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The Qualitative Theory of Control Processes

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Notas de Matemática

Serie: Pre-Print

No. 215

Mérida - Venezuela

2001

Notas de Matemática, No. 215
Mérida, 2001.

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1. Introduction

Controllability and observability problems for discrete and ordinary differential systems were formulated and solved originally by R.Kalman in 1960. These problems as before play a central role in modern control theory, in particular for nonautonomous ordinary differential systems, functional-differential systems, singularly perturbed dynamic systems (SPDS), SPDS with delay (SPDSD). The objects of study become more complicated, the range of controllability and observability problems is widened, definitions, the notions related to their solutions undergo refinement, known approaches and investigation methods are classified, new approaches are suggested, and their relationships are analyzed.

There are several approaches to study the controllability and observability problems for linear nonstationary dynamic systems. The efficiency of introducing *the defining equations* (i.e. matrix algebraic recurrence equations) for studying the controllability problem for nonstationary differential systems with delay was shown in [1].

In this paper *the unified method* of investigating controllability and observability problems for various types of systems (autonomous and nonstationary ordinary differential systems, SPDS, linear functional-differential systems, SPDSD etc.) is suggested. This method combines the state space method and the method of the defining equations, takes into account the specific character of the objects being investigated (their nonstationarity, singularity, the presence or lack of delay) and does not require the investigation of conjugate systems in the observability problem. In terms of the components of the defining equations we formulate all controllability and observability conditions. The rules for constructing the defining equations are very simple and reflect the type of the object being investigated by a natural way.

The present paper is a mini-course, which has been read for postgraduated students and professors of the Department of Mathematics (Science Faculty) and Department of Control Systems (Engineering Faculty) of the University de Los Andes during October, 30 - November, 21, 2000. It includes such directions of qualitative theory of control processes as controllability and observability problems of linear time-invariant and time-varying systems (of ordinary differential equations, with constant delay, with the deviatng argument of neutral type), singularly perturbed dynamic systems (without delay, with constant delay, with the deviatng argument of neutral type), the stabilization problem.

2. Controllability of Linear Time-Invariant Systems

2.1. Complete Controllability

Let us consider a control system for which the equation of controlled motion has the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq t_0, \quad x \in R^n, \quad u \in R^m, \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$ are constant matrices. There are no restrictions on the value of the control function $u(t)$.

Definition 2.1. *System (1) is called controllable on the segment $T = [t_0, t_1]$, $t_1 > t_0$, if for any vectors $x_0, x_1 \in R^n$ there exists such a control function $u(t)$, that the corresponding solution of the equation (1) satisfies the condition $x(t_0) = x_0$, $x(t_1) = x_1$.*

In other words, the controllable system can be transferred from any arbitrary state x_0 at the moment t_0 to another arbitrary prescribed final state x_1 at the moment t_1 by choosing the corresponding control $u(t)$. It is well known that the controllability condition of system (1) depends on so-called *the controllability matrix*

$$K = \{B, AB, \ldots, A^{n-1}B\}. \tag{2}$$

A pair of matrices (A, B) is called controllable if rank of (2) is equal to n.

Theorem 2.1. *System (1) is controllable if and only if rank of the controllability matrix is equal to n :*

$$rank\{B, AB, \ldots, A^{n-1}B\} = n. \tag{3}$$

Let us denote

$$W(t_0, t) = \int_{t_0}^{t} F(t_0, \tau)BB'F'(t_0, \tau)d\tau,$$

where $F(t,\tau)$ is a fundamental Cauchy matrix, satisfying the homogeneous part of the equation (1):

$$\frac{dF(t,\tau)}{dt} = AF(t,\tau),$$

$$\frac{dF(t,\tau)}{d\tau} = -F(t,\tau)A$$

with the initial condition

$$F(t,t) = E_n,$$

where E_n is identity $n \times n$-matrix.

Then Theorem 2.1 is equivalent to the following one.

