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On Hilbert extensions of Weierstrass' theorem with weights

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Resumen

En este artículo estudiamos el conjunto de las funciones a valores en $\mathcal{G}$ que pueden ser aproximadas por funciones continuas a valores en $\mathcal{G}$ en la norma $L^{\infty}_{\mathcal{G}}(I,w)$, donde $I \subset \mathbb{R}$ es un intervalo compacto, $\mathcal{G}$ es un espacio de Hilbert real separable y w es cierta función peso a valores en $\mathcal{G}$, débilmente medible. Así, obtenemos una nueva extensión del celebrado teorema de aproximación de Weierstrass.

Palabras Claves: teorema de Weierstrass, pesos a valores en $\mathcal{G}$, polinomios a valores en $\mathcal{G}$, funciones continuas a valores en $\mathcal{G}$.

Abstract

In this paper we study the set of $\mathcal{G}$-valued functions which can be approximated by $\mathcal{G}$-valued continuous functions in the norm $L^{\infty}_{\mathcal{G}}(I,w)$, where $I \subset \mathbb{R}$ is a compact interval, $\mathcal{G}$ is a separable real Hilbert space and w is a certain $\mathcal{G}$-valued weakly measurable weight. Thus, we obtain a new extension of the celebrated Weierstrass approximation theorem.

key words. Weierstrass' theorem, $\mathcal{G}$-valued weights, $\mathcal{G}$-valued polynomials, $\mathcal{G}$-valued continuous functions.

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1 Introduction.

If $I \subset \mathbb{R}$ is any compact interval, Weierstrass' approximation theorem says that $C(I)$ is the largest set of functions which can be approximated by polynomials in the norm $L^{\infty}(I)$, if we identify, as usual, functions which are equal almost everywhere. Weierstrass proved this theorem in 1885. He also proved the density of trigonometric polynomials in the class of 2π-periodic continuous real-valued functions. These results were, in a sense, a counterbalance to Weierstrass' famous example given in 1861 about the existence of a continuous nowhere differentiable function (see [1]).

The result obtained in his paper in 1885 titled *On the possibility of giving an analytic representation to an arbitrary function of real variable* [12], shows that he suspected that any analytic functions could be represented by power series. Weierstrass' approximation theorem can be stated as follows.

THEOREM 1.1 *(K. Weierstrass).*
Given $f : [a, b] \to \mathbb{R}$ *continuous and an arbitrary* $\epsilon > 0$ *there exists an algebraic polynomial* p *such that*

$$|f(x) - p(x)| \le \epsilon, \ \forall\ x \in [a, b]. \tag{1.1}$$

There have been many improvements, generalizations and extensions of this theorem; such results may be found in [2], [8], [9] and [22]. Further, we should recall the Bernstein's problem on approximation by polynomials on the whole real line (see [12], [13] and [14]), and the approximation problem for unbounded functions in I (see for example, [7]).

Some recent generalizations of Weierstrass' approximation theorem use weighted approximation. More precisely, if $I \subset \mathbb{R}$ is a compact interval, the approximation problem is studied with the norm $L^\infty(I,\, w)$ defined by

$$\|f\|_{L^\infty(I,w)} := \operatorname{ess\,sup}_{x\in I} |f(x)| w(x)\,, \tag{1.2}$$

where w is a weight, i.e., a non-negative measurable function. The convention $0 \cdot \infty = 0$ is used as well. Observe that (1.2) is not the usual definition of the L^∞ norm in the context of measure theory, although it is the correct definition when we work with weights (see e.g. [3] and [4]). The reader may find in [18], [19] and [20] recent and detailed results on this subject.

Other kinds of approximation problems can arise when we consider simultaneous approximation including derivatives of certain functions; this is the case for Weierstrass' theorem in the context of weighted Sobolev spaces. About this subject we refer to [19] and [20].

