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Approximate Controllability of a System of Parabolic Equations with Delay

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Abstract

In this paper we give a necessary and sufficient conditions for the approximate controllability of the following system of parabolic equations with delay

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} &= D\Delta z + Lz_t + Bu(t), \quad t > 0, \\ \dfrac{\partial z}{\partial \eta} &= 0, \quad t > 0, \quad x \in \partial\Omega \\ z(0,x) &= \phi_0(x), \quad x \in \Omega, \\ z(s,x) &= \phi(s,x), \quad s \in [-\tau, 0), \quad x \in \Omega \end{cases}$$

where Ω is a bounded domain in $I\!R^N$, D is a $n \times n$ non diagonal matrix whose eigenvalues are semi-simple with non negative real part, the control u belong to $L^2([0,r];U)$ $(U = L^2(\Omega, I\!R^m))$ and $B \in L(U,Z)$ with $Z = L^2(\Omega; I\!R^n)$. The standard notation $z_t(x)$ defines a function from $[-\tau, 0]$ to $I\!R^n$ (with x fixed) by $z_t(x)(s) = z(t+s,x), -\tau \leq s \leq 0$. Here $\tau \geq 0$ is the maximum delay, which is suppose to be finite. We assume that the operator $L : L^2([-\tau,0];Z) \longrightarrow Z$ is linear and bounded and $\phi_0 \in Z$, $\phi \in L^2([-\tau,0];Z)$.

Resumen

En este artículo se dan condiciones necesarias y suficientes para la controlabilidad aproximada del siguiente sistema de ecuaciones parabólicas con retardo:

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} &= D\Delta z + Lz_t + Bu(t), \quad t > 0, \\ \dfrac{\partial z}{\partial \eta} &= 0, \quad t > 0, \quad x \in \partial\Omega \\ z(0,x) &= \phi_0(x), \quad x \in \Omega, \\ z(s,x) &= \phi(s,x), \quad s \in [-\tau, 0), \quad x \in \Omega \end{cases}$$

donde Ω es un dominio acotado en $I\!R^N$, D es una matriz $n \times n$ no diagonal, cuyos autovalores son semisimple con parte real no negativa, el control u pertenece a $L^2([0,r];U)$ $(U = L^2(\Omega, I\!R^m))$ y $B \in L(U,Z)$ con $Z = L^2(\Omega; I\!R^n)$. La notación estandar $z_t(x)$ define una función de $[-\tau, 0]$ en $I\!R^n$ (con x fijo) dada por $z_t(x)(s) = z(t+s,x), -\tau \leq s \leq 0$. Aquí $\tau \geq 0$ es el máximo retardo, el cual se supone finito. Se supone que el operador $L : L^2([-\tau,0];Z) \longrightarrow Z$ es lineal y acotado y $\phi_0 \in Z$, $\phi \in L^2([-\tau,0];Z)$.

Key words. functional partial parabolic equations, variation constant formula, strongly continuous semigroups, approximate controllability.

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Running Title:APPROXIMATE CONTROLLABILITY FOR FPD Eqs.

1 Introduction

In this paper we give a necessary and sufficient conditions for the approximate controllability of the following system of parabolic equations with delay

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} & = & D\Delta z + Lz_t + Bu(t), \quad t, > 0, \\ \dfrac{\partial z}{\partial \eta} & = & 0, \quad t > 0, \quad x \in \partial\Omega, \\ z(0,x) & = & \phi_0(x), \quad x \in \Omega, \\ z(s,x) & = & \phi(s,x), \quad s \in [-\tau, 0), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $I\!R^N$, D is a $n \times n$ non diagonal matrix whose eigenvalues are semi-simple with non negative real part, the control u belong to $L^2([0,r];U)$ $(U = L^2(\Omega, I\!R^m))$ and $B \in L(U,Z)$ with $Z = L^2(\Omega; I\!R^n)$. The standard notation $z_t(x)$ defines a function from $[-\tau, 0]$ to $I\!R^n$ (with x fixed) by $z_t(x)(s) = z(t+s,x), -\tau \leq s \leq 0$. Here $\tau \geq 0$ is the maximum delay, which is suppose to be finite. We assume that the operator $L : L^2([-\tau,0]; Z) \longrightarrow Z$ is linear and bounded and $\phi_0 \in Z,\ \phi \in L^2([-\tau,0]; Z)$.