Theorem 2.2. *The system (1) is controllable on the segment* $[t_0, t_1]$ *if and only if the matrix* $W(t_0, t_1)$ *is positively defined.*

Then the control function which transfers the state of system (1) from any initial point x_0 to any prescribed final point x_1 has the form

$$u(t) = -B'F'(t_0,t)W^{-1}(t_0,t_1)[x_0 - F(t_0,t_1)x_1].$$

Example 2.1. Consider the controllable motion of the material point on the line under the influence of scalar controlling force $u(t)$. The equation of such motion has the form

$$\ddot{x}(t) = u(t).$$

Let $x_1(t)$ be the coordinate of the point, $x_2(t)$ is its velocity. Then the equation of the motion has the form

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t). \tag{4}$$

Equations (4) will have the form (1) if suppose

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The matrix K due to the equality (2) has the form

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So $rank\, K = 2$ and the motion of the material point is controllable.

Remarks.

1. If $rankB = k$, then necessary and sufficient condition for controllability of (4) is

$$rank(B,\ AB,\ \ldots,A^{n-k}) = n.$$

2. The controllability criterion

$$rank\{B, AB, \ldots, A^{n-1}B\} = n \tag{5}$$

is established in the assumption that there are no restrictions on the value of the control function $u(t)$. Controllability criteria with the restrictions on the value of control are of more interest for engineers. For example, the criterion (5) is not sufficient under the restriction $|u(t)| \leq C$, where C is a given constant. Really, if all eigenvalues of matrix A lie at the left half-plane then for any initial state x_0 and any control system (1) remains inside of some restrict domain. On the contrary, if all eigenvalues of the matrix A lie at the right half-plane then there is no one control function that transfers the point x_0 to the origin if x_0 is sufficiently far from the origin of coordinates. In both these cases the system (1) is not controllable.

Theorem 2.3. *In order that the system (1) be controllable under the restriction $|u(t)| \leq C$, $C > 0$, it is necessary and sufficient that*

$$rank\{B, AB, \ldots, A^{n-1}B\} = n$$

and besides all eigenvalues of the matrix A have to lie at the imaginary axis.

(see: Ovseevich A.I. On the complete controllability of linear systems// Prikl. Math. and Mech., 1989, Vol. 53. Vyp. 5.)

2.2. Relative Controllability

Definition 2.2. *The dynamical system (1) is called controllable relative to the subspace H, $Hx = 0$ (relatively controllable) if for each state x_0 there exists a number $t^* < \infty$, and a piece-wise continuous control $u(t)$, $t_0 \leq t \leq t^*$, such that $Hx(t^*) = 0$.*

Theorem 2.4. *The dynamical system (1) is relatively controllable if and only if*

$$rank\{HB, HAB, \ldots, HA^{n-1}B\} = rankH. \tag{6}$$

2.3. Conditional Controllability

Definition 2.3. *The dynamical system (1) is called controllable in the space M (conditionally controllable) if each initial state from the subspace M, $x_0 = My$, $y \in R^n$ is controllable.*

Theorem 2.5. *The dynamical system (1) is conditionally controllable if and only if*

$$rank\{M, B, AB, \ldots, A^{n-1}B\} = rank\{B, AB, \ldots, A^{n-1}B\} \tag{7}$$

Example 2.2. Consider the 4-th order control system with 4 inputs

$$\begin{cases} \dot{x}_1 = 3x_1 + x_2 + u_1 + 2u_2 + 5u_4 \\ \dot{x}_2 = -4x_1 - x_2 + u_2 + 4u_4 \\ \dot{x}_3 = 6x_1 - x_2 + 2x_3 + x_4 + u_3 + 3u_4 \\ \dot{x}_4 = -14x_1 - 5x_2 - x_3 \end{cases} \tag{8}$$

with the matrices

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 6 & -1 & 2 & 1 \\ -14 & -5 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let the initial state of this system lies in the plane $x_3 = x_4 = 0$. It is required to determine whether this system is conditionally controllable separately by the first control.