In this paper we give a new result on Weierstrass' approximation theorem with weights for approximation in Hilbert spaces. We consider a separable real Hilbert space $\mathcal{G}$, a compact interval $I \subset \mathbb{R}$, the space of all the $\mathcal{G}$-valued essentially bounded functions $L^\infty_{\mathcal{G}}(I)$, a weakly measurable function $w : I \to \mathcal{G}$, the space of all $\mathcal{G}$-valued continuous functions $C(I; \mathcal{G})$, and the space of all the $\mathcal{G}$-valued functions $L^\infty_{\mathcal{G}}(I, w)$, which are bounded with respect to the norm defined by

$$\|f\|_{L^\infty_{\mathcal{G}}(I,w)} := \operatorname{ess\,sup}_{t\in I} \|(fw)(t)\|_{\mathcal{G}}\,. \tag{1.3}$$

The paper is organized as follows. In Section 2 we provide some notation, necessary preliminaries and auxiliary results which will be often used throughout the text. Usually we shall use standard notation, and it will be properly introduced whenever needed. In Section 3 we present the main result about approximation in $L^{\infty}_{\mathcal{G}}(I,w)$.

2 Preliminaries.

In what follows, I stands for any compact interval in $\mathbb{R}$. By $l^2(\mathbb{R})$ we denote the real linear space of all sequences $\{x_n\}_{n\in\mathbb{Z}_+}$ with $\sum_{n=0}^{\infty}|x_n|^2<\infty$, and $(\mathcal{G},\langle\cdot\,,\,\cdot\rangle_{\mathcal{G}})$ stands for a separable real Hilbert space with associated norm denoted by $\|\cdot\|_{\mathcal{G}}$.

It is well-known that every separable real Hilbert space $\mathcal{G}$ is isomorphic either to $\mathbb{R}^n$ for some $n\in\mathbb{N}$ or to $l^2(\mathbb{R})$. In each case, $\mathcal{G}$ has the structure of a commutative Banach algebra with the coordinatewise operations. In the first case, we have a commutative Banach algebra with identity and the second case, a commutative Banach algebra without identity. The reader is referred to [10] or [24] for more details about these statements.

However, in practice the representation which we obtain by means of this isomorphism is not always interesting, because the properties of the individual elements of $\mathcal{G}$ can be in many cases more fruitful. This happens when $\mathcal{G}$ is a Hilbert space of analytic functions or of differentiable functions. Despite this, it is very valuable to know the representation given by this isomorphism because it allows us to determine how useful the properties of the Hilbert space by itself can be.

2.1 On weighted spaces.

A detailed discussion about properties of weighted spaces may be found in [6], [11], [15] or [17]. We recall here some important tools and definitions which will be used throughout this paper.

Definition 2.1 *A scalar weight w is a measurable function $w:\mathbb{R}\longrightarrow[0,\infty]$. If w is only defined in $A\subset\mathbb{R}$, we set $w:=0$ in $\mathbb{R}\setminus A$.*

Definition 2.2 *Given a measurable set $A\subset\mathbb{R}$ and a scalar weight w, we define the space $L^{\infty}(A,w)$ as the space of equivalence classes of measurable functions $f:A\longrightarrow\mathbb{R}$ with respect to the norm*

$$\|f\|_{L^{\infty}(A,w)}:=\operatorname{ess\,sup}_{x\in A}|f(x)|w(x)\,.$$

This space inherits some properties from the classical Lebesgue space $L^{\infty}(A)$ and it allows us to study new functions, which do not fit in the classical $L^{\infty}(A)$ (see, for example [21]). Other properties of $L^\infty(A,w)$ have a strong relation with the nature of the weight w: in fact, if $A = I$ and w has a multiplicative inverse, (i.e. there exists a weight $w^{-1} : I \longrightarrow \mathbb{R}$, such that $w(t)w^{-1}(t) = 1, \quad \forall t \in I$) then, it is easy to see that $L^\infty(I,w)$ and $L^\infty(I)$ are isomorphic, since the map $\Psi_w : L^\infty(I,w) \to L^\infty(I)$ given by $\Psi_w(f) = fw$ is a linear and bijective isometry, and therefore, Ψ_w is also homeomorphism, or equivalently, for all $Y \subseteq L^\infty(I,w)$, we have $\Psi_w(\overline{Y}) = \overline{\Psi_w(Y)}$, where we take each closure with respect to the norms $L^\infty(I,w)$ and $L^\infty(I)$, respectively. Also, for all $A \subseteq L^\infty(I)$, $\Psi_w^{-1}(\overline{A}) = \overline{\Psi_w^{-1}(A)}$ and $\Psi_w^{-1} = \Psi_{w^{-1}}$. Then using Weierstrass' theorem we have,

$$\Psi_w^{-1}(\overline{\mathbb{P}}) = \overline{\Psi_w^{-1}(\mathbb{P})} \;\; = \;\; \{f \in L^\infty(I,w) : fw \in C(I)\}. \tag{2.4}$$

