Universidad de Los Andes
Facultad de Ciencias

NOTAS DE MATEMÁTICA

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No. 225
Serie: Pre-Print

Departamento de Matemáticas
Mérida - Venezuela
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Floquet Torus for Perturbations of Product of Quadratic Maps*

Leonardo Mora

Here we show the existence of Floquet invariant torus for a perturbation of product of quadratic maps. In order to do that we extend the Hopf bifurcation theorem to multiparameter families.

1 Introduction

In this paper we consider the family of mappings

$$h_{(a,\mathbf{b})}(x_1,\ldots,x_n)=(a-x_n^2+\sum_1^{n-1}b_ix_i,x_1,...,x_{n-1}),$$

where the parameters $(a,\mathbf{b})=(a,b_1,...,b_{n-1})\in\mathbb{R}^n$. For the parameter $(a,\mathbf{b})=(a,\mathbf{0})$, it turns out to be that $h^n_{(a,\mathbf{0})}$ is a product of quadratic mappings $x_i\mapsto a-x_i^2$. Here we address the following problem: There exists invariant tori for this family? Even more, are these invariant tori of Floquet Type?

These questions arise in [M], where it is proved that the family $h_{(a,\mathbf{b})}$ exhibits invariant curves for certain parameters near the value $(a,\mathbf{b})=(3/4,\mathbf{0})$. The mechanism to produce these invariant curves was that of finding parameters where Hopf's bifurcation appears. In turn, that was done observing that at the parameter $(3/4,\mathbf{0})$, the mapping $h_{(a,\mathbf{b})}$ has a fixed point all of whose eigenvalues belong to the unit circle. This fact suggests that we could try to find invariant tori of higher dimensions. A numerical experiment, shows that for the case $n=4$ and the value of the parameter $a=0.745$, $b_1=0.01$, $b_2=-0.01$, and $b_3=0.01$ the mapping exhibits a set which seems to be an invariant torus of dimension two and not simply an invariant curve. This set is showed in Figure 1, the picture was made with Mathematica.

In this paper we prove the following

THEOREM 1.1

For the family of mappings $h_{(a,\mathbf{b})}$ and for each $k\geq 1$, there exists an open set $\mathcal{U}_k$ with $(3/4,\mathbf{0})\in\overline{\mathcal{U}_k}$ such that for each $(a,\mathbf{b})\in\mathcal{U}$ the mapping $h_{(a,\mathbf{b})}$ exhibits a C^k-normally attracting invariant torus whose dimension is $s=[n/2]$. Moreover there exits inside $\mathcal{U}_k$ a set $\mathcal{T}_k$ of positive Lebesgue measure, where the invariant torus are of Floquet type.

*This work was partially supported by CDCHT project No. C-1060-01-05-B. Also the author would like to thanks the support of IMPA-Brasil and ICTP-Italy where part of this work was done.

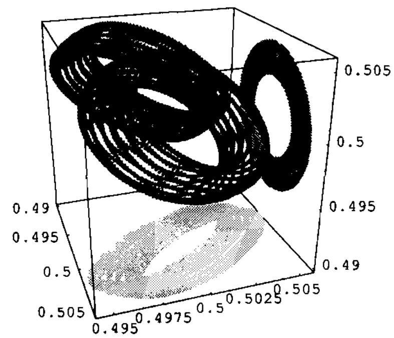

Figure 1. To plot this set we take the point $x_1 = .49, x_2 = .5, x_3 = 0.4995$ and $x_4 = .49$, and the piece of orbit $\{h^n_{(\mathbf{a},\mathbf{b})}(x_1, ..., x_4) : n = 85000, ..., 90000.\}$

Here an invariant torus for a map h is a set V, which is invariant under h $(h(V) = V)$ and C^k-diffeomorphic to $\mathbb{T}^s = \mathbb{R}^s/\mathbb{Z}^s$, the s-dimensional torus. We say that the dynamics on the torus is *parallel* if it is conjugated to a rotation $R_{\boldsymbol{\phi}} : \boldsymbol{\theta} \in \mathbb{T}^s \mapsto \boldsymbol{\theta} + \boldsymbol{\phi} \in \mathbb{T}^s$. It is *quasi-periodic* if the components of $\boldsymbol{\phi}$ are rationally independent over $\mathbb{Z}^s$. The torus V is of *Floquet type* if there can found coordinates $(\boldsymbol{\theta}, \mathbf{r})$ such that h can be written in a neighborhood of V as $h(\boldsymbol{\theta}, \mathbf{r}) = (\boldsymbol{\theta} + \boldsymbol{\phi} + O(\mathbf{r}), \Lambda\mathbf{r} + O(\mathbf{r}))$.

The phenomenon presented above does not appears to be a one parameter phenomenon, to show it we need many parameters parameter. So in order to prove the theorem above, we extend the Hopf's bifurcation theorem to the multiparameter case (see theorem 3.2 in section 3) with all eigenvalues of the fixed point belonging to the unit circle. To the best of our knowledge, this is the first time this version appears with the requested conditions. For the one parameter version please see [B]. The numerical experiment show us that a one parameter family it is not enough to catch the invariant torus.

This result is important because joined with the examples in [M] and [T] shows, on the best of our knowledge, the first example of a homoclinic tangency (higher codimension) which when unfolding it exhibits invariant torus of Floquet type. We have the hope that this example could be useful to find open sets in the space of diffeomorphisms which exhibits infinitely many invariant Floquet Torus in a persistence way, phenomenon which is widely known to hold for the conservative world.

This paper is structured as follows: in section 2 we recall normal forms for mappings having a fixed point whose all eigenvalues belong to the unit circle. In section 3 we prove an extension of Hopf's bifurcation theorem to the multiparameter case. In section 4 we show the proof of theorem 1.1, using the extension's of Hopf's bifurcation theorem.

2 Invariant Torus

Let $g_{\boldsymbol{\mu}} : \mathbb{T}^n \times \mathbb{R}^n_\delta \to \mathbb{T}^n \times \mathbb{R}^n$ be a family of C^∞ maps depending on the parameter $\boldsymbol{\mu} \in V \subset \mathbb{R}^l$. Let $g_{\boldsymbol{\mu}}(\boldsymbol{\theta}, \mathbf{r}) = N_{\boldsymbol{\mu}}(\boldsymbol{\theta}, \mathbf{r}) + O(|\mathbf{r}|^5)$ and $N_{\boldsymbol{\mu}}(\boldsymbol{\theta}, \mathbf{r}) = (\boldsymbol{\Theta}, \mathbf{R})$ with

$$\mathbf{R}_j(\boldsymbol{\theta},\mathbf{r}) = \alpha_j(\boldsymbol{\mu})r_j(1+\sum_1^n c_{jl}(\boldsymbol{\mu})r_l^2) \tag{1}$$

$$\boldsymbol{\Theta}_j(\boldsymbol{\theta},\mathbf{r}) = \theta_j + \phi_j(\boldsymbol{\mu}) + \sum_1^n d_{jl}(\boldsymbol{\mu})r_l^2. \tag{2}$$

Let us put $m_j = \sum_1^n c_{jl}(\boldsymbol{\mu})$ and define the map

$$\Phi_1 : \boldsymbol{\mu} \mapsto (\alpha_1(\boldsymbol{\mu}),...,\alpha_n(\boldsymbol{\mu})).$$

THEOREM 2.1

Let $g_{\boldsymbol{\mu}}(\boldsymbol{\theta},\mathbf{r})$ as above. Assume that Φ is a submersion and that for all $j = 1,...,n$, at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ we have that $\alpha_j(\boldsymbol{\mu}) = 1$, $m_j < 0 (> 0)$ and $c_{jj} < 0 (> 0)$. Then:

- *i) There exists an open set $\mathcal{U}_{\boldsymbol{\mu}_0 k}$ in the parameter space with $\boldsymbol{\mu}_0 \in \overline{\mathcal{U}}_{\boldsymbol{\mu}_0 k}$ such that if $\boldsymbol{\mu} \in \mathcal{U}_{\boldsymbol{\mu}_0}$, then has a C^k-normally attracting (repelling) invariant torus of dimension n.*
- *ii) Moreover, if the map $\Phi_2 : \boldsymbol{\mu} \mapsto (\phi_1(\boldsymbol{\mu}),...,\phi_n(\boldsymbol{\mu}))$ is a submersion, then inside $\mathcal{U}_{\boldsymbol{\mu}_0 k}$ there exists a set $\mathcal{T}$ of positive Lebesgue measure such that the invariant torus obtained are of Floquet type. Also we get that $\boldsymbol{\mu}_0 \in \overline{\mathcal{T}}$.*

PROOF *Let us consider the case $m_j < 0$ and $c_{jj} < 0$. The other one being treated in a similar way considering $g_{\boldsymbol{\mu}}^{-1}$ instead of $g_{\boldsymbol{\mu}}$. Also let us assume without lose of generality that $\boldsymbol{\mu}_0 = \mathbf{0}$. Since Φ is a submersion, we can take $\alpha_i(\boldsymbol{\mu}) - 1$ as the first coordinates of $\boldsymbol{\mu}$ for $i = 1,...,n$. From now on we assume $\alpha_i(\boldsymbol{\mu}) = 1 + \mu_i$.*

As for the first part, first of all, we look for an invariant torus of the mapping $N_{\boldsymbol{\mu}}(\boldsymbol{\theta},\mathbf{r})$. The simplest way to do this, is looking for solutions of the equations

$$\mathbf{R}_j(\boldsymbol{\theta},\mathbf{r}) = \alpha_j r_j(1+\sum_1^n c_{jl}(\boldsymbol{\mu})r_l^2) = r_j. \tag{3}$$

We look them at the diagonal, i.e. $r_1 = ... = r_n$. Then we need to solve the systems

$$\alpha_j(1+m_j r_j^2) = 1. \tag{4}$$

We observe that for the mapping $F : (\alpha_1,...,\alpha_n) \mapsto ((\frac{1}{\alpha_1}-1)\frac{1}{m_1},...,(\frac{1}{\alpha_n}-1)\frac{1}{m_n})$ we have that $\det DF(1,...,1) = \frac{(-1)^n}{m_1...m_n} \neq 0$. From here, we get the existence of a curve $\gamma(t)$ in the parameter space $(\alpha_1,...,\alpha_n)$, included in the set defined by $\{(\alpha_1,...,\alpha_n) : \frac{m_j}{|m_j|}(\alpha_j - 1)\} \leq 0\}$ with $\gamma(t) \to (1,...,1)$ when $t \to 0$ and such that $(\alpha_1,...\alpha_n) = \gamma(t)$ solves 4 with $r_j = t$.