In this case, the subspace M is given by the expression $x = My$, $y \in R^4$, where

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the vector $b' = (1, 0, 0, 0)$ we have $Ab' = (3, -4, 6, -14)$. The conditions of Theorem 4 are satisfied meaning that each state from M may be transferred to the origin by the control u_1. However, all states from the plane $x_1 = x_2 = 0$ may not be transferred to the origin by this control. In this subspace the system is controllable by the third control u_3.

2.4. Conditionally-Relative Controllability

Let matrices $M \in R^{n\times q}$ and $H \in R^{m\times n}$ be given.

Definition 2.4. *System (1), $t > 0$, is called completely conditionally-relatively controllable if for any q-vector y_0 there exists such a time moment $t_1 < \infty$ and a piece-wise control $u(t)$, $t > 0$, that for this equation and for the corresponding solution $x(t)$ of the system (1) with the initial condition $x(0) = My_0$ the following relations $Hx(t) \equiv 0$, $u(t) \equiv 0$, $t \geq t_1$, take place.*

Theorem 2.6. *System (1) is completely conditionally-relatively controllable if and only if*

$$rank \left\{ \begin{matrix} HA^k \\ k = \overline{0, n-1} \end{matrix} \right\} \cdot \left\{ A^k B,\ k = \overline{0, n-1}, \quad M \right\} =$$

$$= rank \left\{ \frac{HA^k}{k = \overline{0, n-1}} \right\} \cdot \left\{ A^k B,\, k = \overline{0, n-1} \right\}. \tag{9}$$

2.5. Canonical Form of Linear Time-Invariant Systems

Consider linear system (1). Suppose that $rankK = j$, $j \leq n$. Let us show that there exists the change of variables

$$x = Ty, \quad detT \neq 0,$$

such that last $n - j$ coordinates $y_{j+1}, \dots, y_n$ of the vector y don't depend on neither control function no the previous coordinates $y_1, \dots, y_j$. Let $k_1, \dots, k_j$ be linear independent columns of K. Let matrix T has vectors $k_1, \dots, k_j$ as first columns and the rest $(n - j)$ columns are arbitrary with the condition $detT \neq 0$. We have then from equation (1)

$$\dot{y}(t) = T^{-1}ATy + T^{-1}Bu. \tag{10}$$

Theorem 2.7. *For matrices $T^{-1}AT$ and $T^{-1}B$ the following relations are valid*

$$T^{-1}AT = \begin{pmatrix} A_1 & A_2 \\ 0_{(n-j)\times j} & A_3 \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} B_1 \\ 0_{(n-j)\times m} \end{pmatrix}, \tag{11}$$

where $A_1 \in R^{j\times j}$, $A_2 \in R^{j\times(n-j)}$, $A_3 \in R^{(n-j)\times(n-j)}$, $B_1 \in R^{j\times m}$. Besides

$$rank(B_1, A_1B_1, \dots, A_1^{j-1}B_1) = j. \tag{12}$$

From this theorem follows that if vector y can be presented in the form $y = (z_1, z_2)$ where $z_1 \in R^j$, $z_2 \in R^{n-j}$, then due to (10), (11) the following relations

$$\dot{z}_1 = A_1z_1 + A_2z_2 + B_1,$$

$$z_2 = A_3z_2$$

with the initial states

$$\begin{pmatrix} z_1(t_0) \\ z_2(t_0) \end{pmatrix} = T^{-1}x(t_0)$$

are valid for the components z_1, z_2. So the component z_2 of the vector y doesn't depend neither control u no z_1 and can be calculated in advance as function of time t. If we substitute it to the equation for z_1 we shall obtain nonhomogeneous differential equation.