Unfortunately, the last equality in (2.4) does not allow us to obtain information on local behavior of the functions $f \in L^\infty(I,w)$ which can be approximated. Furthermore, if $f \in L^\infty(I,w)$, then in general fw is not a continuous function, since its continuity also depends of the singularities of weight w (see [13], [18], [20]).

The next definition presents the classification of the singularities of a scalar weight w done in [20] to show the results about density of continuous functions in the space $L^\infty(supp(w), w)$.

Definition 2.3 *Given a scalar weight w we say that $a \in \operatorname{supp}(w)$ is a singularity of w (or singular for w) if*

$$\operatorname{ess\,lim\,inf}_{x\in \operatorname{supp}(w),\, x\to a} w(x) = 0\,.$$

We say that a singularity a of w is of type 1 if $\operatorname{ess\,lim}_{x\to a} w(x) = 0$.

We say that a singularity a of w is of type 2 if $0 < \operatorname{ess\,lim\,sup}_{x\to a} w(x) < \infty$.

We say that a singularity a of w is of type 3 if $\operatorname{ess\,lim\,sup}_{x\to a} w(x) = \infty$.

We denote by S and S_i $(i = 1,2,3)$ respectively, the set of singularities of w and the set of singularities of w of type i.

We say that $a \in S_i^+$ (respectively $a \in S_i^-$) if a verifies the property in the definition of S_i when we take the limit as $x \to a^+$ (respectively $x \to a^-$). We define $S^+ := S_1^+ \cup S_2^+ \cup S_3^+$ and $S^- := S_1^- \cup S_2^- \cup S_3^-$.

Definition 2.4 *Given a scalar weight w, we define the right regular and left regular points of w, respectively, as*

$$R^+ := \left\{a \in \operatorname{supp}(w) :\, \operatorname{ess\,lim\,inf}_{x\in \operatorname{supp}(w),\, x\to a^+} w(x) > 0\right\},$$

$$R^- := \left\{a \in \operatorname{supp}(w) :\, \operatorname{ess\,lim\,inf}_{x\in \operatorname{supp}(w),\, x\to a^-} w(x) > 0\right\}.$$

The following result was proved in [20] and it states a characterization for the functions in $L^\infty(supp(w), w)$ which can be approximated by continuous functions in norm $L^\infty(supp(w), w)$ for every w.

THEOREM 2.1 *(Portilla et al. [[20], Theorem 1.2]). Let w be any scalar weight and*

$$H_0 := \left\{ \begin{array}{r} f \in L^\infty(\mathrm{supp}(w), w) : f \text{ is continuous to the right at every point of } R^+, \\ f \text{ is continuous to the left at every point of } R^-, \\ \text{for each } a \in S^+, \ \operatorname{ess}\lim_{x\to a^+} |f(x) - f(a)|\, w(x) = 0\,, \\ \text{for each } a \in S^-, \ \operatorname{ess}\lim_{x\to a^-} |f(x) - f(a)|\, w(x) = 0 \end{array} \right\}.$$

Then:

(a) The closure of $C(\mathbb{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is H_0.

(b) If $w \in L^\infty_{loc}(\mathbb{R})$, then the closure of $C^\infty(\mathbb{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is also H_0.

(c) If $\mathrm{supp}(w)$ is compact and $w \in L^\infty(\mathbb{R})$, then the closure of the space of polynomials is H_0 as well.

Theorem 2.1 is going to be an important tool which will allow us to obtain the key for the result about Hilbert extensions of Weierstrass' theorem with weights in the present paper.

2.2 $\mathcal{G}$-valued functions.