From here on, we restrict to the parameters given by $\boldsymbol{\mu}(t) = (\gamma(t),\mu_{n+1},...,\mu_l)$ where $\mu_i, i = n+1,...l$ are assumed small enough. So we have a family of mappings $g_t = g_{\boldsymbol{\mu}(t)} : (\boldsymbol{\theta},\mathbf{r}) \mapsto$

$(\boldsymbol{\Theta}_t, \mathbf{R}_t) + O(| \mathbf{r} |^5)$ *with* $\boldsymbol{\Theta}_t, \mathbf{R}_t$ *having the form of 1. Now we introduce the following change of coordinates:* $z_l \mapsto r_l = t(1 + tz_l)$ *and get that*

$$\begin{aligned}
\mathbf{R}_{tj}(\boldsymbol{\theta}, \mathbf{z}) &= [\alpha_j(1 + tz_j)(1 + \sum_1^n c_{jl}t^2(1 + tz_l)^2) - 1]\frac{1}{t} \\
&= [\alpha_j(1 + \sum_1^n c_{jl}t^2) + 2\alpha_j \sum_1^n c_{jl}t^3 z_l + \alpha_j \sum_1^n c_{jl}t^4 z_l^2 + \\
&\quad \alpha_j t z_j(1 + \sum_1^n c_{jl}t^2(1 + tz_j)^2) - 1]\frac{1}{t} \\
&= [2\alpha_j \sum_1^n c_{jl}t^3 z_l + \alpha_j \sum_1^n c_{jl}t^4 z_l^2 + \alpha_j t z_j(1 + \sum_1^n c_{jl}t^2(1 + tz_l)^2]\frac{1}{t} \\
&= \alpha_j z_j(1 + \sum_1^n c_{jl}t^2) + 2\alpha_j \sum_1^n c_{jl}t^2 z_l + O(t^3) \\
&= z_j + 2\alpha_j \sum_1^n c_{jl}t^2 z_l + O(t^3)
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\boldsymbol{\Theta}_{tj}(\boldsymbol{\theta}, \mathbf{z}) &= \theta_j + \phi_j(t) + \sum_1^n d_{jl}t^2(1 + tz_l)^2 \\
&= \theta_j + \phi_j(t) + (\sum_1^n d_{jl})t^2 + O(t^3).
\end{aligned} \tag{6}$$

As a consequence, $g_t(\boldsymbol{\theta}, \mathbf{z}) = (\tilde{\boldsymbol{\Theta}}_t(\boldsymbol{\theta}, \mathbf{z}), \tilde{\mathbf{R}}_t(\boldsymbol{\theta}, \mathbf{z})) + O(t^3) = (\bar{\boldsymbol{\Theta}}_t(\boldsymbol{\theta}, \mathbf{z}), \bar{\mathbf{R}}_t(\boldsymbol{\theta}, \mathbf{z}))$.

In order to get the invariant torus we are looking for, we apply the following proposition (see [SSTCh, Theor. 4.2, pag. 242]).

THEOREM 2.2

Let the C^∞*-mapping* $g : (\theta, \mathbf{z}) \mapsto (\overline{\boldsymbol{\Theta}}, \overline{\mathbf{R}})$ *be defined in* $\mathbb{T}^n \times \mathbb{R}^n_\delta$ *and such that it holds the following conditions:*

1. $\|D_{\mathbf{z}}\overline{\mathbf{R}}\| < 1$ *for all* $\boldsymbol{\theta}$*;*
2. $\boldsymbol{\theta} \mapsto \overline{\boldsymbol{\Theta}}$ *is a diffeomorphisms for all* $\mathbf{z}$*;*
3. $2\sqrt{\|(D_{\boldsymbol{\theta}}\overline{\boldsymbol{\Theta}})^{-1}\|.\|D_{\mathbf{z}}\overline{\boldsymbol{\Theta}}\|.\|D_{\boldsymbol{\theta}}\overline{\mathbf{R}}.(D_{\boldsymbol{\theta}}\overline{\boldsymbol{\Theta}})^{-1}\|} < 1 - \|(D_{\boldsymbol{\theta}}\overline{\boldsymbol{\Theta}})^{-1}\|.\|D_{\mathbf{z}}\overline{\mathbf{R}}\|$*;*

then the mapping g *has an invariant torus of dimension* n *in* $\mathbb{T}^n \times \mathbb{R}^n_\delta$ *which is normally attractor. This torus is given as the graph of a function* $h : \mathbb{T}^n \to \mathbb{R}^n_\delta$ *which is a* C^1 *map. Moreover, if we have that*

4.

$$\sqrt[k+1]{\left(\|D_{\mathbf{z}}\overline{\mathbf{R}}\| + \|D_{\mathbf{z}}\overline{\Theta}\|.\|D_{\boldsymbol{\theta}}\overline{\mathbf{R}}.(D_{\boldsymbol{\theta}}\overline{\Theta})^{-1}\|\right) \|(D_{\boldsymbol{\theta}}\overline{\Theta})^{-1}\|^k} < 1 - \sqrt{\|(D_{\boldsymbol{\theta}}\overline{\Theta})^{-1}\|.\|D_{\mathbf{z}}\overline{\Theta}\|.\|D_{\boldsymbol{\theta}}\overline{\mathbf{R}}.(D_{\boldsymbol{\theta}}\overline{\Theta})^{-1}\|}, \quad (7)$$

h is a C^k map, i.e. the invariant torus above is C^k.

So it is enough to check that the mappings g_t accomplish the conditions of the above proposition for t small enough.

In order to check the first condition, we observe that $\|D_{\mathbf{z}}\mathbf{R}_t\| \leq \max_j\{1 + \frac{1}{2}\alpha_j c_{jj} t^2 + O(t^3)\}$ which is less than 1 for t small enough since $c_{jj} < 0$. The second condition is immediately verified since the mapping $\boldsymbol{\theta} \mapsto \boldsymbol{\theta} + \boldsymbol{\phi}(0)$, with $\boldsymbol{\phi}(t) = (\phi_1(t), \ldots, \phi_n(t))$, is a diffeomorphism and for t small enough $\boldsymbol{\theta} \mapsto \Theta$ is so close as we want of $\boldsymbol{\theta} \mapsto \boldsymbol{\theta} + \boldsymbol{\phi}(0)$ as can be seeing from 6. For checking condition 3, we have that : $\|D_{\mathbf{z}}\Theta_t\| = O(t^3)$, $\|D_{\boldsymbol{\theta}}^{-1}\Theta_t\| = 1 - O(t)$,$\|D_{\boldsymbol{\theta}}\mathbf{R}_t\| = O(t^3)$ and $\|D_{\mathbf{z}}\mathbf{R}_t\| = 1 - O(t^2)$. Therefore

$$\begin{aligned} 2\sqrt{\|D_{\boldsymbol{\theta}}^{-1}\Theta_t\|.\|D_{\mathbf{z}}\Theta_t\|.\|D_{\boldsymbol{\theta}}\mathbf{R}_t.D_{\boldsymbol{\theta}}^{-1}\Theta_t\|} &\leq O(t^{3/2}) \\ &< O(t) \leq 1 - \|D_{\boldsymbol{\theta}}^{-1}\Theta_t\|.\|D_{\mathbf{z}}\mathbf{R}\|. \end{aligned} \quad (8)$$

and

$$\begin{aligned} \sqrt[k+1]{\left(\|D_{\mathbf{z}}\overline{\mathbf{R}}\| + \|D_{\mathbf{z}}\overline{\Theta}\|.\|D_{\boldsymbol{\theta}}\overline{\mathbf{R}}.(D_{\boldsymbol{\theta}}\overline{\Theta})^{-1}\|\right) \|(D_{\boldsymbol{\theta}}\overline{\Theta})^{-1}\|^k} &\leq 1 - O(t) \\ < 1 - O(t^3) \leq 1 - \sqrt{\|(D_{\boldsymbol{\theta}}\overline{\Theta})^{-1}\|.\|D_{\mathbf{z}}\overline{\Theta}\|.\|D_{\boldsymbol{\theta}}\overline{\mathbf{R}}.(D_{\boldsymbol{\theta}}\overline{\Theta})^{-1}\|}&. \end{aligned} \quad (9)$$

So the 4 conditions have been checked, and we obtained for each parameter t near enough to 0 the existence of an invariant torus C^k-normally attracting. So we can conclude the existence of a set $\mathcal{U}_{\mu_0 k}$ as stated in the item i) of the proposition because for each t as above we get an open set around $\boldsymbol{\mu}(t)$ for which the invariant torus persists.

