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# Distances, structured profiles and Arrow's Theorem

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**Abstract**

We study alternative spaces structured by a distance $d$. With the help of $d$, many functions can be defined for which the input is a pair formed by an alternative and a set of alternatives. We shall call these functions "distances" between an alternative and a set of alternatives. The usual way to construct these distances is via an aggregation function. These distances allow the construction of structured profiles. We propose a natural condition on these distances called *richness property*, which allows us to prove Arrow's Theorem for the class of profiles structured by distances satisfying the condition. Then we study two distances $d^{min}$ and $d^{\sigma}$ when $d$ is the Hamming distance. We prove that $d^{\sigma}$ satisfies the richness property but $d^{min}$ does not.

**key words.** Social Choice Theory, Arrow's Theorem, distances, structured profiles, aggregation functions.

**Resumen**

Estudiamos espacios de alternativas estructurados por una distancia $d$. Con ayuda de $d$ podemos construir funciones con dominio el producto cartesiano de las alternativas por los subconjuntos no vacíos de alternativas. Llamaremos a esta funciones "distancias.entre alternativas y conjuntos de alternativas. La manera estándard de construir estas distancias es usando una función de agregación. Estas distancias permiten construir perfiles estructurados. En este trabajo proponemos una condición natural sobre esta distancias, llamada *propiedad de riqueza*, la cual permite probar el Teorema de Arrow para la clase de los perfiles estructurados por distancias que satisfacen la propiedad de riqueza. En particular estudiamos las distancias $d^{min}$ y $d^{\sigma}$ cuando $d$ es la distancia de Hamming. Probamos que $d^{\sigma}$ satisface la propiedad de riqueza y que $d^{min}$ no la satisface.

**Palabras claves:** Teoría de elección social, Teorema de Arrow, distancias, perfiles estructurados, funciones de agregación.

## 1 Introduction

In the last years, Logic has been used in many successful ways for representing and modelling knowledge. One of the most inspiring approaches was proposed by Alchourrón, Gärdenfors and Makinson and is known as the AGM belief revision framework [1, 4, 8].

With the work of Groves [7] and, in particular, the work of Katsuno and Mendelzon [8] concerning the representation theorems of revision operators, becomes clear that the problem of revising a knowledge base by a new piece of information is a problem of rational choice: choosing the models of the new piece of information that best fit the old knowledge base. This "best fitting" has a full meaning when we have a distance between the models of a piece of information and the old knowledge base: the models that best fit are those which minimalize such a distance.

The tight relations between rational choice, in particular the Social Choice Theory studied by Economists [2, 9], and the logical models of the knowledge dynamics, become more striking with the framework for logical merging proposed by Konieczny and Pino-Perez in [10, 11]. Actually, with a distance $d$ between worlds, we can define a translation of the framework used in Social Choice Theory to the framework in logical merging: alternatives (candidates) correspond to the worlds (models); a preference corresponds to a knowledge base; a profile corresponds to a multi-set of knowledge bases; an agenda corresponds to an integrity constraint; a social choice function corresponds to a merging operator.

Recently, the strategy-proofness property -coming from Social Choice Theory- has been studied in the framework of logical merging [3]. Unfortunately, in the framework of logical merging, there is not yet a general result in the style of Gibbard and Satterthwaite's theorem [6, 14, 15] in the framework of Social Choice Theory. It is well known that the proof of Gibbard and Satterthwaite's theorem is based on the proof of Arrow's Impossibility Theorem. Thus, in a first step towards the establishment of a general result in the framework of logical merging, we define the class of profiles structured by a distance and an aggregation function. Then, we find a natural condition (called richness property) on these distances which allow us to prove the Arrow's Impossibility Theorem for this class of profiles.