DEFINITION 2.5 *Let $\mathcal{G}$ be a separable real Hilbert space and we consider any sequence $\{x_n\} \subset \mathcal{G}$. We say that the support of $\{x_n\}$ is the set of n for which $x_n \neq 0$. We denote to support of $\{x_n\}$ by $\mathrm{supp}(x_n)$.*

Let $\mathcal{G}$ be a separable real Hilbert space. A $\mathcal{G}$-valued polynomial on I is a function $\phi : I \to \mathcal{G}$, such that

$$\phi(t) = \sum_{n\in\mathbb{Z}_+} \xi_n t^n,$$

where $(\xi_n)_{n\in\mathbb{Z}_+} \subset \mathcal{G}$ has finite support.

Let $\mathbb{P}(\mathcal{G})$ be the space of all $\mathcal{G}$-valued polynomials on I. It is well-known that $\mathbb{P}(\mathcal{G})$ is a subalgebra of the space $C(I; \mathcal{G})$ of all continuous $\mathcal{G}$-valued functions on I.

For $1 \leq p \leq \infty$, $L^p_{\mathcal{G}}(I)$ denotes the set of all weakly measurable functions $f : I \to \mathcal{G}$ such that

$$\int_I \|f(t)\|^p_{\mathcal{G}} dt < \infty, \text{ if } 1 \leq p < \infty,$$

or

$$\operatorname{ess\,sup}_{t\in I}\|f(t)\|_{\mathcal{G}} < \infty, \text{ if } p = \infty.$$

Then $L^2_{\mathcal{G}}(I)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{L^2_{\mathcal{G}}(I)} = \int_I \langle f(t), g(t)\rangle_{\mathcal{G}}\, dt.$$

$\mathbb{P}(\mathcal{G})$ is also dense in $L^p_{\mathcal{G}}(I)$, for $1 \le p < \infty$.

More details about these spaces may be found in [23].

DEFINITION 2.6 *Let $\mathcal{G}$ be a separable real Hilbert space, a weight w on $\mathcal{G}$ is a weakly measurable function $w : I \longrightarrow \mathcal{G}$.*

DEFINITION 2.7 *Let w be a weight on $\mathcal{G}$, we define the space $L^\infty_{\mathcal{G}}(I, w)$ as the space of equivalence classes of all the $\mathcal{G}$-valued weakly measurable functions $f : I \longrightarrow \mathcal{G}$ with respect to the norm*

$$\|f\|_{L^\infty_{\mathcal{G}}(I,w)} := \operatorname{ess\,sup}_{t\in I}\|(fw)(t)\|_{\mathcal{G}},$$

where $fw : I \longrightarrow \mathcal{G}$ is defined as follows: If $\dim \mathcal{G} < \infty$; we have the functions f and w can be expressed by $f = (f_1, \dots , f_{n_0})$ and $w = (w_1, \dots , w_{n_0})$, respectively, where f_j, $w_j : I \longrightarrow \mathbb{R}$, for $j = 1, \dots , n_0$, with $n_0 = \dim \mathcal{G}$. Then

$$(fw)(t) := (f_1(t)w_1(t), \dots , f_{n_0}(t)w_{n_0}(t)), \text{ for } t \in I.$$

If $\dim \mathcal{G} = \infty$, let $\{\tau_j\}_{j\in\mathbb{Z}_+}$ be a complete orthonormal system, then for $t \in I$ the functions f and w can be expressed as $f(t) = \sum_{j=0}^\infty \langle f(t), \tau_j\rangle_{\mathcal{G}}\, \tau_j$ and $w(t) = \sum_{j=0}^\infty \langle w(t), \tau_j\rangle_{\mathcal{G}}\, \tau_j$, respectively. So, we can define

$$(fw)(t) := \sum_{j=0}^{\infty} \langle f(t), \tau_j\rangle_{\mathcal{G}}\, \langle w(t), \tau_j\rangle_{\mathcal{G}}\, \tau_j, \text{ for } t \in I.$$

In this way, we can study our approximation problem using the properties of commutative Banach algebra of $l^2(\mathbb{R})$.

The next Proposition shows a result about algebraic properties and density of $\mathbb{P}(\mathcal{G})$ in $C(I; \mathcal{G})$. The analogous result, when $\mathcal{G}$ is a separable complex Hilbert space, appears in [23].

PROPOSITION 2.1

i) $\mathbb{P}(\mathcal{G})$ *is a subalgebra of the space of all $\mathcal{G}$-valued continuous functions on I.*

ii) The closure of $\mathbb{P}(\mathcal{G})$ *in* $L^\infty_{\mathcal{G}}(I)$ *is* $C(I; \mathcal{G})$.