As for the second part, we choose k' large enough in item i) so the Central Manifold Theorem allows us to write the invariant tori $\mathbb{T}_{\boldsymbol{\mu}}$ found in item i) for each μ around $\boldsymbol{\mu}(t)$ as the graph of a $C^{k'}$ function $\psi(\boldsymbol{\theta}, \boldsymbol{\mu})$ and, in turn, from the form of $g_{\boldsymbol{\mu}}$ we get that $g_{\boldsymbol{\mu}}$ can be written in a neighborhood of each $\mathbb{T}_{\boldsymbol{\mu}}$ as follows

$$\begin{aligned} \theta_j &\mapsto \theta_j + \phi_j(\boldsymbol{\mu}) + \sum_1^n d_{j\,l}\psi_l^2(\boldsymbol{\theta}, \boldsymbol{\mu}) + O(\mathbf{r}), \\ r_j &\mapsto \alpha_j r_j + 2\alpha_j\psi_j(\boldsymbol{\theta}, \boldsymbol{\mu}) \sum_1^n c_{j\,l}\psi_l(\boldsymbol{\theta}, \boldsymbol{\mu}) r_l + \alpha_j r_j \sum_1^n c_{j\,l}\psi_l^2(\boldsymbol{\theta}, \boldsymbol{\mu}). \end{aligned} \quad (10)$$

We observe that for each $\boldsymbol{\mu}(t)$, $\|\psi(\boldsymbol{\theta}, \boldsymbol{\mu})\|$ can be done small enough uniformly in θ, just taking $\boldsymbol{\mu}$ near enough $\boldsymbol{\mu}(t)$. This fact implies the map above can be viewed as a perturbation of

$$\begin{aligned} \theta_j &\mapsto \theta_j + \phi_j(\boldsymbol{\mu}), \\ r_j &\mapsto \alpha_j r_j, \end{aligned} \tag{11}$$

for which we have that the map $\boldsymbol{\mu} \mapsto (\alpha_1, ..., \alpha_n, \phi_1, ..., \phi_n)$ is a submersion as well as $|\alpha_j| \neq 1$ for each $\boldsymbol{\mu}(t)$ small. Since the form of the r_j's part of the map we can apply the diffeomorphism's version of the quasi-periodic stability theorem [BHS, Theor. 2.3, pag. 46] in order to get the conclusion of the item ii). We remark that finite differentiability version of this theorem implies the loose of differentiability for the invariant torus, see [BHTS, pag. 79], that is why we choose k' large enough.

◇

3 Normal Forms

In this section we recall normal forms for mappings f which have a fixed point p with all of eigenvalues of $Df(p)$ belongs to the unit circle. So, consider $f_{\boldsymbol{\mu}}: U \to \mathbb{R}^n$, a family of C^∞-diffeomorphisms defined on an open set $U \subset \mathbb{R}^n$ with the parameter $\boldsymbol{\mu}$ varying in the open set $V \subset \mathbb{R}^l$. Suppose that $p = \mathbf{0} \in U$ and $f_{\boldsymbol{\mu}}(\mathbf{0}) = \mathbf{0}$ for all $\boldsymbol{\mu}$. The Taylor expansion of $f_{\boldsymbol{\mu}}$ around $\mathbf{0}$ is given by

$$f_{\boldsymbol{\mu}}(\mathbf{x}) = A(\boldsymbol{\mu})\mathbf{x} + f^2_{\boldsymbol{\mu}} + f^3_{\boldsymbol{\mu}} + O(|\mathbf{x}|^4),$$

with $A(\boldsymbol{\mu}) = Df_{\boldsymbol{\mu}}(\mathbf{0})$ and where $f^2_{\boldsymbol{\mu}}, f^3_{\boldsymbol{\mu}}$ are mappings whose components are homogeneous polynomials of degree 2 and 3 respectively.

Assume that $A(\boldsymbol{\mu})$ has n complex eigenvalues

$$\{\lambda_1(\boldsymbol{\mu}), ..., \lambda_{2s}(\boldsymbol{\mu})) = (\lambda_1(\boldsymbol{\mu}), ..., \lambda_s(\boldsymbol{\mu}), \quad \overline{\lambda}_1(\boldsymbol{\mu}), ..., \overline{\lambda}_s(\boldsymbol{\mu}))\},$$

which for $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ belong to the unit circle. After a basis change, we have that $A(\boldsymbol{\mu})$ at $\boldsymbol{\mu}_0$ is in its real Jordan normal form, i.e. if $\lambda_j(\boldsymbol{\mu}) = \alpha_j(\boldsymbol{\mu}) + i\beta_j(\boldsymbol{\mu})$, then

$$A(\boldsymbol{\mu}) = \left(\begin{array}{cccc|cccc} \alpha_1(\boldsymbol{\mu}) & 0 & \cdots & 0 & -\beta_1(\boldsymbol{\mu}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_s(\boldsymbol{\mu}) & 0 & 0 & \cdots & -\beta_s(\boldsymbol{\mu}) \\ \hline \beta_1(\boldsymbol{\mu}) & 0 & \cdots & 0 & \alpha_1(\boldsymbol{\mu}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_s(\boldsymbol{\mu}) & 0 & 0 & \cdots & \alpha_s(\boldsymbol{\mu}) \end{array}\right)$$

In order to work the normal form, we introduce the following change of coordinates $P\mathbf{z} = \mathbf{x}$ with $\mathbf{z} = (z_1, ..., z_{2s})$, where

$$P = \frac{1}{2}\left(\begin{array}{c|c} I_s & I_s \\ \hline -iI_s & iI_s \end{array}\right)$$

with I_s the identity matrix of size s.

Now we assume that $\mathbf{x}, \mathbf{z}$ have complex coordinates. We remark that when $\mathbf{x}$ is restricted to get real values we have that $z_{s+j} = \overline{z}_j$. In these new coordinates $f_{\boldsymbol{\mu}}$ is written as

$$f_{\boldsymbol{\mu}}(\mathbf{z}) = J\mathbf{x} + f^2_{\boldsymbol{\mu}}(\mathbf{z}) + f^3_{\boldsymbol{\mu}}(\mathbf{z}) + O(|\mathbf{z}|^4),$$

where J is a diagonal matrix with diagonal given by $(\lambda_1(\boldsymbol{\mu}), ..., \lambda_{2s}(\boldsymbol{\mu}))$. Here, $f^2_{\boldsymbol{\mu}}(\mathbf{z}), f^3_{\boldsymbol{\mu}}(\mathbf{z})$ are again mappings whose coordinates are polynomials of degree 2 and 3 respectively.

We assume now, nonresonance conditions of order 2 for the eigenvalues $\lambda_j(\boldsymbol{\mu})$, i.e. $\lambda_j(\boldsymbol{\mu}) \neq \lambda_{i_1}(\boldsymbol{\mu})^{m_1}\lambda_{i_2}(\boldsymbol{\mu})^{m_2}$, with $m_1 + m_2 = 2$. So with a change of variables we get that, in these new variables $f^2_{\boldsymbol{\mu}}(\mathbf{z}) = \mathbf{0}$.

With regard to order 3 terms, we observe that the following terms $z_j|z_l|^2$ which appears in the j- component cannot be eliminated never, since they are resonant ($\lambda_j = \lambda_j\lambda_l\overline{\lambda}_l = \lambda_j|\lambda_l|^2$). The others one, do can be eliminated assuming nonresonant conditions on it, i.e. $\lambda_j(\boldsymbol{\mu}) \neq \lambda_{i_1}(\boldsymbol{\mu})^{m_1}\lambda_{i_2}(\boldsymbol{\mu})^{m_2}\lambda_{i_3}(\boldsymbol{\mu})^{m_2}$ where $i_3 \neq i_2{+}s$ if $i_1 = j, m_1 = m_2 = m_3 = 1$ with $j, i_1, i_2, i_3 = 1, ..., s$ and $m_1 + m_2 + m_3 = 3$.

So with these assumptions, the j component of the map $f_{\boldsymbol{\mu}}$ can be written as

$$f_{\boldsymbol{\mu}j}(\mathbf{z}) = \lambda_j(\boldsymbol{\mu})z_j + \sum_1^s \gamma_{j\,l} z_j|z_l|^2 + O(|\mathbf{z}|^4) \tag{12}$$

Also remember, that normal form theory says to us that $f_{\boldsymbol{\mu}j}(\mathbf{z}) = \overline{f_{\boldsymbol{\mu}j+s}}(\mathbf{z})$ for $j = 1, ..., s$.

To find the normal form for $f_{\boldsymbol{\mu}}$, which we are going to use, we introduce polar coordinates en each coordinate z_j. So let θ_j, r_j be such that $z_j = r_j \exp i\theta_j$. If $\boldsymbol{\theta} = (\theta_1, ..., \theta_s)$, $\mathbf{r} = (r_1, ..., r_s)$ and $f_{\boldsymbol{\mu}j}(\boldsymbol{\theta}, \mathbf{r}) = \mathbf{R}_j \exp i\boldsymbol{\Theta}_j$ and $f_{\boldsymbol{\mu}j+s}(\boldsymbol{\theta}, \mathbf{r}) = \mathbf{R}_j \exp -i\boldsymbol{\Theta}_j$ for $j = 1, ..., s$. Then $\boldsymbol{\Theta}_j, \mathbf{R}_j$ have the following expression

$$\mathbf{R}_j(\boldsymbol{\theta}, \mathbf{r}) = |\lambda_j(\boldsymbol{\mu})| r_j (1 + \sum_1^s \Re(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}) r_l^2) + O(|\mathbf{r}|^5), \tag{13}$$

$$\boldsymbol{\Theta}_j(\boldsymbol{\theta}, \mathbf{r}) = \theta_j + \phi_j(\boldsymbol{\mu}) + \sum_1^s \Im(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}) r_l^2 + O(|\mathbf{r}|^5)), \tag{14}$$

where $\lambda_j(\boldsymbol{\mu}) = |\lambda_j(\boldsymbol{\mu})| \exp i\phi_j(\boldsymbol{\mu})$ and $j = 1, ...s$.

Summing up, we get the following proposition

THEOREM 3.1

Let $f_{\boldsymbol{\mu}} : U \to \mathbb{R}^n$ be a family of C^∞ diffeomorphisms. Suppose that this family satisfy the following conditions:

1. $f_{\mu}(\mathbf{0}) = \mathbf{0}$;
2. $Df_{\mu}(\mathbf{0})$ *has* n *complex eigenvalues* $\lambda_1(\mu), ..., \lambda_s(\mu), \lambda_{s+1}(\mu) = \overline{\lambda_1(\mu)}, ..., \lambda_{2s}(\mu) = \overline{\lambda_s(\mu)}$ *such that* $|\lambda_j(\mu)| = 1$ *for* $\mu = \mu_0$;
3. *The eigenvalues* λj *accomplish the nonresonant conditions:*
   (a) $\lambda_j(\mu) \neq \lambda_{i_1}(\mu)^{m_1}\lambda_{i_2}(\mu)^{m_2}$ *with* $m_1 + m_2 = 2$ *and* $j, m_1, m_2 = 1, ..., 2s$;
   (b) $\lambda_j(\mu) \neq \lambda_{i_1}(\mu)^{m_1}\lambda_{i_2}(\mu)^{m_2}\lambda_{i_3}(\mu)^{m_2}$ *where* $i_3 \neq i_2+s$ *if* $i_1 = j$ *and* $m_1 = m_2 = m_3 = 1$ *with* $j, i_1, i_2, i_3 = 1, ..., 2s$ *and* $m_1 + m_2 + m_3 = 3$.