Our work is motivated by the papers du to Borisovic J.U.G and Turbabin A.S. (see [2]) and H.T. Banks (see [1]). There they found a variational constant formula for the following system of nonhomogeneous differential equation with delay

$$\begin{cases} z'(t) & = & Lz_t + f(t), \quad t > 0, \quad z \in I\!R^n, \\ z(0) & = & z_0, \\ z(s) & = & \phi(s), \ s \in [-\tau, 0), \end{cases} \tag{1.2}$$

where $f : I\!R^+ \to I\!R^n$ is a suitable function. The function z_t is defined from $[-\tau, 0]$ to $I\!R^n$ by $z_t(s) = z(t+s), -\tau \leq s \leq 0$. Here $\tau \geq 0$ is the maximum delay, which is suppose to be finite. We assume that the operator $L : L^2([-\tau,0]; I\!R^n) \longrightarrow I\!R^n$ is linear and bounded, and $z_0 \in I\!R^n,\ \phi \in L^2([-\tau,0]; I\!R^n)$. Under some conditions they prove the existence and the uniqueness of solutions for this system and associate to it a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in the Banach space $\mathbb{M}_2([-\tau,0]; I\!R^n) = I\!R^n \oplus L_2([-\tau,0]; I\!R^n)$.

Therefore, the systems (1.2) is equivalent to the following systems of ordinary differential equations in $\mathbb{M}_2$:

$$\begin{cases} \dfrac{dW(t)}{dt} = \Lambda W(t) + \Phi(t), \quad t > 0, \\ W(0) \;= W_0 = (z_0, \phi(\cdot)), \end{cases} \tag{1.3}$$

where Λ is the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$ and $\Phi(t) = (f(t), 0)$.

Hence, the solution of system (1.2) is given by the variational constant formula o mild solution:

$$W(t) = T(t)W_0 + \int_0^t T(t-s)\Phi(s)ds. \tag{1.4}$$

This formula has been extended in [3] for a system of parabolic equation with delay and we will use it to define approximate controllability and prove our main results.

To the best of our knowledge, this is the first time that this formula is used to characterize the approximate controllability of systems of parabolic equation with delay, and could be applied to those system of PDEs that can be rewritten in the form $\frac{\partial}{\partial t}u = D\Delta u$, like damped nonlinear vibration of a string or a beam, thermoplastic plate equation, etc; for information about this, one can see the paper by Luiz de Oliveira ([8]).

As a particular case we shall consider the following controlled system of parabolic equations with delay

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} &= D\Delta z + \sum_{i=1}^{p} A_i z(t-h_i, x) + Bu(t), \quad t > 0, \\ \dfrac{\partial z}{\partial \eta} &= 0, \quad t > 0, \quad x \in \partial\Omega \\ z(0,x) &= \phi_0(x), \quad x \in \Omega, \\ z(s,x) &= \phi(s,x), \quad s \in [-\tau, 0), \quad x \in \Omega \end{cases} \tag{1.5}$$

where $0 < h_1 < h_2 < \cdots < h_p$ represent the point delays, $\tau = \text{máx}\{h_i : i = 1, 2, \ldots p\}$, $B, A_i \in L(\boldsymbol{C}^n)$, $i = 1, 2, \ldots p$, u belong to $L^2([0, r]; U)$ $(U = L^2(\Omega, I\!R^n))$ and $\phi_0 \in Z,\ \phi \in L^2([-\tau, 0]; Z)$ with $Z = U$.

2 Abstract Formulation of the Problem

In this section we choose a Hilbert Space where system (1.1) can be written as an abstract functional differential equation, to this end, we consider the following hypothesis:

H1). The matrix D is semi simple (block diagonal) and the eigenvalues $d_i \in \boldsymbol{C}$ of D satisfy $\text{Re}(d_i) \geq 0$. Consequently, if $0 = \lambda_1 < \lambda_2 < ... < \lambda_n \longrightarrow \infty$ are the eigenvalues of $-\Delta$ with

homogeneous Neumann boundary conditions, then there exists a constant $M \geq 1$ such that : $\|e^{-\lambda_n Dt}\| \leq M, \quad t \geq 0, \quad n = 1, 2, 3, ...$
H2). For all $I > 0$ and $z \in L^2_{loc}([-\tau, 0); Z)$ we have the following inequality

$$\int_0^t \mid Lz_s \mid ds \leq M_0(t) \mid z \mid_{L^2([-\tau,t),Z)}, \ \ \forall t \in [0, I],$$

where $M_0(\cdot)$ is a positive continuous function on $[0, \infty)$.