Suppose that system (1) is controllable, $u(t)$ is a scalar control, $B = b$ is a column vector. Consider a characteristic polynomial of the matrix A :

$$det(\lambda E - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_n$$

and a set of vectors

$$\gamma_1 = A^{n-1}b + \alpha_1 A^{n-2}b + \ldots + \alpha_{n-1}b,$$

$$\gamma_2 = A^{n-2}b + \alpha_1 A^{n-2}b + \ldots + \alpha_{n-2}b,$$

$$\ldots$$

$$\gamma_{n-1} = Ab + \alpha_1 b,$$

$$\gamma_n = b.$$

Due to the controllobility of system (1) it follows that vectors $\gamma_1, \ldots, \gamma_n$ form basis in the space R^n.

Let L be nonsingular a transfer matrix from the initial basis to the basis $\gamma_1, \ldots, \gamma_n$. Suppose $x(t) = Ly(t)$. Then

$$\dot{y}(t) = L^{-1}ALy(t) + L^{-1}Bu(t), \tag{13}$$

where

$$L^{-1}AL = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & -\alpha_{n-3} & \ldots & -\alpha_1 \end{pmatrix}, \quad L^{-1}B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{14}$$

In other words system (1) is equivalent to the differential equation of the order n with respect to $y_1(t)$ that has the form

$$y_1^{(n)}(t) + \alpha_1 y_1^{(n-1)}(t) + \ldots + \alpha_n y_1(t) = u(t). \tag{15}$$

So system (1) with the scalar control ($B = b$) which satisfies the controllability condition (5) can be transformed to a single equation of the n-order (15).

2.6. Hautus's Controllability Criterion

To check controllability of a system it is useful to have Hautus's criterion controllability.

Theorem 2.8. *The system (1) is controllable if and only if for any complex number* $\lambda \in C$

$$rank(\lambda E_n - A,\ B) = n.$$

As all proper numbers of the matrix A satisfy the equality

$$det \quad (\lambda E_n - A) \neq 0,$$

then the condition

$$rank(\lambda E_n - A,\ B) = n$$

holds automatically. Therefore it is necessary to check Theorem 2.8 only for all eigenvalues of the matrix A and Theorem 2.8 may be reformulated by the following way:

Theorem 2.8.1. *The system (1) is controllable if and only if for all eigenvalues* λ_i, $i = 1, 2, \ldots, n$ *of the matrix* A

$$rank(\lambda E_n - A\,B) = n.$$

Now we shall give some examples of the application of Theorem 2.8.

2.7. Examples

Example 2.3. Let A be a diagonal matrix with λ_i on the main diagonal. Show that system $\dot{x} = Ax + Bu$ is controllable if and only if all rows of B are nonzero and rows of B, which correspond to the same diagonal elements of matrix A, are linear independent.

Example 2.4. Show that from the controllability of the pare (A, B) follows the controllability of the pair of $(A + \eta E_n), B$ for any number η.

Example 2.5. Let us consider two-links manipulator consisting from two rigid bodies Q_1, Q_2 with masses m_1, m_2 which are fasten together with help of hinge O_2 and with the fixed base with the help of hinge O_1. The axes of hinges are parallel. The manipulator can move in the plane which is perpendicular to the axes of hinges (see Fig.1).

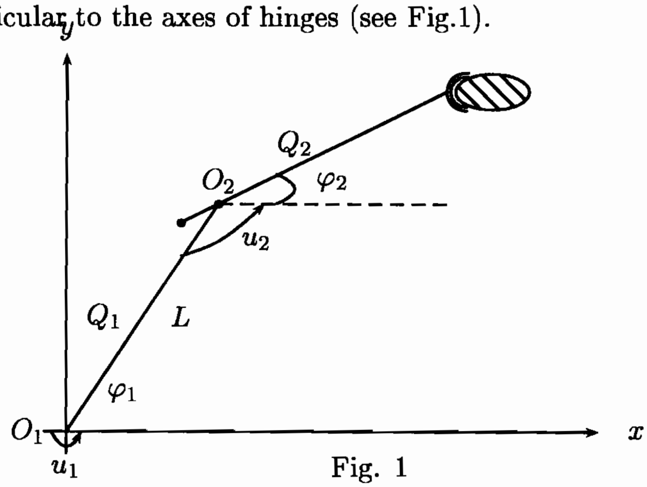

Fig. 1

We can control by the manipulator due to moments u_1, u_2, which are applied to the axes O_1, O_2. Suppose that the centre of masses is disposed on the axis O_2. Then the equations for the motion of the manipulator have the form:

$$(I_1 + m_2 L^2)\ddot{\varphi}_1 = u_1 - u_2,$$
$$I_2 \ddot{\varphi}_2 = u_2, \tag{16}$$

where φ_1 is the angle between link Q_1 and the axis $O_1 x$, φ_2 is the angle between link Q_2 and the axis $O_1 x$, L is the distance between the axes of hinges O_1, O_2; I_1, I_2 are the moments of inertia of hinges Q_1, Q_2 with respect to the axes O_1, O_2 correspondingly.

Introduce new variables

$$x_1 = (I_1 + m_2 L^2)^{1/2}\varphi_1, \quad x_2 = I_2^{1/2}\varphi_2, \quad x_3 = (I_1 + m_2 L^2)^{1/2}\dot{\varphi}_1, \quad x_4 = I_2^{1/2}\dot{\varphi}_2.$$

Then system (16) can be written as

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u_1 - u_2,$$
$$\dot{x}_3 = x_4, \quad \dot{x}_4 = u_2 \tag{17}$$

System (17) has the form of system (1) with the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The controllability matrix of the system (17) is

$$K = (B,\ AB,\ A^2B,\ A^3B) = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can see that

$$det \begin{pmatrix} 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -1,$$

so the system (17) is controllable.

We can obtain the same result from the Hautus Theorem 8. Really all eigenvalues λ_i of matrix A are equal to zero, so the matrix $(A - \lambda E, B)$ for $\lambda = 0$ has the form

$$(A, B) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As $rank(A, B) = 4$ then system (17) is controllable.

Let us show that system (17) is not controllable with the help of one control u_2, where u_2 is a rotating moment. In this case matrices B and $K = (B,\ AB,\ A^2B,\ A^3B)$ have the forms

$$B = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is obviously that $rankK = 2$. It means that system (17) is not controllable with help of only single control u_2.

3. Controllability of Linear Time-Varying Systems

3.1. Complete State (Output) Controllability

Consider the n-dimensional linear system defined by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{18}$$

$$y(t) = C(t)x(t), \tag{19}$$

where $x(t) \in R^n$ is the state, $y(t) \in R^m, m \leq n$, is the output, $u(t) \in R^r$ is the input of (18). The matrices $A(t)$, $B(t)$, $C(t)$ are of the order compatible to the vectors, and they are assumed to be at least piecewise continuous. The basic definitions for controllability of the system (18), (19) may be stated as follows:

Definition 3.1. *The system (18), (19) is completely state (output) controllable at t_0 if for any points $x_0, x_1 \in R^n$ $(y_0, y_1 \in R^m)$ there exists an input $u(t)$, defined on some finite interval $[t_0,\ t_1]$, such that the corresponding solution $x(t) = x(t, t_0, x_0, u)$ $(y(t) = y(t, u))$ with the initial condition $x(t_0) = x_0$ satisfies the equality $x(t_1) = x_1$ $(y(t_1) = y_1)$.*

In general $u(t)$ depends on both t_0, $x(t_0)$.

Definition 3.2. *The system (18), (19) is completely state (output) controllable if the conditions of Definition 1 hold for all* t_0.

Definition 3.3. *The system (18), (19) is totally state (output) controllable if for all* t_0 *and almost all* $t_1 > t_0$ *and any state* $x(t_0)$ *at* t_0, *there exists an input* $u(t)$, *defined on* $[t_0, t_1]$, *such that* $x(t_1) = x_1 \quad (y(t_1) = y_1)$.