Then there exists new coordinates $(\theta, \mathbf{r})$*, which depend in a smooth way of* μ *such that*

$$f_{\mu}(\theta, \mathbf{r}) = g_{\mu}(\theta, \mathbf{r}) + O(|\mathbf{r}|^5),$$

with $g_{\mu}(\theta, \mathbf{r}) = (\Theta, \mathbf{R})$ *with the following form*

$$\mathbf{R}_j(\theta, \mathbf{r}) = |\lambda_j(\mu)| r_j (1 + \sum_1^s c_{jl}(\mu) r_l^2) \tag{15}$$

$$\Theta_j(\theta, \mathbf{r}) = \theta_j + \phi_j(\mu) + \sum_1^s d_{jl}(\mu) r_l^2 \tag{16}$$

with $c_{jl}(\mu), d_{jl}(\mu)$ *smooth functions of* μ *which are defined as follows:*

$$\begin{aligned} c_{jl}(\mu) &= \Re(\frac{\gamma_{j\,l}}{\lambda_j(\mu)}) \\ d_{jl}(\mu) &= \Im(\frac{\gamma_{j\,l}}{\lambda_j(\mu)}), \end{aligned} \tag{17}$$

where the $\gamma_{j\,l}$ *are given as in 12 and, in turn, these are defined by terms which are built up with terms up to order three of the mapping* f_{μ} *in the initial coordinates.*

As an immediate corollary we get the following theorem which is a extension of the Hopf's bifurcation theorem to multiparameter families.

THEOREM 3.2

Let $f_{\mu} : U \to \mathbb{R}^n$ *be a family of* C^{∞}*-diffeomorphisms. Suppose that* f_{μ} *satisfies:*

1. $f_{\mu}(\mathbf{0}) = \mathbf{0}$;
2. $Df_{\mu}(\mathbf{0})$ *has* n *complex eigenvalues* $\lambda_1(\mu), ..., \lambda_s(\mu), \lambda_{s+1}(\mu) = \overline{\lambda_1(\mu)}, ..., \lambda_{2s}(\mu) = \overline{\lambda_s(\mu)}$ *such that* $|\lambda_j(\mu)| = 1$ *for* $\mu = \mu_0$;
3. $\lambda_j(\mu) \neq \lambda_{i_1}(\mu)^{m_1}\lambda_{i_2}(\mu)^{m_2}\lambda_{i_3}(\mu)^{m_2}$ *where* $i_3 \neq i_2 + s$ *if* $i_1 = j$ *and* $m_1 = m_2 = m_3 = 1$ *with* $j, i_1, i_2, i_3 = 1, ..., 2s$ *and* $m_1 + m_2 + m_3 = 3$;
4. *the map* $\mu \mapsto (|\lambda_1(\mu)|, \ldots, |\lambda_s(\mu)|)$ *is a submersion.*

As well as we assume that the derivatives up to third order of f_μ accomplish generic conditions, then

1. *for each $k \geq 1$, there exists an open set $\mathcal{U}_k$, in the space the parameters, such that $\mu_0 \in \overline{\mathcal{U}_k}$ and for all $\mu \in \mathcal{U}_k$ f_μ exhibits a C^k invariant torus of dimension $[n/2]$ either normally attracting or repelling.*

2. *Assuming that $\lambda_j(\mu) = |\lambda_j(\mu)| \exp i\phi_j(\mu)$ and that $\mu \mapsto (\phi_1(\mu), \ldots, \phi_1(\mu))$ is a submersion, then inside $\mathcal{U}$ there exists another set $\mathcal{T}_k$ such that the invariant torus obtained for parameters there in is of Floquet type. This set has positive Lebesgue measure and $\mu_0 \in \overline{\mathcal{T}_k}$*

4 The quadratic family $h_{(a,\mathbf{b})}(x_1, \ldots, x_n) = (a - x_n^2, x_1, ..., x_{n-1})$

In this section we want to use the results of the last section in order to show the existence of invariant tori of Floquet type for the family $h_{(a,\mathbf{b})}$ when the parameters $(a, \mathbf{b})$ are near enough to $(3/4, \mathbf{0})$. First of all, we are going to recall how evolves the dynamics of $h_{(a,\mathbf{b})}$ at the fixed points.

4.1 Fixed points

To begin with, we observe that its fixed points are given by the equations

$$a + (\sum_{j=1}^{n-1} b_j)x_1 - x_n^2 = x_1, \quad \text{and} \quad x_1 = x_2, ..., x_{u-1} = x_n, \tag{18}$$

so $x_n^\pm = x_{n,(a,\mathbf{b})}^\pm = \frac{-(1-\sum_{j=1}^{n-1} b_j) \pm \sqrt{(1-\sum_{j=1}^{n-1} b_j)^2 + 4a}}{2}$. For $a \geq -\frac{(1-\sum_{j=1}^{n-1} b_j)^2}{4}$ there exist two fixed points

$$P = P_{(a,\mathbf{b})} = (x_{1,(a,\mathbf{b})}^+, ..., x_{n,(a,\mathbf{b})}^+) \text{ and } Q = Q_{(a,\mathbf{b})} = (x_{1,(a,\mathbf{b})}^-, ..., x_{n,(a,\mathbf{b})}^-). \tag{19}$$

When $a = -\frac{(1-\sum_{j=1}^{n-1} b_j)^2}{4}$, $P = Q = (-(1 - \sum_{j=1}^{n-1} b_j)/2, ..., -(1 - \sum_{j=1}^{n-1} b_j)/2)$.

4.2 Eigenvalues at P

The characteristic polynomial of $Dh_{(a,\mathbf{b})}$ is given by

$$p_{(a,\mathbf{b})}(\lambda) = \lambda^n - \sum_{j=1}^{n-1} b_j \lambda^{n-j} + 2x_n. \tag{20}$$

We want to study this family for the parameters $(a, \mathbf{0})$. So let $h_a = h_{(a,\mathbf{0})}$. From 20 we get that Dh_a has eigenvalues given by $\lambda^n = -2x_n$. For P if $a \in [-1/4, 3/4)$ then $|-2ax_{n,a}^+| < 1$ so P is an attractor. At $a = 3/4$, $Dh_a(P)$ has n eigenvalues which are solutions of $\lambda^n = -1$. After this value, that is for $a \in (3/4, \infty)$, P became a repeller fixed point.

For $(a, \mathbf{b}) = (3/4, \mathbf{0})$ at the point P, Dh_a has the eigenvalues $\lambda = \exp i\pi(2k+1)/n$, for $k = 0, ..., n-1$. When n is odd -1 is included in that set.

For $s = [n/2]$, let $\{\lambda_1(a,\mathbf{b}), \overline{\lambda_1(a,\mathbf{b})}, ..., \lambda_s(a,\mathbf{b}), \overline{\lambda_s(a,\mathbf{b})}\}$ and $\{\lambda_1(a,\mathbf{b}), \overline{\lambda_1(a,\mathbf{b})}, ..., \lambda_s(a,\mathbf{b}), \overline{\lambda_s(a,\mathbf{b})}, \lambda_{s+1}(a,\mathbf{b})\}$ be the eigenvalues of $Dh_{(a,\mathbf{b})}(P)$ for n even and odd respectively. We choose the index in such a way that $\lambda_{s+1}(a,\mathbf{b}) \in \mathbb{R}$. It is easily seen that $v_\lambda = (\lambda^{n-1}, \lambda^{n-2}, ..., \lambda, 1)$ is an eigenvector associated to the eigenvalue λ. Also the eigenvalues of $Dh_{3/4}(P)$ are simple so it is diagonalizable. Since the set formed by the eigenvectors is a base of the space over $\mathbb{C}$, then the the set formed by $\mathrm{Re} v_{\lambda_i}, \mathrm{Im} v_{\lambda_i}$ for all i it is also a base of the space. Put $\lambda_i(a,\mathbf{b}) = |\lambda_i(a,\mathbf{b})| \exp i\theta_i(a,\mathbf{b})$. Observe that when n is odd then $\theta_{s+1}(a,\mathbf{b}) = 0$. The following proposition is proved in [M].

LEMMA 4.1 *Let $f_i(a,\mathbf{b}) = |\lambda_i(a,\mathbf{b})|$ for $i = 1, ..., s$ or $i = 1, ..., s, s+1$ depending on the parity of n. Then the vectors $\{\nabla f_i(3/4,\mathbf{0}), \nabla\theta_i(3/4,\mathbf{0})\}_{i=1}^{s}$ ($\{\nabla f_i(3/4,\mathbf{0}), \nabla\theta_i(3/4,\mathbf{0})\}_{i=1}^{s} \cup \{\nabla f_{s+1}(3/4,\mathbf{0})\}$) are linearly independent.*

4.3 Preliminary Normal Form

Now we are going to bring $h_{(a,\mathbf{0})}$ into the form 12 around the point P and for $(a,\mathbf{b})$ near enough to $(3/4,\mathbf{0})$, so we can apply theorem to this family. Firstly, let us move P to zero. So let $l_{(a,\mathbf{b})}(\boldsymbol{\eta}) = \boldsymbol{\eta} + P$ then, using 18, $h_{(a,\mathbf{b})} = l_{(a,\mathbf{b})}^{-1} \circ h_{(a,\mathbf{b})} \circ l_{(a,\mathbf{b})}$ is given by

$$\begin{aligned} h_{(a,\mathbf{b})}(\boldsymbol{\eta}) &= (a - P_1 + \sum_{j=1}^{n-1} b_j(\eta_j + P_j) - (\eta_n + P_n)^2, \eta_1, ..., \eta_{n-1}) \\ &= (\sum_{j=1}^{n-1} b_j\eta_j - \eta_n^2 - 2\eta_n P_n, \eta_1, ..., \eta_{n-1}) = H\boldsymbol{\eta} + (-\eta_n^2, 0, ..., 0) \end{aligned} \tag{21}$$

We observe that $Dh_{(a,\mathbf{b})}(\mathbf{0})$ is diagonalizable for the parameters under consideration. Let S be the matrix which brings $Dh_{(a,\mathbf{b})(\mathbf{0})}$ into diagonal form. It is easily seen that

$$S = \begin{pmatrix} \lambda_1^{n-1} & \cdots & \lambda_s^{n-1} & \overline{\lambda}_1^{n-1} & \cdots & \overline{\lambda}_s^{n-1} & \lambda_n^{n-1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_1 & \cdots & \lambda_s & \overline{\lambda}_1 & \cdots & \overline{\lambda}_s & \lambda_n \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \tag{22}$$

where $\lambda_1, \overline{\lambda_1}, ..., \lambda_s, \overline{\lambda_s}$ and $\lambda_1, \overline{\lambda_1}, ..., \lambda_s, \overline{\lambda_s}, \lambda_n$ are the eigenvalues of $Dh_{(a,\mathbf{b})}$ accordingly whether n is even or odd and where $s = [n/2]$. Here, when n is even it must be understood that the last column of S is missing.