Consider $H = L^2(\Omega, I\!R)$ and $0 = \lambda_1 < \lambda_2 < ... < \lambda_n \longrightarrow \infty$ the eigenvalues of $-\Delta$, each one with finite multiplicity γ_n equal to the dimension of the corresponding eigenspace. Then :

(i) There exists a complete orthonormal set $\{\phi_{n,k}\}$ of eigenvectors of $-\Delta$.

(ii) For all $\xi \in D(-\Delta)$ we have

$$-\Delta\xi = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < \xi, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n \xi, \tag{2.1}$$

where $< \cdot, \cdot >$ is the inner product in H and

$$E_n x = \sum_{k=1}^{\gamma_n} < \xi, \phi_{n,k} > \phi_{n,k}. \tag{2.2}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in H and
$\xi = \sum_{n=1}^{\infty} E_n \xi, \quad \xi \in H.$
(iii) Δ generates an analytic semigroup $\{T_\Delta(t)\}$ given by

$$T_\Delta(t)\xi = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n \xi. \tag{2.3}$$

Now, we denote by Z the Hilbert space $L^2(\Omega, I\!R^n)$ and define the following operator

$$A : D(A) \subset Z \longrightarrow Z, \quad A\psi = -D\Delta\psi$$

with $D(A) = H^2(\Omega, I\!R^n) \cap H^1_0(\Omega, I\!R^n)$.
Therefore, for all $z \in D(A)$ we obtain,

$$Az = \sum_{n=1}^{\infty} \lambda_n D P_n z,$$

and

$$z = \sum_{n=1}^{\infty} P_n z, \quad \| z \|^2 = \sum_{n=1}^{\infty} \| P_n z \|^2, \quad z \in Z,$$

where

$$P_n = diag(E_n, E_n, ..., E_n),$$

is a family of complete orthogonal proyections in Z.

Consequently, system (1.1) can be written as an abstract functional differential equation in Z:

$$\begin{cases} \dfrac{dz(t)}{dt} = -Az(t) + Lz_t + Bu(t), \quad t > 0, \\ z(0) = \phi_0, \\ z(s) = \phi(s), \ \ s \in [-\tau, 0), \end{cases} \tag{2.4}$$

where $u \in L^2([0, r]; U)$ and $B : U \to Z$.

We shall use the following result from [3]: The equation (2.4) can be written as an ordinary differential equation in the Hilbert space $\mathbb{M}_2([-\tau, 0]; Z) = Z \oplus L^2([-\tau, 0]; Z)$ as follows:

$$\begin{cases} \dfrac{dW(t)}{dt} = \Lambda W(t) + \mathcal{B}u(t), \quad t > 0, \\ W(0) \ \ = W_0, \end{cases} \tag{2.5}$$

where Λ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ define by:

$$T(t) \begin{pmatrix} \phi_0 \\ \phi(\cdot) \end{pmatrix} = \begin{pmatrix} w(t) \\ w(t + \cdot) \end{pmatrix}, \tag{2.6}$$

where $w(\cdot)$ is the only Mild Solution of the system

$$\begin{cases} \dfrac{dz(t)}{dt} = -Az(t) + Lz_t, \quad t > 0, \\ z(0) = \phi_0, \\ z(s) = \phi(s), \quad s \in [-\tau, 0), \end{cases} \tag{2.7}$$

and $\mathcal{B} : U \longrightarrow \mathbb{M}_2$, is given by $\mathcal{B}u = (Bu, 0)^T$.

3 Approximate Controllability of the System

For all $W_0 \in \mathbb{M}_2$ and a control $u \in L^2([0, r]; U)$ the equation (2.5) admits only one mild solution given by:

$$W(t) = T(t)W_0 + \int_0^t T(t - s)\mathcal{B}u(t)ds, \ 0 \leq t \leq r \tag{3.1}$$

Definition 3.1 *The system (2.5) is said to be approximately controllable on* $[0, r]$*, if for all* $W_0, W_1 \in \mathbb{M}_2$ *and* $\varepsilon > 0$*, there exists a control* $u \in L^2(0, r; U)$ *such that the corresponding solution* $W(t)$ *of (3.1) satisfies*

$$\| W(r) - W_1 \| < \epsilon$$

Consider the following linear operators $\mathcal{B}^r : L^2(0,r;U) \to \mathbb{M}_2, \quad L_{\mathcal{B}} : \mathbb{M}_2 \to \mathbb{M}_2$, define by

$$\mathcal{B}^r u = \int_0^r T(s)\mathcal{B}u(s)ds \ \text{ and } \ L_{\mathcal{B}}W = \mathcal{B}^r\mathcal{B}^{r*}W = \int_0^r T(s)\mathcal{B}\mathcal{B}^*T^*(s)W ds.$$

Then, the following theorem can be found in a general form for evolution equation in [4].