Some interesting questions arise when we have a concrete alternative set $X$, a concrete distance $d$ on the alternative set $X$ and a concrete aggregation function $g$ and we would like to know if the distance $d^g$ between an alternative and a set of alternatives satisfies the richness property. In particular, we studied the concrete case when $X$ is the set $\{0, 1\}^n$, *i.e.* the set of vectors of zeros and ones of size $n$ (this set corresponds to set of worlds of $n$ propositional variables); $d$ is the Hamming distance, *i.e.* the number of positions in which two vectors differ; and the aggregation functions are the *min* and the *sum* (denoted here by $\sigma$). We will prove that for all $n \geq 2$, $d^{min}$ has not the richness property. Concerning $d^\sigma$, we will prove that it is rich for all $n$, except $n = 2$.

## 2 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be a set of individuals. Let $X$ be a finite set of alternatives. The preferences of the individual $i \in N$ are given by a total pre-order $\preceq_i$ on $X$, that is a total and transitive binary relation on $X$.

The relation $\preceq_i$ expresses when an alternative is *at least as good as* another one. Thus, $x \preceq_i y$ means that for the individual $i$, $x$ is at least as good as $y$. The relation of strict preference $\prec_i$ is defined by $x \preceq_i y$ and $y \not\preceq_i x$. Thus, $x \prec_i y$ means that for the individual $i$, $x$ is strictly preferred to $y$.

The set of total pre-orders over $X$ will be denoted $P$. An element $u$ of $P^n$ (the cartesian product of $P$, $n$ times) is called a *profile*. In the profile $u = (\preceq_1, \ldots, \preceq_n)$, the preference $\preceq_i$ denotes the preference of the individual $i$. A nonempty set of $X$ is called an *agenda* (the names profile and agenda are the technical terms used by economists in Social Choice Theory). The set of agendas will be denoted $\mathcal{P}^*(X)$.

If $V$ is an agenda and $\preceq$ is a total pre-order over $X$, we define the set of minimal elements of $V$ with respect to $\preceq$, denoted $\text{mín}(V, \preceq)$ as follows:

$$\text{mín}(V, \preceq) = \{x \in V : \forall y(y \prec x \Rightarrow y \notin V)\}$$

**Definition 1** *A* social choice function *is a function* $f : P^n \times \mathcal{P}^*(X) \longrightarrow \mathcal{P}^*(X)$ *such that* $f(u, V) \subseteq V$. *Often* $f(u, V)$ *will be denoted* $f_u(V)$.

Let $\preceq$ be a preference relation. The relation $\sim$, called indifference relation, is defined by putting $x \sim y$ iff $x \preceq y \ \wedge \ y \preceq x$.

The graphical representation of preference relations by levels is very useful. In one particular level are all the indifferent alternatives. The lower the level, the more preferred are the alternatives. For instance when $X = \{x, y, z, w\}$, the total pre-order $x \sim y \prec z \sim w$ will be represented by

$$\begin{matrix} z & w \\ x & y \end{matrix}$$

and the total pre-order $y \prec x \prec z \prec w$ will be represented by

$$\begin{matrix} w \\ z \\ x \\ y \end{matrix}$$

### 2.1 Postulates and Arrow's Theorem

We set the postulates that a good social choice function $f$ has to satisfy:

*Standard Domain condition (SD)* There are at least $3$ elements in $X$ and $f$ is defined for all the pairs in $P^n \times \mathcal{P}^*(X)$. The totality of the function is a desirable property because we want to have a procedure that gives a result in any given situation.

The individual $i$ is a dictator for $f$ if for all $u \in P^n$ and all $V \in \mathcal{P}^*(X)$, if $x \prec_i y$ and $x \in V$ then $y \notin f_u(V)$.

*No Dictator condition (ND)* There is no individual $i \in N$ such that $i$ is a dictator for $f$. The absence of dictator is also a desirable property.

*Weak Pareto Condition (WP)* For all profile $u$ and for all agenda $V$ if $x \in V$ and $\forall i, x \prec_i y$ then $y \notin f_u(V)$. In particular, if $f$ satisfies the Domain Standard Condition and $V = \{x, y\}$, the Weak Pareto Condition says that if for all the individuals $x$ is preferred to $y$, then selecting the best elements of $V$, will give only $x$.

