = a_{21}^{ij} E_j z_1(x) + a_{22}^{ij} E_j z_2(x) = 0, \quad \forall x \in \Omega.$$

Hence

$$B_{\Omega}^* T^*(t)z = \sum_{j=1}^{\infty} B^* e^{A_j^* t} P_j^* z = \sum_{j=1}^{\infty} \sum_{s=1}^3 e^{\sigma_s(j)t} B_{\Omega}^* P_{s,j}^* z = 0, \quad \forall t \in [0, \tau].$$

From Theorem 3.2 we know that system (2.13) is approximately controllable, then from part iii) of Theorem 3.1 we conclude that $z = 0$.

Case2. $\eta^2 < 4\gamma$. In this case the eigenvalues of the matrix R_j are complex and given by $\alpha_j \pm i\gamma_j$, with:

$$\alpha_j = -\lambda_j^{\frac{1}{2}} \frac{\eta}{2} \quad \text{and} \quad \gamma_j = \lambda_j^{\frac{1}{2}} \frac{\sqrt{4\gamma - \eta^2}}{2} \quad j = 1, 2, 3, \dots$$

Then, there exists a family of complete complementary projections $\{q_1(j), q_2(j)\}$ on \mathbb{R}^2 such that

$$\begin{aligned} B_{\omega}^* T^*(t)z &= \sum_{j=1}^{\infty} B_{\omega}^* e^{A_j^* t} P_j^* z = \sum_{j=1}^{\infty} \{e^{(\alpha_j + i\gamma)t} B_{\omega}^* P_{1,j}^* z + e^{(\alpha_j - i\gamma)t} B_{\omega}^* P_{2,j}^* z\} \\ &= \sum_{j=1}^{\infty} e^{\alpha_j t} \{e^{i\gamma t} B_{\omega}^* P_{1,j}^* z + e^{-i\gamma t} B_{\omega}^* P_{2,j}^* z\} \end{aligned}$$

where $P_{s,j} = q_s(j)P_j = P_j q_s(j)$. From here, applying Lemma 4.2, the approximate controllability of system (2.12) follows in the same way as the case 1 .

Now, given the initial and the final states z_0 and z_1 , we consider the sequence of controls

$$\begin{aligned} u_{\alpha}(\cdot) &= B_{\omega}^* T(\tau - \cdot)(\alpha I + L_{\mathcal{B}^{\tau}})^{-1}(z_1 - T(\tau)z_0) \\ &= \mathcal{B}^{\tau*}(\alpha I + L_{\mathcal{B}^{\tau}})^{-1}(z_1 - T(\tau)z_0), \quad \alpha > 0. \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{B}^{\tau} u_{\alpha} &= L_{\mathcal{B}^{\tau}}(\alpha I + L_{\mathcal{B}^{\tau}})^{-1}(z_1 - T(\tau)z_0) \\ &= (\alpha I + L_{\mathcal{B}^{\tau}} - \alpha I)(\alpha I + L_{\mathcal{B}^{\tau}})^{-1}(z_1 - T(\tau)z_0) \\ &= z_1 - T(\tau)z_0 - \alpha(\alpha I + L_{\mathcal{B}^{\tau}})^{-1}(z_1 - T(\tau)z_0). \end{aligned}$$

From Corollary 3.1 we know that

$$\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + L_{\mathcal{B}^{\tau}})^{-1}(z_1 - T(\tau)z_0) = 0.$$

Therefore,

$$\lim_{\alpha \rightarrow 0^+} \mathcal{B}^{\tau} u_{\alpha} = z_1 - T(\tau)z_0.$$

i.e.,

$$\lim_{\alpha \rightarrow 0^+} \{T(\tau)z_0 + \int_0^{\tau} T(\tau - s)B_{\omega}u(s)ds\} = z_1.$$

This completes the proof of the theorem. □

Corollary 4.1 *Under the conditions of Theorem 4.1, the approximate controllability of the system (2.12) is equivalent to the approximate controllability of the system (2.13)*

5 Final remarks

Our result can be formulated in a more general setting. Indeed, we can consider the following second order evolution equation in a general Hilbert space Z

$$\begin{cases} y'' + \eta A^{1/2} y' + \gamma A y = B u(t), & t \in (0, \tau], \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases} \quad (5.26)$$

where, $A : D(A) \subset Z \rightarrow Z$ is an unbounded linear operator in Z with the following spectral decomposition:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k},$$

with the eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \dots < \lambda_n \rightarrow \infty$ of A having finite multiplicity γ_j equal to the dimension of the corresponding eigenspaces, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenfunctions of A . The operator $-A$ generates a strongly continuous semigroup $\{T_A(t)\}_{t \geq 0}$ given by

$$T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}.$$

The control $u \in L^2(0, \tau; Z)$, and $B : Z \rightarrow Z$ is a linear and bounded operator (linear and continuous). In this case the characteristic function set is a particular operator B , and the following theorem is a generalization of Theorem 4.1.

Theorem 5.1 *If $\eta^2 \neq 4\gamma$ and the vectors $B^* \phi_{j,k}$ are linearly independent in Z , then the system (5.26) is approximately controllable on $[0, \tau]$.*

An example of those kind of systems is the following partial differential equations modeling the structural damped vibrations of a string or a beam:

$$\begin{cases} y_{tt} - 2\beta \Delta y_t + \Delta^2 y = 1_{\omega} u(t, x), & \text{on } (0, \tau) \times \Omega, \\ y = \Delta y = 0, & \text{on } (0, \tau) \times \partial\Omega, \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & \text{in } \Omega, \end{cases} \quad (5.27)$$

where Ω is a bounded domain in \mathbb{R}^n , ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control $u \in L^2(0, \tau; L^2(\Omega))$ and $y_0, y_1 \in L^2(\Omega)$.

Another interesting point in this work is that, we could exhibit a sequence of controls steering the system from an initial state to a final state in a prefixed time, which is very important from a practical and numerical point of view. Finally, we want to point out that this work adds to the current literature a rigorous and at the same time simple technique that can be applied to a wide range of problems, including heat, wave and thermoelasticity equations.

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This work was supported by the CDHT-ULA-project: C-1667-09-05-AA

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