As it is well known, the output of the system (18), (19) can be expressed in the form

$$y(t) = C(t)F(t,t_0)x(t_0) + \int_{t_0}^{t} C(t)F(t,\tau)B(\tau)u(\tau)d\tau, \tag{20}$$

where $F(t,\tau)$ is Cauchy matrix satisfying the homogeneous part of (18) with respect to the first argument t:

$$\frac{dF(t,\tau)}{dt} = A(t)F(t,\tau), \quad F(\tau,\tau) = E_n.$$

Suppose

$$H_s(t,\tau) = F(t,\tau)B(\tau),$$

$$H_o(t,\tau) = C(t)F(t,\tau)B(\tau) = C(t)H_s(t,\tau).$$

The necessary and sufficient conditions for various types of controllability can then be summarized in the following theorems (see Kalman, Kreindler and Sarachik).

Theorem 3.1. it The system (18), (19) is completely state (output) controllable at t_0 if and only if there exists a finite time $t_1 > t_0$ such that the rows of the matrix $H_s(t_1,\tau) \quad (H_0(t,\tau))$ are linearly indep$[t_0, t_1]$.

Theorem 3.2. *The system (18), (19) is totally state (output) controllable if and only if for all* t_0 *and for almost all* $t_1 > t_0$ *the rows of the matrix* $H_s(t_1,\tau) \quad (H_0(t,\tau))$ *are linearly independent functions of* τ *on* $[t_0, t_1]$.

Corollary 3.1. *System (18) is completely state controllable on the interval* $[t_0, t_1]$ *if and only if*

$$det W(t_0, t_1) \neq 0,$$

where

$$W(t_0, t) = \int_{t_0}^{t} H_s(t_0,\tau)H_s'(t_0,\tau)d\tau.$$

Corollary 3.2. *System (18), (19) is completely output controllable on the interval* $[t_0, t_1]$ *if and only if*

$$detV(t_0, t_1) \neq 0,$$

where

$$V(t_0, t) = \int_{t_0}^{t} H_o(t, \tau) H_o'(t, \tau) d\tau.$$

If the system (18), (19) is fixed one (i.e. we have time-invariant system), controllability is independent of the initial time, and complete controllability implies total controllability. The converse is clearly always true.

The necessary and sufficient conditions for controllability in the fixed case can be formulated directly in terms of the matrix coefficients of the system, as in the following theorem.

Theorem 3.3.*A fixed system (18), (19) is state (output) controllable if and only if*

$$rank\, K_s = n$$

where

$$K_s = \{B, AB, \ldots, A^{n-1}B\}$$

or

$$rank(K_o) = m,$$

where

$$K_o = CK_s = \{CB, CAB, \ldots, CA^{n-1}B\}.$$

Now criteria, which seem to be natural generalizations of Theorem 3.3 for time-varying systems, are derived and discussed.

3.3. Generalization of Theorem 3.3

Let $F(t, \tau)$ be some fundamental matrix of system (18),

$$G(t) = F(t, \tau)^{-1} B(t)$$

and p be nonnegative entire number. We shall say that *system (18) has a class* p *on the open set* $\Delta \in R$, if matrix function $G(t)$ is p times differentiable on Δ.

As the function $G(t)$ is given with the help of fundamental matrix $F(t, \tau)$, which is unknown in explicit form as a rule, then it is desirable to have some conditions of belonging system (18) to the class p which are expressed directly through parameters $A(t), B(t)$.