Let $V$ be a nonempty subset of $X$. Let $\preceq$ a preference. We denote by $\preceq\restriction_V$ the restriction to $V$ of the relation $\preceq$. If $u = (\preceq_1, \ldots, \preceq_n)$ then $u \restriction_V = (\preceq_1\restriction_V, \ldots, \preceq_n\restriction_V)$. A social choice function $f$ satisfies the *Independence of Irrelevant Alternatives (IAI)* Property if and only if for all $V \in \mathcal{P}^*(X)$ and for all $u, u' \in P$ if $u \restriction_V = u' \restriction_V$ then $f_u(V) = f_{u'}(V)$. This condition states that the result of selecting on an agenda $V$ depends only on the individual preferences on $V$.

A social choice function $f$ satisfies *Transitive Explanations (TE)* if for all profile $u$ there exists a total pre-order $\preceq_u$ such that $f_u(V) = \text{mín}(V, \preceq_u)$, for any agenda $V$. This is a very interesting property. It says that there is a very uniform way for choosing the best elements of agendas when the profile is fixed. In other words, the social choice function can be seen operating in two steps: the first step in the process consists in giving an aggregation total pre-order $\preceq_u$ to the input $u$ and the second step consists in taking the minimal elements (the preferred ones) of the agenda $V$ with respect to this relation $\preceq_u$.

Now, having stated the previous properties, we can formulate the Arrow's Impossibility Theorem [2]. It tells us that it is impossible to have a function for which these five good properties hold (for a proof we can also see [9, 5] or [13]; in the last reference one can find a very interesting analysis of the proof). More precisely, it can be stated as follows.

**Theorem 1** *If a social choice function $f$ satisfies the Domain Standard Condition, the Independence of Irrelevant Alternatives Property, the Weak Pareto Condition and Transitive Explanations, then $f$ has a dictator.*

It is interesting to note the following proposition that is a useful tool in the proof and it is reminiscent of representation theorems in knowledge dynamics:

**Proposition 1** *If $f$ satisfies Transitive explanations ($f_u(V) = min(V, \preceq_u)$) then $\preceq_u$ is unique and it is defined by putting*

$$x \preceq_u y \Longleftrightarrow x \in f_u(\{x, y\})$$

## 2.2 Structured profiles

We are going to define a subset of the the set of profiles over a set of alternatives $X$. Such a subset will have a sort of structure given by a distance over $X$. First, let us recall the notion of *distance*.

**Definition 2** *Let $X$ be a set. The function $d : X \times X \longrightarrow \mathbb{R}^+$, with $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, is a distance over $X$ if :*

- $d(x, y) = d(y, x)$,
- $d(x, y) = 0 \Longleftrightarrow x = y$,
- $d(x, y) \leq d(x, z) + d(z, y)$.

**Example 1** *Consider $X = \{0, 1\}^n$ with $n \in \mathbb{N}$. Define $d : X \times X \longrightarrow \mathbb{R}^+$ by putting*

$$d(x, z) = \# \text{ of positions in which } x \text{ and } z \text{ are different}.$$

*It is easy to see that this function is a distance. It is called the Hamming distance.*

**Definition 3** $g : \bigcup_{n\geq 1} (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$ *is an aggregation function*[1] *if the following conditions hold:* $g(\overline{0}) = 0$*, and* $g(\overline{x}) = g(\overline{y})$ *if* $\overline{y}$ *is a permutation of* $\overline{x}$*.*

Let $d : X \times X \longrightarrow \mathbb{R}^+$ be a distance over $X$. Let $g : \bigcup_{n\geq 1} (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$ be an aggregation function. Now we can define $d^g : X \times \mathcal{P}^*(X) \longrightarrow \mathbb{R}^+$ a "distance"[2] between elements of $X$ and non empty subsets of $X$ in the following way:

$$d^g(x, V) = g(d(x, v_1), ..., d(x, v_n)), \text{ where } V = \{v_1, ..., v_n\}$$

Note that due to the fact $g$ is an aggregation function (in particular, the output does not depend upon the order in which the arguments are presented), $d^g$ is well defined.