Assume from now on that n is even. Consider $\lambda_j = \alpha_j + i\beta_j$ and $\lambda_j^k = \alpha_j^k + i\beta_j^k$, then if we take the following base

$$\begin{pmatrix}\alpha_1^{n-1}\\ \vdots\\ \alpha_1\\ 1\end{pmatrix},\dots,\begin{pmatrix}\alpha_s^{n-1}\\ \vdots\\ \alpha_s\\ 1\end{pmatrix},\begin{pmatrix}-\beta_1^{n-1}\\ \vdots\\ -\beta_1\\ 0\end{pmatrix},\dots,\begin{pmatrix}-\beta_s^{n-1}\\ \vdots\\ -\beta_s\\ 0\end{pmatrix}, \tag{23}$$

H has the real Jordan canonical form $H_{\mathbb{R}}$ when written in this base. Let

$$S_{\mathbb{R}} = \begin{pmatrix}\alpha_1^{n-1} & \cdots & \alpha_s^{n-1} & -\beta_1^{n-1} & \cdots & -\beta_s^{n-1}\\ \vdots & & \vdots & \vdots & & \vdots\\ \alpha_1 & \cdots & \alpha_s & -\beta_1 & \cdots & -\beta_s\\ 1 & & 1 & 0 & & 0\end{pmatrix}, \tag{24}$$

then $H_{\mathbb{R}} = S_{\mathbb{R}}^{-1} \circ H \circ S_{\mathbb{R}}$. Now put $S_{\mathbb{R}}\mathbf{y} = \boldsymbol{\eta}$ so $h_{(a,\mathbf{b})}$ in this new coordinates is written as

$$\begin{aligned} h_{(a,\mathbf{b})}(\mathbf{y}) &= H_{\mathbb{R}}\mathbf{y} - (S_{\mathbb{R}}\mathbf{y})_n^2\, S_{\mathbb{R}}^{-1}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\\ &= H_{\mathbb{R}}\mathbf{y} - (y_1 + \cdots + y_s)^2\, S_{\mathbb{R}}^{-1}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}. \end{aligned} \tag{25}$$

In order to get a normal form we make the variables y_i complex. Let us introduce the following coordinates $P\mathbf{z} = \mathbf{y}$ con $\mathbf{z} = (z_1, ..., z_{2s})$, as in section 3. In this new coordinates we get that

$$\begin{aligned} h_{(a,\mathbf{b})}(\mathbf{z}) = (\lambda_1 z_1, ..., \lambda_s z_s, \overline{\lambda}_1 z_{s+1}, ..., \overline{\lambda}_s z_{2s}) - \\ \left(\frac{z_1 + z_{s+1}}{2} + \cdots + \frac{z_s + z_{2s}}{2}\right)^2 P^{-1} S_{\mathbb{R}}^{-1}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}. \end{aligned} \tag{26}$$

We need to compute

$$(B_1, ..., B_s, B_{s+1}, ..., B_{2s}) = P^{-1}S_{\mathbb{R}}^{-1}\left(\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\right).$$

In order to do that, we observe that $S_{\mathbb{R}} \circ P = S$. Let $p_n(\lambda) = (\lambda - \lambda_i)q_i(\lambda)$. In [M], it was computed that $B_i = 1/q_i(\lambda_i)$ and at $(3/4, \mathbf{0})$, $B_i(3/4, \mathbf{0}) = \frac{1}{n\lambda_i^{n-1}}$ for $i = 1, ..., s$.

4.4 Existence of invariant Torus:

In this section we want to apply Theorem 3.2 in order to get the existence of invariant torus as stated in Theorem 1.1. For doing that we will use the preliminary normal form computed in the las subsection and the formula developed in the appendix to compute the coefficients of the resonant cubic terms. We observe that the eigenvalues λ_i of $Dh_{(a,\mathbf{b})}(P)$ satisfies non resonant conditions of order two for all $(a, \mathbf{b})$ near $(3/4, \mathbf{0})$, but it does not happen the same with nonresonant conditions of order three different from those as $\lambda_i = \lambda_i \lambda_j \overline{\lambda}_j$.

We are going to consider two cases:

- Case i) n is even.

We are going to proof the following proposition

THEOREM 4.1

There exists a surface S in the parameter space with $(3/4, \mathbf{0}) \in S$ such that in a small neighborhood $B \subset S$ of $(3/4, \mathbf{0})$ there exists a dense set $T \subset B$ with $(3/4, \mathbf{0}) \in \overline{T}$ satisfying the following: for each parameter $(3/4, \mathbf{0}) \in T$ the map $h_{(a,\mathbf{0})}$ satisfies the conditions of the Theorem 3.2.

From this the proof of Theorem 1.1 follows immediately.

PROOF *[Proof of Proposition 4.1]*

We know by the last subsection that

$$h_{(a,\mathbf{b})}(\mathbf{z}) = (\lambda_1 z_1, ..., \lambda_s z_s, \overline{\lambda}_1 z_{s+1}, ..., \overline{\lambda}_s z_{2s}) - (\frac{z_1 + z_{s+1}}{2} + \cdots + \frac{z_s + z_{2s}}{2})^2 \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \\ B_{s+1} \\ \vdots \\ B_{2s} \end{pmatrix} \tag{27}$$

with $B_i = \overline{B}_{i+s}$ for all $i = 1, ..., s$. Now putting $Z = (z_1, ..., z_s)$ and $\overline{Z} = (z_{s+1}, ..., z_{2s})$ we get that

$$\begin{aligned} h_{(a,\mathbf{b})}(Z, \overline{Z}) =& (h^1_{(a,\mathbf{b})}(Z, \overline{Z}), h^2_{(a,\mathbf{b})}(Z, \overline{Z}), ..., \\ & h^s_{(a,\mathbf{b})}(Z, \overline{Z}), h^{s+1}_{(a,\mathbf{b})}(Z, \overline{Z}), ..., h^{2s}_{(a,\mathbf{b})}(Z, \overline{Z})), \end{aligned}$$

with

$$h^i_{(a,\mathbf{b})}(Z, \overline{Z}) = \lambda_i z_i - B_i(Z, \overline{Z}) M \begin{pmatrix} Z \\ \overline{Z} \end{pmatrix}$$

where $M = \frac{1}{4}\left(\begin{array}{c|c} I_s & I_s \\ \hline I_s & I_s \end{array}\right)$ *and* I_s *is the identity matrix of size s. We remark that* $h^i_{(a,\mathbf{b})} = \overline{h^{i+s}_{(a,\mathbf{b})}}$ *for all* $i = 1, ..., s$.

We let $S = \{(a, \mathbf{b}) : |\lambda_1(a, \mathbf{b})| = \cdots = |\lambda_s(a, \mathbf{b})| = 1\}$*, by lemma 4.1 we get that* S *is a codimension s surface in a neighborhood of* $(a, \mathbf{b})$*. Now let* T *the set inside* S *where the* λ_i *satisfy nonresonant conditions of order three (except the non evitable resonant terms) and four.*

So we can write, after a change of coordinates, which we follow calling $Z, \overline{Z}$*,*

$$h^i_{(a,\mathbf{b})}(Z, \overline{Z}) = \lambda_i(\boldsymbol{\mu})z_i + \sum_1^s \gamma_{ij} z_i |z_j|^2 + O(|\mathbf{z}|^5). \tag{28}$$

Lemma 5.2 permits us to compute γ_{ij}*. What we want to check then is that* $c_{ij} = \Re(\frac{\gamma_{ij}}{\lambda_i}) < 0$.

But these coefficients depend continuously on the parameters $(a, \mathbf{b})$*, so it is enough to check the inequality when* $(a, \mathbf{b}) = (3/4, \mathbf{0})$.