Proof.

i) It is straight forward.

ii) It is enough to prove that $C(I;\mathcal{G}) \subset \overline{\mathbb{P}(\mathcal{G})}$, since $\overline{\mathbb{P}(\mathcal{G})} \subset \overline{C(I;\mathcal{G})} = C(I;\mathcal{G})$.

Case 1: $\dim \mathcal{G} < \infty$.

Let us assume that $\dim \mathcal{G} = n_0$. Given an orthonormal basis $\{\tau_1, \dots, \tau_{n_0}\}$ of $\mathcal{G}$, $\epsilon > 0$ and $f \in C(I;\mathcal{G}) = C(I, \mathbb{R}^{n_0})$, then $f \sim (f_1, \dots, f_{n_0})$ with $f_j \in C(I)$, $j = 1, \dots, n_0$. The Weierstrass' theorem guarantees that there exists $p_k \in \mathbb{P}$ such that

$$\|f_j - p_j\|_{L^\infty(I)} < \frac{\epsilon}{\sqrt{n_0}}, \quad j = 1, \dots, n_0.$$

If we consider the polynomial $p \in \mathbb{P}(\mathcal{G})$ such that $p \sim (p_1, \dots, p_{n_0})$, then we have that

$$\begin{aligned}
\|f - p\|_{L^\infty_{\mathcal{G}}(I)} &= \operatorname{ess\,sup}_{t\in I} \|(f - p)(t)\|_{\mathcal{G}} \\
&= \operatorname{ess\,sup}_{t\in I} \left[\sum_{j=1}^{n_0} |\langle f(t) - p(t), \tau_j \rangle_{\mathcal{G}}|^2 \right]^{1/2} \\
&\leq \operatorname{ess\,sup}_{t\in I} \|((f_1 - p_1)(t), \dots, (f_{n_0} - p_{n_0})(t))\|_{\mathbb{R}^{n_0}} < \epsilon.
\end{aligned}$$

Case 2: $\mathcal{G}$ is infinite-dimensional.

Let $f \in C(I;\mathcal{G})$ and $\{\tau_j\}_{j\in\mathbb{Z}_+}$ a complete orthonormal system, then for each $t \in I$

$$f(t) = \sum_{j=0}^{\infty} \langle f(t), \tau_j \rangle_{\mathcal{G}}\, \tau_j,$$

consequently, given $\epsilon > 0$ there exists $m_0 \in \mathbb{Z}_+$ such that

$$\left\| f(t) - \sum_{j=0}^{n} \langle f(t), \tau_j \rangle \tau_j \right\|_{\mathcal{G}} < \epsilon, \text{ whenever } n \geq m_0.$$

Now, let us consider the functions $f_j : I \to \mathbb{R}$ given by $f_j(t) = \langle f(t), \tau_j \rangle_{\mathcal{G}}$. We have that $f \sim \{f_j\}$ with $\sum_{j\in\mathbb{Z}_+} |f_j(t)|^2 < \infty$, for each $t \in I$ and $f_j \in C(I)$.

So, Weierstrass' approximation theorem guarantees that there exists a sequence $\{p_j\}_{j\in\mathbb{Z}_+} \subset \mathbb{P}$ such that

$$\|f_j - p_j\|_{L^\infty(I)} < \frac{\epsilon}{j+1}, \quad j \in \mathbb{Z}_+.$$

We define the $\mathcal{G}$-polynomials $\tilde{p}_j \in \mathbb{P}(\mathcal{G})$ by $\tilde{p}_j(t) = p_j(t)\tau_j$, for each $j \in \mathbb{Z}_+$. Then for $n \geq m_0$ we have

$$\begin{aligned}
\left\| f(t) - \sum_{j=0}^{n} \tilde{p}_j(t) \right\|_{\mathcal{G}} &\leq \left\| f(t) - \sum_{j=0}^{n} f_j(t)\tau_j \right\|_{\mathcal{G}} + \left\| \sum_{j=0}^{n} f_j(t)\tau_j - \sum_{j=0}^{n} \tilde{p}_j(t) \right\|_{\mathcal{G}} \\
&\leq \epsilon + \left(\sum_{j=0}^{\infty} |f_j(t) - p_j(t)|^2 \right)^{1/2} \\
&< \epsilon + \left(\sum_{j=0}^{\infty} \left(\frac{\epsilon}{j+1} \right)^2 \right)^{1/2} = \epsilon \left(1 + \left(\sum_{j=0}^{\infty} \frac{1}{(j+1)^2} \right)^{1/2} \right).
\end{aligned}$$