Define $S(n,r)$ as a set of systems (18) with continuous matrices $A(t) \in R^{n\times n}, B(t) \in R^{n\times r}$ on some interval $\Delta \subset R$. Identify this set with the set of pairs

$$(A,B) = (A(t), B(t)).$$

Define a map

$$P_\Delta : S(n,r) \to S(n,r)$$

according to the rule

$$P_\Delta(A,B)(t) = (A(t), A(t)B(t) - \dot{B}(t)), \quad t \in \Delta.$$

It is clear that the domain $dom(P_\Delta)$ of the operator P_Δ consists of such elements $(A,B) \in S(n,r)$, that matrix function $B(t)$ is continuously differentiable on Δ. If the pair

$$(A,B) \in dom(P_\Delta)$$

is such that matrix

$$Q_1(t) = A(t)B(t) - \dot{B}(t)$$

is also continuously differentiable on Δ then there exists an operator P_Δ^2 on this pair for which

$$P_\Delta^2(A,B) = P_\Delta(A,Q_1).$$

Denote the domain of the map P_Δ^2 as $dom(P_\Delta^2)$. We can construct any degree of P_Δ^k, $k = 0,1,2,\ldots$ of the operator P_Δ by induction. We assume that P_Δ^0 is the identical map in $S(n,r)$ and that

$$dom(P_\Delta^0) = S(n,r).$$

Lemma 3.1. *System (18) belongs to the class p on the set Δ if and only if*

$$(A,B) \in dom(P_\Delta^p).$$

(see: Gaishun I.V. *Introduction to the Theory of Linear Nonstationaty Systems.* Minsk, 1999.)

Thus system (18) belongs to the class p on the set Δ if and only if the following matrix functions

$$Q_0(t) = B(t), \quad Q_i(t) = A(t)Q_{i-1}(t) - \dot{Q}_{i-1}(t), \quad i = 1,2,\ldots,p, \tag{21}$$

are defined and continuous on this set Δ.

Note. If matrices $A(t)$, $B(t)$ are continuously differentiable $p-1$ and p times correspondingly then system (18) belongs to the class p. The converse is not true.

Definition 3.4. *If the pair of matrices (A, B) belongs to the class $n-1$ then $(n \times nr)$-matrix function*

$$Q(t) = \{Q_0(t), Q_1(t), \ldots, Q_{n-1}(t)\}$$

is called the controllability matrix of system (18).

Theorem 3.4. *System (18) of the class $n-1$ on the open set $\Delta \supset [t_0, t_1]$ is completely state controllable on the interval $[t_0, t_1]$ if there exists such a point $t^* \in [t_0, t_1]$ that*

$$rank\, Q(t^*) = n. \tag{22}$$

Theorem 3.5. *System (18) of the class $n-1$ on the open set $\Delta \supset [t_0, t_1]$ is completely output controllable with respect to output (19) on the interval $[t_0, t_1]$ if there exists such a point $t^* \in [t_0, t_1]$ that*

$$rank\, C(t_1)Q(t^*) = m. \tag{23}$$

Theorem 3.4 gives us only sufficient conditions for complete state controllability, i.e. there exist completely state controllable systems which don't satisfy (22).

Theorem 3.6. *System (18) with analytical matrices on R is completely state controllable on any interval $[t_0, t_1] \subset R$ if and only if for some $t^* \in [t_0, t_1]$ the equality (22) hold.*

3.4. Canonical Form of Linear Time-Varying Systems

Consider now the canonical form for linear time-varying control systems for the scalar function $u(t)$, $B(t) = b(t)$ and $rank\, Q(t) = n$ for all $t \in [t_0, t_1]$.

Define the following operators and matrices:

$$\Delta = A(t) - \frac{d}{dt}, \quad \alpha = Q^{-1}(t)\Delta^n B(t),$$

$$e_1 = (1, 0, \ldots, 0)', \ldots, e_{n-1} = (0, \ldots, 0, 1, 0)',$$

$$e_i \in R^n, \quad i = \overline{1, n-1},$$

$$A_1 = (\alpha, e_1, \ldots, e_{n-1}), \quad \Delta_1 = A_1' + \frac{d}{dt},$$

$$D = (e_1, \Delta_1 e_1, \ldots, \Delta_1^{n-1} e_1), \quad G(t) = DQ^{-1}(t).$$