Associated to the most common aggregation functions we have the following distances:

[1] Actually, we are not asking for all the properties of aggregation functions in the literature.

[2] Strictly speaking this is not a distance but, by abuse, we will call this kind of functions distances.

- $d^{\sigma}(x, V) := \sum_{y \in V} d(x, y)$,

  where $\sigma =: \bigcup_{n \geq 1} (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$ is defined by $\sigma(x_1, ..., x_n) = \sum_{i=1}^{n} x_i$

- $d^{min}(x, V) := min\{d(x, y) : y \in V\}$,

  where $min : \bigcup_{n \geq 1} (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$ is defined by $min(x_1, ..., x_n) = min\{x_i : 1 \leq i \leq n\}$

- $d^{max}(x, V) := max\{d(x, y) : y \in V\}$,

  where $max : \bigcup_{n \geq 1} (\mathbb{R}^+)^n \longrightarrow \mathbb{R}^+$ is defined by $max(x_1, ..., x_n) = max\{x_i : 1 \leq i \leq n\}$

**Definition 4** *Let $X$ be a set of alternatives. Let $d$ be a distance over $X$. Let $g$ be an aggregation function. A preference (a total pre-order) $\preceq$ is $d^g-$consistent if there exists $A \in \mathcal{P}^*(X)$ such that*

$$x \preceq y \Longleftrightarrow d^g(x, A) \leq d^g(y, A)$$

**Example 2** *Let $X = \{0, 1\}^3$ and Let $d$ be the Hamming distance.*

- *The following total preorder is $d^{min}-$consistent, with $A = \{(0, 0, 0)\}$.*

$$\begin{array}{c} (1,1,1) \\ (1,1,0)\ (1,0,1)\ (0,1,1) \\ (0,0,1)\ (0,1,0)\ (1,0,0) \\ (0,0,0) \end{array}$$

- *The following total preorder is $d^{\sigma}-$consistent, with $A = \{(0, 0, 0), (0, 0, 1)\}$.*

$$\begin{array}{c} (1,1,1)\ (1,1,0) \\ (1,0,1)\ (0,1,1)\ (0,1,0)\ (1,0,0) \\ (0,0,1)\ (0,0,0) \end{array}$$

- *The following total preorder is $d^{\sigma}-$consistent, with $A = \{(0, 0, 0), (1, 0, 0)\}$.*

$$\begin{array}{c} (0,0,1)\ (0,1,1)\ (0,1,0)\ (0,0,0) \\ (1,1,1)\ (1,1,0)\ (1,0,1)\ (1,0,0) \end{array}$$

**Definition 5** *A profile $u = (\preceq_1, \preceq_2, \ldots, \preceq_n)$ is $d^g-$consistent if for all $i$, $\preceq_i$ is $d^g-$consistent.*

In [12] appears the notion of $d$-consistent preference (profile). Is easy to see that the notion of $d$-consistent preference is a particular case of $d^{min}-$consistent in which the set $A$ is the lowest level of the preference $\preceq$. Actually, the profiles used in logical merging ($\Delta^{\Sigma}$ or $\Delta^{G}max$) are all $d^{min}-$consistent, in fact $d$-consistent (see [10]).

## 3 Arrow's Theorem for structured profiles

Because we want to study the social choice functions for $d^g-$consistent profiles we will modify consequently some of the postulates.

A social choice function $f$ satisfies the $d^g$*-consistent domain condition* if $X$ has at least 3 elements and $f$ is defined for all $d^g-$consistent profiles.

A social choice function $f$ satisfies the $d^g-$*consistent Transitive Explanations* if for any $d^g-$consistent profile $u$ there exists a total pre-order $\preceq_u$ such that for all agenda $V$, $f_u(V) = min(V, \preceq_u)$.