From these inequalities we can deduce that for a large enough n there exists $q_n(t) = \sum_{j=0}^{n} \tilde{p}_j(t)$ such that

$$\|f - q_n\|_{L^\infty_{\mathcal{G}}(I)} < C\epsilon.$$

This completes the proof. □

3 Approximation in $L^\infty_{\mathcal{G}}(I, w)$

In this section, we only deal with weights w such that $supp(w) = I$.

DEFINITION 3.1 *Let $\mathcal{G}$ be a real and separable Hilbert space and let w be a weight on $\mathcal{G}$. We say that w is admissible* if one of the following conditions is satisfied*

i) If $\dim \mathcal{G} < \infty$ then each one of the components w_j, $1 \leq j \leq \dim \mathcal{G}$, is a scalar weight.

ii) If $\dim \mathcal{G} = \infty$, $\{\tau_j\}_{j \in \mathbb{Z}_+}$ is a complete orthonormal system, and $w(t) = \sum_{j=0}^{\infty} \langle w(t), \tau_j \rangle_{\mathcal{G}}\, \tau_j$, then each one of the functions $\langle w(t), \tau_j \rangle_{\mathcal{G}}$ is a scalar weight.

Let us observe that if $\dim \mathcal{G} = \infty$ and w is admissible*, then it induces a family of weighted $l^2(\mathbb{R})$ spaces, $\{l^2_t(\mathbb{R}; w) : t \in I\}$ given by

$$l^2_t(\mathbb{R}; w) = \left\{ \{x_j\}_{j \in \mathbb{Z}_+} : \sum_{j=0}^{\infty} \langle w(t), \tau_j \rangle_{\mathcal{G}}\, |x_j|^2 < \infty \right\}.$$

For each $t \in I$, the function $w_j(t) = \langle w(t), \tau_j \rangle_{\mathcal{G}}$ also induces a linear isometry

$\Psi^t_{w_j} : l^2_t(\mathbb{R}; w_j) \to l^2(\mathbb{R})$ given by

$$\Psi^t_{w_j}\left(\{x_j\}_{j\in\mathbb{Z}_+}\right) = \{w_j(t)x_j\}_{j\in\mathbb{Z}_+} = \{\langle w(t), \tau_j\rangle_{\mathcal{G}}\, x_j\}_{j\in\mathbb{Z}_+}.$$

The reader is referred to [5] where weighted $l^2(\mathbb{R})$ spaces are studied. In order to characterize the $\mathcal{G}$-valued functions which can be approximated in $L^\infty_{\mathcal{G}}(I,w)$ by functions in $C(I;\mathcal{G})\cap L^\infty_{\mathcal{G}}(I,w)$, our argument requires an admissible* weight w. It is clear that in the one-dimensional case an admissible* weight is an arbitrary scalar weight on I, and therefore the Theorem 2.1 in [20] holds in this case.

THEOREM 3.1 *Let $\mathcal{G}$ be a real and separable Hilbert space and let w be an admissible* weight on $\mathcal{G}$. Let us define*

$$H := \left\{ \begin{array}{c} f \in L^\infty_{\mathcal{G}}(I,w) : f \sim (f_1,\dots,f_{n_0}) \text{ and } f_j \in H_j,\, 1\le j\le n_0 \text{ with } n_0 = \dim\mathcal{G}, \\ \text{or } f\sim\{f_j\} \text{ and } f_j\in H_j,\, j\in\mathbb{Z}_+ \text{ if } \dim\mathcal{G}=\infty. \end{array} \right\},$$

where

$$H_j := \left\{ \begin{array}{ll} f_j \in L^\infty(I,w_j) : & f_j \text{ is continuous to the right at every point of } R^+, \\ & f_j \text{ is continuous to the left at every point of } R^-, \\ & \text{for each } a\in S^+,\ \operatorname{ess\,lim}_{x\to a^+} |f_j(x)-f_j(a)|\, w_j(x) = 0\,, \\ & \text{for each } a\in S^-,\ \operatorname{ess\,lim}_{x\to a^-} |f_j(x)-f_j(a)|\, w_j(x) = 0 \end{array} \right\}.$$