Theorem 3.2 *System (2.5) is approximately controllable on* $[0,r]$ *if, and only if, any one of the following conditions hold:*
a) $\overline{Rang(\mathcal{B}^r)} = \mathbb{M}_2$.
b) $\mathcal{B}^*T^*(s)z = 0, \ \forall s \in [0,r] \Longrightarrow z = 0$.
c) $L_{\mathcal{B}} > 0$.

In [3] it was proved that the semigroup $\{T(t)\}_{t\geq 0}$, associated to (2.5) can be represented as follows

$$T(t)W = \sum_{n=1}^{\infty} T_n(t)Q_nW, \ \ W \in \mathbb{M}_2, \ \ t \geq 0, \tag{3.2}$$

where

$$Q_n = \begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix},$$

with $(\widetilde{P}_n\phi)(s) = P_n\phi(s), \phi \in L^2([-\tau,0];Z), \ s \in [-\tau,0]$, and $\{T_n(t)\}_{t\geq 0}$ is a family of strongly continuous semigroups in $\mathbb{M}_2^n = Q_n\mathbb{M}_2$ define by

$$T_n(t)\begin{pmatrix} w_n^0 \\ w_n \end{pmatrix} = \begin{pmatrix} W^n(t) \\ W^n(t+\cdot) \end{pmatrix}, \ \begin{pmatrix} w_n^0 \\ w_n \end{pmatrix} \in \mathbb{M}_2^n,$$

where $W^n(\cdot)$ is the only solution of the initial value problem

$$\begin{cases} \dfrac{dw(t)}{dt} = -\lambda_n Dw(t) + L_n w_t, \quad t > 0, \\ w(0) = w_n^0, \\ w(s) = w_n(s), \quad s \in [-\tau, 0), \end{cases} \tag{3.3}$$

and $L_n = L\widetilde{P}_n$. The infinitesimal generator of this semigroup is given by

$$\Lambda_n \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix} = \begin{pmatrix} -\lambda_n D w_n^0 + L_n w_n(\cdot) \\ \dfrac{\partial w_n(\cdot)}{\partial s} \end{pmatrix}$$

with

$$D(\Lambda_n) = \left\{ \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix} \in \mathbb{M}_2^n : w_n \ es\ a.c., \ \frac{\partial w_n(\cdot)}{\partial s} \in L_2([-\tau,0];P_nZ) \ \text{ and } \ w_n = w_n^0 \right\}.$$

Theorem 3.3 *(necessary condition for approximate controllability) If the system (2.5) is approximately controllable on* $[0, r]$*, then each of the following systems is approximately controllable on* $[0, r]$

$$y' = \Lambda_j Q_j y + Q_j \mathcal{B} u(t); \;\; j = 1, 2, \ldots \,. \tag{3.4}$$

Proof . For the purpose of contradiction, let us assume that system (2.5) is approximately controllable on $[0, r]$ and there exists J such that the system

$$y' = \Lambda_J Q_J y + Q_J \mathcal{B} u(t); \;\; y \in Rang(Q_J).$$

is not approximately controllable on $[0, r]$. Then, there exists $V_J \in Rang(Q_J)$ such that:

$$(Q_J \mathcal{B})^* T_J^*(t) V_J = 0, \;\; t \in [0, r] \; and \; V_J \neq 0. \tag{3.5}$$

On the other hand, from part b) of Theorem 3.2 we have that:

$$\mathcal{B}^* T^*(t) W = 0, \;\; \forall t \in [0, r] \Longrightarrow W = 0.$$

Now, letting $W = Q_J V_J = V_J$, we obtain:

$$\begin{array}{rcl} \mathcal{B}^* T^*(t) W & = & \mathcal{B}^* \sum_{n=1}^{\infty} T_n^* Q_n W \\ & = & \mathcal{B}^* T_J^*(t) Q_J V_J \\ & = & (Q_J \mathcal{B})^* T_J^*(t) W \\ & = & 0 \end{array}$$

This implies that $V_J = 0$, which contradicts the assumption (3.5). $\square$

Theorem 3.4 *(sufficient condition for approximate controllability)*
Suppose that $P_j BB^* = BB^* P_j, \;\; j = 1, 2, \ldots .$ *Then, the approximate controllability of all the following systems on* $[0, r]$

$$y' = \Lambda_j Q_j y + Q_j \mathcal{B} u(t); \;\; j = 1, 2, \ldots \,, \tag{3.6}$$

implies the approximate controllability of the system (2.5) on $[0, r]$.