For this value of the parameter we have that $B_i = \dfrac{1}{n\lambda_i^{n-1}} = -\dfrac{1}{n}\lambda_i = -\dfrac{1}{n\overline{\lambda}_i}$*, and so, by lemma 5.2,*

$$\begin{aligned}
c_{ij} = &- B_i \sum_k \left(\frac{B_k}{\lambda_k - \lambda_i\overline{\lambda}_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \overline{\lambda}_i\lambda_j} + \frac{B_k}{\lambda_k - \lambda_i\lambda_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\lambda_j}\right) \\
&- 4\lambda_1\left(\frac{B_iB_1\lambda_j}{(\lambda_i - \lambda_1\lambda_j)(\lambda_1 - \lambda_i\overline{\lambda}_j)} + \frac{B_iB_1\overline{\lambda}_j}{(\lambda_1 - \lambda_i\lambda_j)(\lambda_i - \lambda_1\overline{\lambda}_j)}\right) \\
&\vdots \\
&- 4\lambda_s\left(\frac{B_iB_s\lambda_j}{(\lambda_i - \lambda_s\lambda_j)(\lambda_s - \lambda_i\overline{\lambda}_j)} + \frac{B_iB_s\overline{\lambda}_j}{(\lambda_s - \lambda_i\lambda_j)(\lambda_i - \lambda_s\overline{\lambda}_j)}\right) \\
&\vdots \\
&- 4\overline{\lambda}_s\left(\frac{B_i\overline{B}_s\lambda_j}{(\lambda_i - \overline{\lambda}_s\lambda_j)(\overline{\lambda}_s - \lambda_i\overline{\lambda}_j)} + \frac{B_i\overline{B}_s\overline{\lambda}_j}{(\overline{\lambda}_s - \lambda_i\lambda_j)(\lambda_i - \overline{\lambda}_s\overline{\lambda}_j)}\right) \\
&+ B_1\left(\frac{B_i\lambda_j}{(\lambda_i - \lambda_1\lambda_j)} + \frac{B_i\overline{\lambda}_j}{(\lambda_i - \lambda_1\overline{\lambda}_j)}\right) \\
&\vdots \\
&+ B_s\left(\frac{B_i\lambda_j}{(\lambda_i - \lambda_s\lambda_j)} + \frac{B_i\overline{\lambda}_j}{(\lambda_i - \lambda_s\overline{\lambda}_j)}\right) \\
&\vdots \\
&+ \overline{B}_s\left(\frac{B_i\lambda_j}{(\lambda_i - \overline{\lambda}_s\lambda_j)} + \frac{B_i\overline{\lambda}_j}{(\lambda_i - \overline{\lambda}_s\overline{\lambda}_j)}\right).
\end{aligned} \tag{29}$$

Introducing the value of B_i above, we get

$$n^2\frac{c_{ij}}{\lambda_i} = -\sum_k\Big(\frac{\lambda_k}{\lambda_k-\lambda_i\overline{\lambda}_j}+\frac{\overline{\lambda}_k}{\overline{\lambda}_k-\overline{\lambda}_i\lambda_j}+\frac{\lambda_k}{\lambda_k-\lambda_i\lambda_j}+\frac{\overline{\lambda}_k}{\overline{\lambda}_k-\lambda_i\lambda_j}\Big) \Big\} \quad (1)$$

$$\left.\begin{aligned}
&-4\Big(\frac{\lambda_1^2\lambda_j}{(\lambda_i-\lambda_1\lambda_j)(\lambda_1-\lambda_i\overline{\lambda}_j)}+\frac{\lambda_1^2\overline{\lambda}_j}{(\lambda_1-\lambda_i\lambda_j)(\lambda_i-\lambda_1\overline{\lambda}_j)}\Big)\\
&\vdots\\
&-4\Big(\frac{\lambda_s^2\lambda_j}{(\lambda_i-\lambda_s\lambda_j)(\lambda_s-\lambda_i\overline{\lambda}_j)}+\frac{\lambda_s^2\overline{\lambda}_j}{(\lambda_s-\lambda_i\lambda_j)(\lambda_i-\lambda_s\overline{\lambda}_j)}\Big)\\
&\vdots\\
&-4\Big(\frac{\overline{\lambda}_s^2\lambda_j}{(\lambda_i-\overline{\lambda}_s\lambda_j)(\overline{\lambda}_s-\lambda_i\overline{\lambda}_j)}+\frac{\overline{\lambda}_s^2\overline{\lambda}_j}{(\overline{\lambda}_s-\lambda_i\lambda_j)(\lambda_i-\overline{\lambda}_s\overline{\lambda}_j)}\Big)
\end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned}
&+\Big(\frac{\lambda_1\lambda_j}{(\lambda_i-\lambda_1\lambda_j)}+\frac{\lambda_1\overline{\lambda}_j}{(\lambda_i-\lambda_1\overline{\lambda}_j)}\Big)\\
&\vdots\\
&+\Big(\frac{\lambda_s\lambda_j}{(\lambda_i-\lambda_s\lambda_j)}+\frac{\lambda_s\overline{\lambda}_j}{(\lambda_i-\lambda_s\overline{\lambda}_j)}\Big)\\
&\vdots\\
&+\Big(\frac{\overline{\lambda_s}\lambda_j}{(\lambda_i-\overline{\lambda}_s\lambda_j)}+\frac{\overline{\lambda_s}\overline{\lambda}_j}{(\lambda_i-\overline{\lambda}_s\overline{\lambda}_j)}\Big)
\end{aligned}\right\} \quad (3).$$

Now we collect together the terms of the above expression as follows: from each group (1),(2) or (3) in the above equation we select one term so the above equation become a sum of terms of the following form

$$\begin{aligned}
T =&\frac{\lambda_k}{\lambda_k-\lambda_i\overline{\lambda}_j}+\frac{4\lambda_k^2\lambda_j}{(\lambda_i-\lambda_k\lambda_j)(\lambda_k-\lambda_i\overline{\lambda}_j)}-\Big(\frac{\lambda_k\lambda_j}{(\lambda_i-\lambda_k\lambda_j)}\Big)\\
=&\frac{2}{(1-\lambda_i\overline{\lambda}_j\overline{\lambda}_k)}-\frac{4}{(1-\lambda_i\overline{\lambda}_j\overline{\lambda}_k)^2},
\end{aligned} \quad (30)$$

which by lemma 5.1 each one has positive real part. In particular when $i=j$ we get that $\Re(c_{ii})<0$ and so we are in the conditions of Theorem 3.2, thus we are done in this case. ◇

- Case *ii)* n is odd: In this case consider the surface $S=\{(a,\mathbf{b}):|\lambda_1(a,\mathbf{b})|=\cdots=|\lambda_s(a,\mathbf{b})|=|\lambda_{s+1}=1\}|$, as in the above case this is a codimension one surface in a neighborhood of $(3/4,\mathbf{0})\in S$. This surface is foliated by the surfaces $\{\lambda_{s+1}=c\}$ with c near enough to -1. We are going to get the theorem applying Theorem 3.2 to the maps $h_{(a,\mathbf{b})}$ restricted to the central manifold associated to the eigenvalues λ_i with $i = 1, 2, \ldots, 2s$. In order to do that we need to write first $h_{(a,\mathbf{b})}$ as follows

$$\begin{aligned} h_{(a,\mathbf{b})}(\mathbf{y}) &= H_{\mathbb{R}}\mathbf{y} - (S_{\mathbb{R}}\mathbf{y})_n^2\, S_{\mathbb{R}}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= H_{\mathbb{R}}\mathbf{y} - (y_1 + \cdots + y_s + y_n)^2\, S_{\mathbb{R}}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned} \tag{31}$$

where $H_{\mathbb{R}}$ is the real canonical Jordan form of H as in section 4.3 and $S_{\mathbb{R}}\mathbf{y} = \eta$ with

$$S_{\mathbb{R}} = \begin{pmatrix} \alpha_1^{n-1} & \cdots & \alpha_s^{n-1} & \beta_1^{n-1} & \cdots & -\beta_s^{n-1} & 1 \\ \vdots & & \vdots & \vdots & & \vdots & \\ \alpha_1 & \cdots & \alpha_s & -\beta_1 & \cdots & -\beta_s & -1 \\ 1 & & 1 & 0 & & 0 & 1 \end{pmatrix}. \tag{32}$$

Now letting the coordinates $y_1, \ldots y_s, y_{s+1}, \ldots y_{2s}$ be complex coordinates and putting as in section 3,

$$(z_1, \ldots, z_s, z_{s+1}, \ldots, z_{2s}) = P(y_1, \ldots, y_s, y_{s+1}, \ldots, y_{2s})$$

and $z_n = y_n$ we get that in this new coordinates $\mathbf{z} = (z_1, \ldots, z_s, z_{s+1}, \ldots, z_{2s}, z_n)$

$$\begin{aligned} h_{(a,\mathbf{b})}(\mathbf{z}) &= (\lambda_1 z_1, ..., \lambda_s z_s, \overline{\lambda}_1 z_{s+1}, ..., \overline{\lambda}_s z_{2s}, \lambda_n z_n) \\ &- (\frac{z_1 + z_{s+1}}{2} + \cdots + \frac{z_s + z_{2s}}{2} + z_n)^2 \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_s \\ B_{s+1} \\ \vdots \\ B_{2s} \\ B_n \end{pmatrix} \end{aligned} \tag{33}$$

with $B_i = \overline{B}_{i+s}$ for all $i = 1, ..., s$.

Once done this, we compute $h_{(a,\mathbf{b})}$ restricted to the central manifold C associated to the eigenvalues

$$\lambda_1, ..., \lambda_s, \lambda_{s+1}, ..., \lambda_{2s}$$

. In order to do that we start putting the central manifold as the graph of

$$\phi(Z, \overline{Z}) = \sum \alpha_{ij} z_i z_j + \sum \beta_{ij} z_i \overline{z}_j + \sum \gamma_{ij} \overline{z}_i \overline{z}_j,$$

where $Z = (z_1, \ldots, z_s)$ and $\overline{Z} = (z_{s+1}, \ldots, z_{2s})$.