What is interesting here is that if we modify the hypotheses of Arrow's Theorem, changing Standard Domain by $d^g$-consistent domain and Transitive Explanations by $d^g-$consistent Transitive Explanations, then the theorem will be true when $d^g$ satisfies some properties. In particular, the richness property, which we define as follows:

**Definition 6 (Richness Property)** *A distance function $d^g : X \times \mathcal{P}^*(X) \longrightarrow \mathbb{R}^+$ (where $d$ is a distance and $g$ is an aggregation function) satisfies the richness property if for every triple $x, y, z \in X$ all different between them, the following conditions hold:*

*i)* $\exists Y \subseteq X \left[d^g(x,Y) < d^g(y,Y) < d^g(z,Y)\right]$,

*ii)* $\exists Y \subseteq X \left[d^g(x,Y) = d^g(y,Y) < d^g(z,Y)\right]$,

*iii)* $\exists Y \subseteq X \left[d^g(x,Y) < d^g(y,Y) = d^g(z,Y)\right]$

*In such a case, we will say that $d^g$ is a rich distance.*

Now we are ready to set the modified Arrow's Theorem:

**Theorem 2 (Arrow's Theorem for $d^g$-consistent profiles)** *Let $d^g$ be a rich distance between elements of $X$ and non empty subsets of $X$ and let $f$ be a social choice function satisfying the following conditions:*

1. *$d^g-$consistent domain,*
2. *$d^g-$consistent Transitive Explanations ,*
3. *Independence of Irrelevant Alternatives and*
4. *Weak Pareto condition.*

*Then $f$ has a dictator.*

The proof of this theorem is not difficult. It is enough to follow a standard proof of the classical Arrow's Theorem and to remark that the richness property allows to build the preferences required in such a classical proof.

It is interesting to remark that, in general, the class of $d^s$-consistent profiles is a proper class of the class of all the profiles. Thus, the previous theorem is not a trivial one. In order to see that, we establish the following theorem.

**Theorem 3** *Let $X = \{0,1\}^3$ and $d$ the Hamming distance over $X$. Then for each non empty $A \subseteq X$, there are $y_1, y_2 \in X$, $y_1 \neq y_2$ such that*

$$d^s(y_1, A) = d^s(y_2, A) \tag{1}$$

The following observations concerning $X = \{0,1\}^n$, $N = \{1,\ldots,n\}$ and $d$ -the Hamming distance on $X$- are very useful:

**O**1 $x = (x_1, ..., x_n) \in X$ iff $x_i \in \{0,1\}, \forall i \in N$.

For each $i \in N$ we define

$$\overline{x_i} = \begin{cases} 0 & \text{si } x_i = 1 \\ 1 & \text{si } x_i = 0 \end{cases}$$

If $x = (x_1, ..., x_n) \in X$ then we define $\overline{x} = (\overline{x_1}, ..., \overline{x_n})$

**O**2 If $d(x,y) = t$ then $d(\overline{x}, y) = n - t$.

**O**3 For any vector $x = (x_1, \ldots, x_n) \in X$ there is a unique vector $y \in X$ such that $d(x,y) = n$. Actually, $y = \overline{x}$.

The following lemma summarizes some observations expressing the symmetry in the hypercube structure given by the Hamming distance and the sum.

**Lemma 1** *Let $X = \{0,1\}^n$ and let $d$ be the Hamming distance on $X$; then the following conditions hold:*

*(i) For any $A, B \in \mathcal{P}^*(X)$ and any $x \in X$ such that $A \cap B = \emptyset$, $d^\sigma(x, A \cup B) = d^\sigma(x, A) + d^\sigma(x, B)$.*

*(ii) For any $x, y \in X$, $d^\sigma(x, X) = d^\sigma(y, X)$. More precisely, for any $x \in \{0,1\}^n$ we have*

$$d(x, X) = \sum_{i=1}^{n} \binom{n}{i} \cdot i \tag{2}$$

*In particular, if $n = 3$, then $d^\sigma(x, X) = 12$, for any $x \in X$.*

*(iii) Let $A \in \mathcal{P}^*(X)$ and $x, y \in X$. If $d^\sigma(x, A) = d^\sigma(y, A)$ then $d^\sigma(x, X \setminus A) = d^\sigma(y, X \setminus A)$.*

*Proof.* (i) is straightforward by definition of $d^\sigma$.