Then the closure of $C(I;\mathcal{G})\cap L^\infty_{\mathcal{G}}(I,w)$ in $L^\infty_{\mathcal{G}}(I,w)$ is H. Furthermore, if $w\in L^\infty_{\mathcal{G}}(I)$ then the closure of the space of $\mathcal{G}$-valued polynomials is H as well.

Proof. Let us assume first that $\dim\mathcal{G} = n_0$. If $f\in \overline{C(I;\mathcal{G})\cap L^\infty_{\mathcal{G}}(I,w)}^{L^\infty_{\mathcal{G}}(I,w)}$, then $f\sim (f_1,\dots,f_{n_0})$, with $f_j : I\longrightarrow \mathbb{R}$, $1\le j\le n_0$. Given $\epsilon>0$, there exists $g\in C(I;\mathcal{G})\cap L^\infty_{\mathcal{G}}(I,w)$ such that $\|f-g\|_{L^\infty_{\mathcal{G}}(I,w)}<\epsilon$. Let us consider $(g_1,\dots,g_{n_0})$ such that $g_j\in C(I)\cap L^\infty(I,w_j)$ and $g\sim(g_1,\dots,g_{n_0})$, then

$$|(f_j(t)-g_j(t))w_j(t)| \le \operatorname{ess\,sup}_{s\in I}\left[\sum_{j=1}^{n_0} |(f_j(s)-g_j(s))w_j(s)|^2\right]^{1/2} \quad \text{a.e.}$$

On other hand, $\operatorname{ess\,sup}_{s\in I}\left[\sum_{j=1}^{n_0}|(f_j(s)-g_j(s))w_j(s)|^2\right]^{1/2} = \|f-g\|_{L^\infty_{\mathcal{G}}(I,w)}$, as consequence of $\mathcal{G}$ is isomorphic to $\mathbb{R}^{n_0}$ and the Parseval identity (see [5] or [24]). Then,

$$\|f_j-g_j\|_{L^\infty(I,w_j)} \le \|f-g\|_{L^\infty_{\mathcal{G}}(I,w)} < \epsilon.$$

Hence, $f_j\in\overline{C(I)\cap L^\infty(I,w_j)}^{L^\infty(I,w_j)}$ for $1\le j\le n_0$, and the part (a) of Theorem 2.1 gives that H contains $\overline{C(I;\mathcal{G})\cap L^\infty_{\mathcal{G}}(I,w)}^{L^\infty_{\mathcal{G}}(I,w)}$.

In order to see that H is contained in $\overline{C(I;\mathcal{G})\cap L^\infty_{\mathcal{G}}(I,w)}^{L^\infty_{\mathcal{G}}(I,w)}$, let us fix $f\in H$ and $\epsilon>0$, and let us consider each one of its component functions $f_j\in H_j$, $j=1,\dots,n_0$. By the part (a) of Theorem 2.1, there exists $g_j\in C(I)\cap L^\infty(I,w_j)$, $j=1,\dots,n_0$, such that

$$\|f_j-g_j\|_{L^\infty(I,w_j)}<\frac{\epsilon}{\sqrt{n_0}}.$$

We consider $g\in C(I;\mathcal{G})$ such that $g\sim(g_1,\dots,g_{n_0})$, then

$$\begin{aligned}\|f-g\|_{L^\infty_{\mathcal{G}}(I,w)} &= \operatorname{ess\,sup}_{t\in I}\|((f-p)w)(t)\|_{\mathcal{G}}\\ &= \operatorname{ess\,sup}_{t\in I}\left[\sum_{j=1}^{n_0}|(f_j(t)-g_j(t))w_j(t)|^2\right]^{1/2}<\epsilon.\end{aligned}$$

If $w\in L^\infty_{\mathcal{G}}(I)$, the closure of the $\mathcal{G}$-valued polynomials is H as well, as a consequence of Proposition 2.1.