LEMMA 4.2 *For ϕ as above we have*

$$\begin{aligned}\alpha_{ij} =& \frac{B_n}{4(\lambda_n - \lambda_i\lambda_j)}\\ \beta_{ij} =& \frac{B_n}{2(\lambda_n - \lambda_i\overline{\lambda}_j)}\\ \gamma_{ij} =& \frac{B_n}{2(\lambda_n - \overline{\lambda}_i\overline{\lambda}_j)}.\end{aligned}$$

PROOF

These formulas follow solving the equation $h_{(a,\mathbf{b})}(Z,\overline{Z},\phi(Z,\overline{Z}) = (W,\overline{W},\phi(W,\overline{W}))$ for the coefficients α_{ij},β_{ij} and γ_{ij}. ◇

In order to apply Theorem 3.2 we need to compute the quadratic and cubic terms of $h_{(a,\mathbf{b})}|C$. Observe that $(h_{(a,\mathbf{b})}|C)_i$, the $i-th$ component of $h_{(a,\mathbf{b})}$, is written as :

$$(h_{(a,\mathbf{b})}|C)_i(Z,\overline{Z}) = \lambda_i z_i + \left(\sum \frac{z_l + z_{l+s}}{2} + \phi(Z,\overline{Z})\right)^2 B_i. \tag{34}$$

From here we get

$$\begin{aligned}(h_{(a,\mathbf{b})}|C)_i(Z,\overline{Z}) =& \lambda_i z_i + \left(\sum \frac{z_l + z_{l+s}}{2}\right)^2 B_i\\ &+ 2\sum\left(\frac{z_l + z_{l+s}}{2}\right)\phi(Z,\overline{Z})B_i + h.o.t\\ =& \lambda_i z_i + \left(\sum \frac{z_l + z_{l+s}}{2}\right)^2 B_i\\ &+ \sum \beta_{ll} z_i z_l \overline{z}_l.B_i + \sum(\alpha_{il} + \alpha_{li}) z_i z_l \overline{z}_l + h.o.t.\end{aligned} \tag{35}$$

As in the above case, we only need to check that certain coefficients are negative. The quadratic part produces coefficients that are treated as when n is even. Indeed we will be worried about just the terms

$$\sum \beta_{ll} z_i z_l \overline{z}_l.B_i + \sum(\alpha_{il} + \alpha_{li}) z_i z_l \overline{z}_l. \tag{36}$$

So we need to check that

$$\begin{aligned}* =& \Re\frac{b_{ll}B_i}{\lambda_i} + \Re\frac{(\alpha_{il} + \alpha_{li})B_i}{\lambda_i}\\ =& \Re\frac{B_nB_i}{2(\lambda_n - \lambda_l\overline{\lambda}_l)\lambda_i} + \Re\left(\frac{1}{\lambda_n - \lambda_i\lambda_l} + \frac{1}{\lambda_n - \lambda_i\lambda_l}\right).\frac{B_nB_i}{4\lambda_i}\\ =& \Re\frac{B_nB_i}{2(\lambda_n - \lambda_l\overline{\lambda}_l)\lambda_i} + \Re\frac{1}{\lambda_n - \lambda_i\lambda_l}.\frac{B_nB_i}{2\lambda_i} < 0.\end{aligned} \tag{37}$$

Since for $(a, \mathbf{b}) = (3/4, \mathbf{0})$ we get $B_i = \dfrac{-1}{n\overline{\lambda}_i}$ and as $\lambda_n \approx -1$ we have that

$$\begin{aligned} * &= \frac{-B_n}{2n}\Re\frac{1}{\lambda_n - 1} - \frac{B_n}{2n}\Re\frac{1}{\lambda_n - \lambda_i\lambda_l} \\ &= \frac{-B_n}{2n}\frac{1}{\lambda_n - 1} - \frac{B_n}{2n}\frac{\lambda_n + \cos(\theta_i + \theta_l)}{|\lambda_n - e^{i(\theta_i+\theta_l)}|^2} \\ &= \frac{-B_n}{2n}\left(\frac{1}{\lambda_n - 1} + \frac{\lambda_n + \cos(\theta_i + \theta_l)}{|\lambda_n - e^{i(\theta_i+\theta_l)}|^2}\right) < 0, \end{aligned} \tag{38}$$

if $(a, \mathbf{b}) \approx (a, \mathbf{0})$. That is what we wanted to proof.

5 Appendix

In this appendix we present two technical result. The first one treats about the positiveness of the real part of certain expression of complex numbers. The second one treats about the formula to compute third order resonant terms.

LEMMA 5.1 *Consider an expression of the form*

$$z = \frac{2}{1 - e^{(a+b+c)i}} - \frac{4}{[1 - e^{(a+b+c)i}]^2},$$

with $a, b, c \in \mathbb{R}$. *Then* $\Re(z) \geq 0$.

PROOF *A little bit of algebra show us that*

$$\begin{aligned} \Re(z) =& \frac{4[|1 - e^{(a+b+c)i}|^2 + 4\cos(a+b+c)]\sin^2((a+b+c)/2)}{|1 - e^{(a+b+c)i}|^4} \\ =& \frac{8[1 + \cos(a+b+c)]\sin^2((a+b+c)/2)}{|1 - e^{(a+b+c)i}|^4} \geq 0. \end{aligned} \tag{39}$$

◇

As for the second result we consider a map

$$h(Z, \overline{Z}) = (E, \overline{E})$$

where $(Z, \overline{Z}) = (z_1, ..., z_s, \overline{z}_1, ..., \overline{z}_s)$, $E = (e_1, ..., e_s)$ and $\overline{E} = (\overline{e}_1, ..., \overline{e}_s)$.

We suppose that the map e_i has the following form:

$$e_i(Z, \overline{Z}) = \lambda_i z_i - B_i(Z, \overline{Z})M\begin{pmatrix} Z \\ \overline{Z} \end{pmatrix}$$

where $M = \frac{1}{4}\left(\begin{array}{c|c} I_s & I_s \\ \hline I_s & I_s \end{array}\right)$ and I_s is the identity matrix of size s. As well as suppose that $\lambda_i, \overline{\lambda}_i$ satisfy nonresonant conditions of order two and three as in Proposition 3.1. Then it can be eliminated with a change of variables the quadratic terms and the nonresonant cubic terms. The next lemma give us a formula to compute the resonant cubic terms.

LEMMA 5.2 *in the above conditions , there exists a change of coordinates $\Phi(Z,\overline{Z})$ such that if $H(Z,\overline{Z}) = (c_1, ..., c_s, \overline{c}_1, ..., \overline{c}_s)$ is the expression in this new coordinates of h, then*

$$c_i = \lambda_i z_i + \sum p^i_j z_i |z_j|^2 + h.o.t.$$

with

$$\begin{aligned}
p^i_j =& - B_i \sum_k \left(\frac{B_k}{\lambda_k - \lambda_i\overline{\lambda}_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \overline{\lambda}_i\lambda_j} + \frac{B_k}{\lambda_k - \lambda_i\lambda_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\lambda_j}\right) \\
&- 4\lambda_1\left(\frac{B_iB_1\lambda_j}{(\lambda_i - \lambda_1\lambda_j)(\lambda_1 - \lambda_i\overline{\lambda}_j)} + \frac{B_iB_1\overline{\lambda}_j}{(\lambda_1 - \lambda_i\lambda_j)(\lambda_i - \lambda_1\overline{\lambda}_j)}\right) \\
&\vdots \\
&- 4\lambda_s\left(\frac{B_iB_s\lambda_j}{(\lambda_i - \lambda_s\lambda_j)(\lambda_s - \lambda_i\overline{\lambda}_j)} + \frac{B_iB_s\overline{\lambda}_j}{(\lambda_s - \lambda_i\lambda_j)(\lambda_i - \lambda_s\overline{\lambda}_j)}\right) \\
&\vdots \\
&- 4\overline{\lambda}_s\left(\frac{B_i\overline{B}_s\lambda_j}{(\lambda_i - \overline{\lambda}_s\lambda_j)(\overline{\lambda}_s - \lambda_i\overline{\lambda}_j)} + \frac{B_i\overline{B}_s\overline{\lambda}_j}{(\overline{\lambda}_s - \lambda_i\lambda_j)(\lambda_i - \overline{\lambda}_s\overline{\lambda}_j)}\right) \\
&+ B_1\left(\frac{B_i\lambda_j}{(\lambda_i - \lambda_1\lambda_j)} + \frac{B_i\overline{\lambda}_j}{(\lambda_i - \lambda_1\overline{\lambda}_j)}\right) \\
&\vdots \\
&+ B_s\left(\frac{B_i\lambda_j}{(\lambda_i - \lambda_s\lambda_j)} + \frac{B_i\overline{\lambda}_j}{(\lambda_i - \lambda_s\overline{\lambda}_j)}\right) \\
&\vdots \\
&+ \overline{B}_s\left(\frac{B_i\lambda_j}{(\lambda_i - \overline{\lambda}_s\lambda_j)} + \frac{B_i\overline{\lambda}_j}{(\lambda_i - \overline{\lambda}_s\overline{\lambda}_j)}\right)
\end{aligned} \tag{40}$$

PROOF

We want to do a change of coordinates which kill all quadratic terms. This change of coordinates has the following form:

$$\Phi(Z,\overline{Z}) = (\Psi_1(Z,\overline{Z}), ..., \Psi_s(Z,\overline{Z}), \overline{\Psi}_1(Z,\overline{Z}), ..., \overline{Z}), \overline{\Psi}_s(Z,\overline{Z}),$$

con $\Psi_i(Z,\overline{Z}) = z_i + (Z,\overline{Z})N^i\begin{pmatrix} Z \\ \overline{Z}\end{pmatrix}$ with $N^i = (n^i_{kj})$ a matrix of size $2s$. From the above equation we conclude that

$$\Phi^{-1}(W,\overline{W}) = (\Psi_1^{-1}(W,\overline{W}), ..., \Psi_s^{-1}(W,\overline{W}), \overline{\Psi}_1^{-1}(W,\overline{W}), ..., \overline{W}), \overline{\Psi}_s^{-1}(W,\overline{W}),$$

with $\Psi_i^{-1}(W,\overline{W}) = w_i - (W,\overline{W})N^i\Big(\begin{array}{c} W \\ \overline{W} \end{array}\Big) + h.o.t$

let $H(Z,\overline{Z})$ *be the map* $\Phi^{-1}\circ h\circ\Phi(Z,\overline{Z})$, *and put* $(\Psi,\overline{\Psi}) = (\Psi_1,...,\Psi_s,\overline{\Psi}_1,...,\overline{\Psi}_s)$, $h\circ\Phi(Z,\overline{Z}) = (d_1,...,d_s,\overline{d}_1,...,\overline{d}_s) = (D,\overline{D})$ *and* $H(Z,\overline{Z}) = (c_1,...,c_s,\overline{c}_1,...,\overline{c}_s)$, *then*

$$\begin{aligned} d_i &= \lambda_i\psi_i - B_i(\Psi,\overline{\Psi})M\Big(\begin{array}{c} \Psi \\ \overline{\Psi} \end{array}\Big) \\ c_i &= d_i - (D,\overline{D})N^i\Big(\begin{array}{c} D \\ \overline{D} \end{array}\Big) + h.o.t. \end{aligned} \tag{41}$$