Proof Suppose that each of the systems (3.6) is approximately controllable on $[0, r]$ and define the operators

$$\mathcal{B}_j^r : L^2(0, r; U) \longrightarrow Rang(Q_j), \;\; L_{\mathcal{B}_j} : Rang(Q_j) \longrightarrow Rang(Q_j),$$

by

$$\mathcal{B}_j^r u = \int_0^r T_j(s)\mathcal{B}_j u(s)ds, \quad L_{\mathbf{B_j}} = \mathcal{B}_j^r(\mathcal{B}_j^r)^*.$$

where $\mathcal{B}_j = Q_j\mathcal{B}$. Then,

$$L_{\mathbf{B_j}} y = \int_0^r T_j(s)\mathcal{B}_j\mathcal{B}_j^* T_j^*(s) y ds, \quad y \in Rang(Q_j).$$

Therefore, from Theorem 3.2 part c) (or Theorem 4.1.7 from [4]) we have that $L_{\mathbf{B_j}} > 0, \ j = 1, 2, \ldots$

On the other hand, if $P_j BB^* = BB^* P_j$, then $Q_j\mathcal{B}\mathcal{B}^* = \mathcal{B}\mathcal{B}^* Q_j$, and hence

$$\begin{aligned} L_{\mathcal{B}} W &= \int_0^r (\sum_{j=1}^{\infty} T_j(s)Q_j)\mathcal{B}\mathcal{B}^*(\sum_{k=1}^{\infty} T_k^*(s)Q_k W)ds \\ &= \int_0^r \sum_{j=1}^{\infty} T_j(s)\mathcal{B}_j\mathcal{B}_j^* T_j^*(s)Q_j W ds \\ &= \sum_{j=1}^{\infty} \int_0^r T_j(s)\mathcal{B}_j\mathcal{B}_j^* T_j^*(s)Q_j W ds \\ &= \sum_{j=1}^{\infty} L_{\mathcal{B}_j} Q_j W. \end{aligned}$$

Consequently, $L_{\mathcal{B}} > 0$. and the system (2.5) is approximately controllable on $[0, r]$. $\square$

As a consequence of Theorem 3.4 and Theorem 4.2.10 from [5] we can prove the following theorem.

Theorem 3.5 *The system (1.5) is approximately controllable on on $[0, r]$ if, and only if,*

$$\begin{aligned} Rank(\Delta_j(\lambda) : B_j) &= n\gamma_j, \\ Rank(P_j A_p : B_j) &= n\gamma_j, \end{aligned}$$

where

$$\Delta_j(\lambda) = \lambda I_{R(P_j)} - \lambda_j P_j D - \sum_{i=1}^{p} P_j A_j e^{-\lambda h_j}, \quad j = 1, 2, \ldots .$$

4 Conclusion

As one can see, this work can be generalized to a broad class of functional reaction diffusion equation in a Hilbert space Z of the form:

$$\begin{cases} \dfrac{dz(t)}{dt} = \mathcal{A}z(t) + Lz_t + Bu(t), \quad z \in Z, u \in U, t > 0, \\ z(0) = \phi_0 \\ z(s) = \phi(s), \quad s \in [-\tau, 0), \end{cases} \tag{4.1}$$

where $\mathcal{A}$ is given by

$$\mathcal{A}z = \sum_{n=1}^{\infty} A_n P_n z, \qquad z \in D(\mathcal{A}), \tag{4.2}$$

Z and U are Hilbert spaces, $L : L^2([-\tau, 0]; Z) \longrightarrow Z$ is linear and bounded $B \in L(U, Z)$, the control u belong to $L^2([0, r]; U)$ and $\phi_0 \in Z$, $\phi \in L^2([-\tau, 0]; Z)$. Some examples of this class are the following well known systems of partial differential equations with delay:

Example 4.1 *The equation modeling the damped flexible beam:*

$$\left\{ \begin{array}{rcl} \dfrac{\partial^2 z}{\partial^2 t} & = & -\dfrac{\partial^3 z}{\partial^3 x} + 2\alpha \dfrac{\partial^3 z}{\partial t \partial^2 x} + z(t-\tau, x) + u(t, x) \quad t \geq 0, \quad 0 \leq x \leq 1 \\[2ex] z(t, 1) & = & z(t, 0) = \dfrac{\partial^2 z}{\partial^2 x}(0, t) = \dfrac{\partial^2 z}{\partial^2 x}(1, t) = 0, \\[2ex] z(0, x) & = & \phi_0(x), \quad \dfrac{\partial z}{\partial t}(0, x) = \psi_0(x), \quad 0 \leq x \leq 1 \\[2ex] z(s, x) & = & \phi(s, x), \quad \dfrac{\partial z}{\partial t}(s, x) = \psi(s, x), \quad s \in [-\tau, 0), \quad 0 \leq x \leq 1 \end{array} \right. \tag{4.3}$$

where $\alpha > 0$, $u \in L^2(0, r; L^2[0, 1])$, $\phi_0, \psi_0 \in L^2[0, 1]$ and $\phi, \psi \in L^2([-\tau, 0]; L^2[0, 1])$.

Example 4.2 *The strongly damped wave equation with Dirichlet boundary conditions*

$$\left\{ \begin{array}{ll} \dfrac{\partial^2 w}{\partial^2 t} + \eta(-\Delta)^{1/2} \dfrac{\partial w}{\partial t} + \gamma(-\Delta) w = L w_t + u(t, x), & t \geq 0, \quad x \in \Omega, \\[2ex] w(t, x) = 0, \quad t \geq 0, \quad x \in \partial\Omega. & \\[2ex] w(0, x) = \phi_0(x), \quad \dfrac{\partial z}{\partial t}(0, x) = \psi_0(x), \quad x \in \Omega, & \\[2ex] w(s, x) = \phi(s, x), \quad \dfrac{\partial z}{\partial t}(s, x) = \psi(s, x), \quad s \in [-\tau, 0), \quad x \in \Omega, & \end{array} \right. \tag{4.4}$$

where Ω is a sufficiently smooth bounded domain in $I\!R^N$, $u \in L^2(0, r; L^2(\Omega))$, $\phi_0, \psi_0 \in L^2(\Omega)$ and $\phi, \psi \in L^2([-\tau, 0]; L^2(\Omega))$ and $\tau \geq 0$ is the maximum delay, which is supposed to be finite. We assume that the operators $L : L^2([-\tau, 0]; Z) \longrightarrow Z$ is linear and bounded and $Z = L^2(\Omega)$.

Example 4.3 *The thermoelastic plate equation with Dirichlet boundary conditions*

$$\begin{cases} \dfrac{\partial^2 w}{\partial^2 t} + \Delta^2 w + \alpha \Delta \theta = L_1 w_t + u_1(t,x) \quad t \geq 0, \quad x \in \Omega, \\[2ex] \dfrac{\partial \theta}{\partial t} - \beta \Delta \theta - \alpha \Delta \dfrac{\partial w}{\partial t} = L_2 \theta_t + u_2(t,x) \quad t \geq 0, \quad x \in \Omega, \\[2ex] \theta = w = \Delta w = 0, \quad t \geq 0, \quad x \in \partial \Omega. \\[2ex] w(0,x) = \phi_0(x), \quad \dfrac{\partial w}{\partial t}(0,x) = \psi_0(x), \quad \theta(0,x) = \xi_0(x) \quad x \in \Omega, \\[2ex] w(s,x) = \phi(s,x), \quad \dfrac{\partial w}{\partial t}(s,x) = \psi(s,x), \quad \theta(0,x) = \xi(s,x), \quad s \in [-\tau, 0), \quad x \in \Omega, \end{cases} \tag{4.5}$$

where Ω is a sufficiently smooth bounded domain in $I\!R^N$, $u_1, u_2 \in L^2(0, r; L^2(\Omega))$, $\phi_0, \psi_0, \xi_0 \in L^2(\Omega)$ and $\phi, \psi, \xi \in L^2([-\tau, 0]; L^2(\Omega))$ and $\tau \geq 0$ is the maximum delay, which is supposed to be finite. We assume that the operators $L_1, L_2 : L^2([-\tau, 0]; Z) \longrightarrow Z$ are linear and bounded and $Z = L^2(\Omega)$.

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