In a similar way, if $\dim\mathcal{G}=\infty$, $\{\tau_j\}_{j\in\mathbb{Z}_+}$ is a complete orthonormal system and $f\in\overline{C(I;\mathcal{G})\cap L^\infty_{\mathcal{G}}(I,w)}^{L^\infty_{\mathcal{G}}(I,w)}$, then $f(t)=\sum_{j=0}^{\infty}\langle f(t),\tau_j\rangle\tau_j$. Given $\epsilon>0$, there exists $g\in C(I;\mathcal{G})\cap L^\infty_{\mathcal{G}}(I,w)$ such that $\|f-g\|_{L^\infty_{\mathcal{G}}(I,w)}<\epsilon$. Let us consider $\{g_j\}_{j\in\mathbb{Z}_+}$ such that $g_j\in C(I)\cap L^\infty(I,w_j)$ and $g\sim\{g_j\}_{j\in\mathbb{Z}_+}$, then

$$|(f_j(t)-g_j(t))w_j(t)|\le\operatorname{ess\,sup}_{s\in I}\left[\sum_{j=0}^{\infty}|(f_j(s)-g_j(s))w_j(s)|^2\right]^{1/2}\quad\text{a.e.}$$

On other hand, $\operatorname{ess\,sup}_{s\in I}\left[\sum_{j=1}^{\infty}|(f_j(s)-g_j(s))w_j(s)|^2\right]^{1/2}=\|f-g\|_{L^\infty_{\mathcal{G}}(I,w)}$, as consequence of $\mathcal{G}$ is isomorphic to $l^2(\mathbb{R})$ and the Parseval identity (see [5] or [24]). Then,

$$\|f_j-g_j\|_{L^\infty(I,w_j)}\le\|f-g\|_{L^\infty_{\mathcal{G}}(I,w)}<\epsilon.$$

Hence, $f_j\in\overline{C(I)\cap L^\infty(I,w_j)}^{L^\infty(I,w_j)}$ for $j\in\mathbb{Z}_+$, and the part (a) of Theorem 2.1 gives that H contains $\overline{C(I;\mathcal{G})\cap L^\infty_{\mathcal{G}}(I,w)}^{L^\infty_{\mathcal{G}}(I,w)}$.

In order to see that H is contained in $\overline{C(I;\mathcal{G})\cap L^\infty_{\mathcal{G}}(I,w)}^{L^\infty_{\mathcal{G}}(I,w)}$, let $f\in H$ and $\epsilon>0$, and let us consider the component functions $f_j\in H_j$ of f, $0\le j<\infty$. Since $w_j(t)=\langle w(t),\tau_j\rangle$ is a weight, by the part (a) of Theorem 2.1, there exists $g_j\in C(I)\cap L^\infty(I,w_j)$, $0\le j<\infty$, such that

$$\|f_j-g_j\|_{L^\infty(I,w_j)}<\frac{\epsilon}{j+1},\qquad j\in\mathbb{Z}_+.$$

We define the function $g : I \longrightarrow \mathcal{G}$ by $g(t) = \sum_{j=0}^{\infty} g_j(t)\tau_j$, then

$$\begin{aligned}
\|f-g\|_{L^{\infty}_{\mathcal{G}}(I,w)} &= \operatorname{ess\,sup}_{t\in I}\|((f-g)w)(t)\|_{\mathcal{G}} \\
&= \operatorname{ess\,sup}_{t\in I}\|\{(f_j(t)-g_j(t))w_j(t)\}\|_{l^2(\mathbb{R})} \\
&= \operatorname{ess\,sup}_{t\in I}\left[\sum_{j=0}^{\infty}|f_j(t)-g_j(t)|^2 w_j^2(t)\right]^{1/2} \\
&\leq \left[\sum_{j=0}^{\infty}\left(\frac{\epsilon}{j+1}\right)^2\right]^{1/2} = \epsilon\left[\sum_{j=0}^{\infty}\frac{1}{(j+1)^2}\right]^{1/2}
\end{aligned}$$

□

This result is similar when $\mathcal{G}$ is a complex separable Hilbert space and it can also be extended to $L^{\infty}_{L(\mathcal{G})}(I,w)$, where $L(\mathcal{G})$ is the space of operators on $\mathcal{G}$.

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