From here we get that

$$\begin{aligned} c_i =&\lambda_i\Psi_i - B_i(\Psi,\overline{\Psi})M\Big(\begin{array}{c} \Psi \\ \overline{\Psi} \end{array}\Big) - (\Lambda\Psi,\overline{\Lambda\Psi})N^i\Big(\begin{array}{c} \Lambda\Psi \\ \overline{\Lambda\Psi} \end{array}\Big) + \\ &(\Lambda\Psi,\overline{\Lambda\Psi})N^i\Big(\begin{array}{c} B \\ \overline{B} \end{array}\Big).(\Psi,\overline{\Psi})M\Big(\begin{array}{c} \Psi \\ \overline{\Psi} \end{array}\Big) + (B,\overline{B})N^i\Big(\begin{array}{c} \Lambda\Psi \\ \overline{\Lambda\Psi} \end{array}\Big).(\Psi,\overline{\Psi})M\Big(\begin{array}{c} \Psi \\ \overline{\Psi} \end{array}\Big) - \\ &(B,\overline{B})N^i\Big(\begin{array}{c} B \\ \overline{B} \end{array}\Big)[(\Psi,\overline{\Psi})M\Big(\begin{array}{c} \Psi \\ \overline{\Psi} \end{array}\Big)]^2. \end{aligned} \tag{42}$$

So we can conclude that:

1. *lineal terms of* c_i: $\lambda_i\ z_i$.
2. *Quadratic terms of* c_i:

$$c_i^2 = \lambda_i(Z,\overline{Z})N^i\Big(\begin{array}{c} Z \\ \overline{Z} \end{array}\Big) - B_i(Z,\overline{Z})M\Big(\begin{array}{c} Z \\ \overline{Z} \end{array}\Big) - (\Lambda Z,\overline{\Lambda Z})N^i\Big(\begin{array}{c} \Lambda Z \\ \overline{\Lambda Z} \end{array}\Big).$$

3. *Cubic terms of* c_i:

$$\begin{aligned} c_i^3 =& -2B_i(Z,\overline{Z})M\Big(\begin{array}{c} \Psi^2 \\ \overline{\Psi}^2 \end{array}\Big) - 2(\Lambda\Psi^2,\Lambda\overline{\Psi}^2)N^i\Big(\begin{array}{c} \Lambda Z \\ \overline{\Lambda Z} \end{array}\Big) + \\ &(\Lambda Z,\overline{\Lambda Z})N^i\Big(\begin{array}{c} B \\ \overline{B} \end{array}\Big).(Z,\overline{Z})M\Big(\begin{array}{c} Z \\ \overline{Z} \end{array}\Big) + \\ &(B,\overline{B})N^i\Big(\begin{array}{c} \Lambda Z \\ \overline{\Lambda Z} \end{array}\Big).(Z,\overline{Z})M\Big(\begin{array}{c} Z \\ \overline{Z} \end{array}\Big) \\ =& -2B_i(Z,\overline{Z})M\Big(\begin{array}{c} \Psi^2 \\ \overline{\Psi}^2 \end{array}\Big) - 2(\Lambda\Psi^2,\Lambda\overline{\Psi}^2)N^i\Big(\begin{array}{c} \Lambda Z \\ \overline{\Lambda Z} \end{array}\Big) + \\ &2(B,\overline{B})N^i\Big(\begin{array}{c} \Lambda Z \\ \overline{\Lambda Z} \end{array}\Big).(Z,\overline{Z})M\Big(\begin{array}{c} Z \\ \overline{Z} \end{array}\Big) \end{aligned}$$

We get, from the vanishing condition on the quadratic terms c_i^2, that the matrix N^i has the following form:

$$\begin{aligned} n^i_{kj} &= \frac{B_i}{\lambda_i - \lambda_k\lambda_j} \\ n^i_{k\,s+j} &= \frac{B_i}{\lambda_i - \lambda_k\overline{\lambda}_j} \\ n^i_{s+k\,s+j} &= \frac{B_i}{\lambda_i - \overline{\lambda}_k\overline{\lambda}_j} \end{aligned} \tag{43}$$

where $j, k = 1, \ldots, s$.

Expanding the formula for c_i^3, we get

$$\begin{aligned} c_i^3 = &- \frac{B_i}{2}\sum_j z_j(\Psi_1^2 + \cdots + \Psi_s^2 + \overline{\Psi}_1^2 + \cdots + \overline{\Psi}_s^2) \\ &- \frac{B_i}{2}\sum_j \overline{z}_j(\Psi_1^2 + \cdots + \Psi_s^2 + \overline{\Psi}_1^2 + \cdots + \overline{\Psi}_s^2) \\ &- 2\lambda_1\Psi_1^2\sum_j n^i_{1j}\lambda_j z_j \\ &- 2\lambda_1\Psi_1^2\sum_j n^i_{1\,s+j}\overline{\lambda}_j\overline{z}_j \\ &\vdots \\ &- 2\lambda_s\Psi_s^2\sum_j n^i_{2sj}\lambda_j z_j \\ &- 2\lambda_s\Psi_1^2\sum_j n^i_{2s\,s+j}\overline{\lambda}_j\overline{z}_j \\ &+ 2B_1\sum_j n^i_{1j}\lambda_j z_j(\frac{z_1+\overline{z}_1}{2} + \frac{z_2+\overline{z}_2}{2} + \cdots + \frac{z_s+\overline{z}_s}{2})^2 \\ &+ 2B_1\sum_j n^i_{1\,s+j}\overline{\lambda}_j\overline{z}_j(\frac{z_1+\overline{z}_1}{2} + \frac{z_2+\overline{z}_2}{2} + \cdots + \frac{z_s+\overline{z}_s}{2})^2 \\ &\vdots \\ &+ 2\overline{B}_s\sum_j n^i_{2sj}\lambda_j z_j(\frac{z_1+\overline{z}_1}{2} + \frac{z_2+\overline{z}_2}{2} + \cdots + \frac{z_s+\overline{z}_s}{2})^2 \\ &+ 2\overline{B}_s\sum_j n^i_{2s\,s+j}\overline{\lambda}_j\overline{z}_j(\frac{z_1+\overline{z}_1}{2} + \frac{z_2+\overline{z}_2}{2} + \cdots + \frac{z_s+\overline{z}_s}{2})^2. \end{aligned} \tag{44}$$

Let p^i_j the term of c_i^3 which match the monomial $z_i z_j \overline{z}_j$. Then we get from the last equation

$$\begin{aligned} p^i_j = \ & -\tfrac{B_i}{2}\sum_k(2n^k_{i\,s+j} + 2\overline{n}^k_{j\,s+i} + 2n^k_{ij} + 2\overline{n}^k_{s+i\,s+j}) \\ & -4\lambda_1(n^i_{1j}\lambda_j n^1_{i\,s+j} + n^i_{1\,s+j}\overline{\lambda}_j n^1_{ij}) \\ & \vdots \\ & -4\lambda_s(n^i_{sj}\lambda_j n^1_{i\,s+j} + n^i_{s\,s+j}\overline{\lambda}_j n^s_{ij}) \\ & \vdots \\ & -4\overline{\lambda}_s(n^i_{2sj}\lambda_j \overline{n}^s_{j\,s+i} + n^i_{2s\,s+j}\overline{\lambda}_j \overline{n}^s_{s+is+j}) \\ & +B_1(n^i_{1j}\lambda_j + n^i_{1\,s+j}\overline{\lambda}_j) \\ & \vdots \\ & +B_s(n^i_{sj}\lambda_j + n^i_{s\,s+j}\overline{\lambda}_j) \\ & \vdots \\ & +\overline{B}_s(n^i_{2sj}\lambda_j + n^i_{2s\,s+j}\overline{\lambda}_j) \end{aligned} \tag{45}$$

and using the formula for the matrix N^i we get

$$\begin{aligned} p^i_j = \ & -B_i\sum_k\left(\frac{B_k}{\lambda_k-\lambda_i\overline{\lambda}_j} + \frac{\overline{B}_k}{\overline{\lambda}_k-\overline{\lambda}_i\lambda_j} + \frac{B_k}{\lambda_k-\overline{\lambda}_i\lambda_j} + \frac{\overline{B}_k}{\overline{\lambda}_k-\lambda_i\lambda_j}\right) \\ & -4\lambda_1\left(\frac{B_iB_1\lambda_j}{(\lambda_i-\lambda_1\lambda_j)(\lambda_1-\lambda_i\overline{\lambda}_j)} + \frac{B_iB_1\overline{\lambda}_j}{(\lambda_1-\lambda_i\lambda_j)(\lambda_i-\lambda_1\overline{\lambda}_j)}\right) \\ & \vdots \\ & -4\lambda_s\left(\frac{B_iB_s\lambda_j}{(\lambda_i-\lambda_s\lambda_j)(\lambda_s-\lambda_i\overline{\lambda}_j)} + \frac{B_iB_s\overline{\lambda}_j}{(\lambda_s-\lambda_i\lambda_j)(\lambda_i-\lambda_s\overline{\lambda}_j)}\right) \\ & \vdots \\ & -4\overline{\lambda}_s\left(\frac{B_i\overline{B}_s\lambda_j}{(\lambda_i-\overline{\lambda}_s\lambda_j)(\overline{\lambda}_s-\lambda_i\overline{\lambda}_j)} + \frac{B_i\overline{B}_s\overline{\lambda}_j}{(\overline{\lambda}_s-\lambda_i\lambda_j)(\lambda_i-\overline{\lambda}_s\overline{\lambda}_j)}\right) \\ & +B_1\left(\frac{B_i\lambda_j}{(\lambda_i-\lambda_1\lambda_j)} + \frac{B_i\overline{\lambda}_j}{(\lambda_i-\lambda_1\overline{\lambda}_j)}\right) \\ & \vdots \\ & +B_s\left(\frac{B_i\lambda_j}{(\lambda_i-\lambda_s\lambda_j)} + \frac{B_i\overline{\lambda}_j}{(\lambda_i-\lambda_s\overline{\lambda}_j)}\right) \\ & \vdots \\ & +\overline{B}_s\left(\frac{B_i\lambda_j}{(\lambda_i-\overline{\lambda}_s\lambda_j)} + \frac{B_i\overline{\lambda}_j}{(\lambda_i-\overline{\lambda}_s\overline{\lambda}_j)}\right) \end{aligned} \tag{46}$$

that is what we want to show. ◇

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