In order to prove (ii), it is enough to see that the equation (2) holds. Notice that for a fix vector $x$ there are hay exactly $\binom{n}{i}$ vectors having distance $i$ from $x$. From this observation the equation (2) follows.

In order to prove (iii), we use (i) and (ii). Actually, by (i) and (ii) for any $x \in X$ and any $A \in \mathcal{P}^*(X)$, $d^\sigma(x, A) + d^\sigma(x, X \setminus A) = d^\sigma(x, X) = \sum_{i=1}^{n} \binom{n}{i} \cdot i$. Now suppose that $d^\sigma(x, A) = d^\sigma(y, A)$. Then,

$$\begin{aligned} d^\sigma(x, X \setminus A) &= \left[\sum_{i=1}^{n} \binom{n}{i} \cdot i\right] - d^\sigma(x, A) \\ &= \left[\sum_{i=1}^{n} \binom{n}{i} \cdot i\right] - d^\sigma(y, A) \\ &= d^\sigma(y, X \setminus A) \end{aligned}$$

∎

*Proof of Theorem 3.* Let $A$ be a nonempty subset of $X$. First we consider the $8$ possible cases according to the cardinality of $A$, *i.e.* $|A| = i$ for $i = 1, \ldots, 8$.

The case $i = 8$, *i.e.* $A = X$, follows from part (ii) of Lemma 1.

By part (iii) of Lemma 1, it is enough to consider $i = 1, \ldots, 4$.

We adopt the following notation: if $a \in \{0, 1\}$ then $\overline{a} \in \{0, 1\}$ is defined by 0 iff $a = 1$. If $x$ is a vector in $\{0, 1\}^3$, say $x = (a_1, a_2, a_3)$, we define $\overline{x} = (\overline{a_1}, \overline{a_2}, \overline{a_3})$

Case $|A| = 1$. That is, $A = \{(a_1, a_2, a_3)\}$. Taking $y_1 = (\overline{a_1}, a_2, a_3)$ and $y_2 = (a_1, \overline{a_2}, a_3)$ the equation (1) holds.

Case $|A| = 2$. That is, $A = \{a, b\}$. Taking $y_1 = a$ and $y_2 = b$ the equation (1) holds.

Case $|A| = 3$. In this case $A = \{a, b, c\}$.

**Subcase 1.** One of the three vectors is equidistant from the other two vectors. Without loss of generality, suppose $d(a, b) = d(a, c)$. Then,

$$\begin{aligned} d^\sigma(b, A) &= d(b, a) + d(b, b) + d(b, c) \\ d^\sigma(c, A) &= d(c, a) + d(c, b) + d(c, c) \end{aligned}$$

Thus, taking $y_1 = b$ and $y_2 = c$, the equation (1) holds.

**Subcase 2.** None of the vectors is equidistant from the other two vectors. That is,

$$d(a, b) \neq d(a, c) \;\wedge\; d(a, b) \neq d(c, b) \;\wedge\; d(a, c) \neq d(c, b)$$

Without loss of generality, we may suppose

$$d(a,b)=1 \ \wedge \ d(b,c)=2 \ \wedge \ d(a,c)=3$$

and that the situation is as follows:

$$a=(a_1,a_2,a_3)$$
$$b=(\overline{a_1},a_2,a_3)$$
$$c=(\overline{a_1},\overline{a_2},\overline{a_3})$$

Consider $y_1, y_2 \in X$ to be the only two vectors such that $d(y_1,a)=1=d(y_2,b)$. Then, by the observation **O**2, $d(y_1,c)=d(y_2,c)=2$.

But then it is clear that

$$d(y_1,A)=1+1+2=d(y_2,A)$$

Therefore, the equation (1) holds.

Case $|A|=4$. We consider subcases mutually exclusive. We use the symmetry of the cube to simplify the reasoning.

Subcase 1 There exists $a \in A$ such that $\overline{a} \in A$.

Then, $A=\{a,\overline{a},b,c\}$ and define $y_1=b,\ y_2=c$.

Thus, we have:

$$\begin{array}{rcll} d^\sigma(y_1,A) & = & d(b,a)+d(b,\overline{a})+d(b,b)+d(b,c) & \\ & = & n+d(b,c) & \text{(by } \mathbf{O}2\text{)} \\ d^\sigma(y_2,A) & = & d(c,a)+d(c,\overline{a})+d(c,b)+d(c,c) & \\ & = & n+d(c,b) & \text{(by } \mathbf{O}2\text{)} \end{array}$$

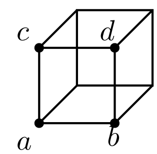
Therefore, the equation (1) holds.

Subcase 2. There is no $a \in A$ such that $\overline{a} \in A$. That is, $d(x,y)<3$, for any pair $x,y \in A$. Put $A=\{a,b,c,d\}$.

Now we consider three possibilities that cover all the possibilities un this subcase:

- The four points of $A$ are in the same face of the cube.

  Without loss of generality, by the symmetry of the cube, we may suppose that the situation is as in the following figure:

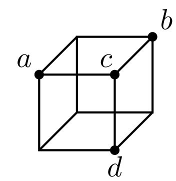



that is, $d(a,b) = d(a,c) = d(b,d) = d(c,d) = 1$ y $d(a,d) = d(b,c) = 2$.

Putting $y_1 = a$ y $y_2 = b$, the equation (3) holds.

- Three points in a face (say $a$, $c$ and $d$ are in the same face).

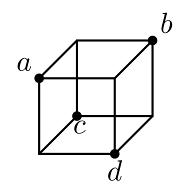
  Without loss of generality, due to the symmetry of the cube, we may suppose that the situation is as in the following figure:




  In this situation, we put $y_1 = d$ and $y_2 = b$. A straightforward verification shows that the equation (3) holds.

- Two points in a face.

  Without loss of generality, due to the symmetry of the cube, we may suppose that the situation is as in the following figure:




  that is, $d(a,c) = d(b,c) = d(c,d) = d(a,b) = d(a,d) = d(b,d) = 2$.

  Putting $y_1 = a$ y $y_2 = b$, the equation (3) holds. ∎

As a straightforward corollary, we obtain the following theorem:

**Theorem 4** *There is no a linear order over $\{0,1\}^3$ that is $d^\sigma$-consistent.*

As a consequence of that, we have the following:

**Theorem 5** *The class of $d^\sigma$-consistent profiles is a proper class of the class of all the profiles.*

## 4 Some rich distances and some not rich distances

In this section we will consider $X = \{0,1\}^n$ with $n \in \mathbb{N} \setminus \{0\}$ and $d$ will be the Hamming distance.

**Remark 1** *It is clear that if $n = 1$, $X$ has only two elements. Therefore, for any aggregation function $g$, $d^g$ is trivially rich.*

However, the following result shows that in case $n \geq 2$ there are distances that don't satisfy the richness property:

**Theorem 6** *If $d$ is the Hamming distance then $d^{min}$ is not rich for every $n \geq 2$.*

Unlike the previous theorem, if we change the aggregation function, we get a rich distance for almost all the cases.

**Theorem 7** *Let $X = \{0,1\}^n$ with $n \geq 3$ and $d$ the Hamming distance on $X$. Then $d^\sigma$ is rich.*

By Remark 1 and Theorem 7, the only integer for which we don't know if $d^\sigma$ is rich is $n = 2$. Actually, for $n = 2$, $d^\sigma$ fails to be rich; this is our next theorem:

**Theorem 8** *Let $X = \{0,1\}^2$ and $d$ the Hamming distance on $X$. Then $d^\sigma$ is not rich.*

## 5 Concluding remarks

In this work, we have set some first bases in order to establish a general impossibility theorem and a general manipulability theorem for structured profiles.

Theorem 2 tells us that the richness property is a sufficient condition on $d^g$ for the impossibility for the class of $d^g$-consistent profiles. Unfortunately, we don't know if the richness property is a necessary condition. Thus, even in presence of Theorem 6 we don't know yet if Arrow's Theorem holds for the class of $d^{min}$-consistent profiles.

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