Suppose $z(t) = G(t)x(t)$. Then the following equation is valid for function $z(t)$:

$$\dot{z}(t) = A_0(t)z(t) + B_0(t)u(t), \quad z \in R^n,\ u \in R^1,$$

where

$$A_0(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_1(t) & a_2(t) & a_3(t) & \dots & a_n(t) \end{pmatrix},$$

$$B_0(t) = (0, 0, \dots, 0, 1)',$$

$$A_0(t) = G(t)A(t)G^{-1}(t) + \dot{G}(t)G^{-1}(t).$$

This system can be written as one equation of n-order with respect to $z_1(t)$:

$$z_1(t)^{(n)}(t) + a_1(t)z_1^{(n-1)}(t) + \dots + a_n(t)z_1(t) = u(t).$$

3.5. Examples

Example 3.1. Consider the system of the second order

$$\dot{\xi} = \xi + u, \quad \dot{\eta} = \eta + u$$

with the output

$$y(t) = \xi(t) + \eta(t).$$

This system is not completely state controllable on any segment $[t_0, t_1]$: we can only transfer any arbitrary state ξ_0, η_0 on some line $\xi - \eta = c$. However it is output controllable because of the following equality takes place:

$$y(t_1) = e^{t_1}(y_0 e^{-t_0} + 2\int_{t_0}^{t_1} e^{-\tau}u(\tau)d\tau),$$

which allow us to choose $u(t)$ and to obtain any desire $y_1 = y(t_1)$.

Example 3.2. Suppose $[t_0, t_1] = [0, 1]$ and

$$\dot{x} = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix} x + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} u, \tag{24}$$

where α, β are real numbers,

$$b_1(t) = \begin{cases} 0, & t \in [0, \frac{1}{3}], \\ \varphi(t), & t \in [\frac{1}{3}, \frac{2}{3}], \\ 1, & t \in [\frac{2}{3}, 1], \end{cases} \qquad b_2(t) = \begin{cases} 1, & t \in [0, \frac{1}{6}], \\ \psi(t), & t \in [\frac{1}{6}, \frac{1}{3}], \\ 0, & t \in [\frac{1}{3}, 1]. \end{cases}$$

Moreover functions $\varphi(t)$, $\psi(t)$ are such ones that $b_1(t), b_2(t)$ are continuously differentiable functions and $\varphi(t) > 0$ for $t \in (1/3, 2/3]$ and $\psi(t) > 0$ for $t \in [1/6, 1/3)$ (see Fig. 2).

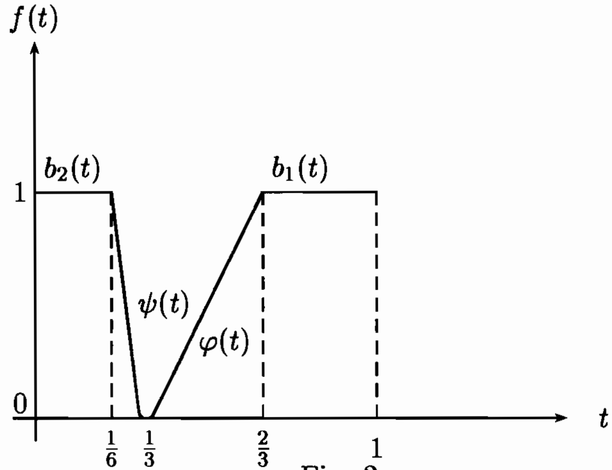

Fig. 2

The controllability matrix has the form

$$Q(t) = \begin{pmatrix} b_1(t) & -\alpha b_1(t) - \dot{b}_1(t) \\ b_2(t) & -\beta b_2(t) - \dot{b}_2(t) \end{pmatrix}$$

and due to the construction of functions $b_1(t)$, $b_2(t)$

$$det Q(t) = (\alpha - \beta) b_1(t) b_2(t) - b_1(t)\dot{b}_2(t) + b_2(t)\dot{b}_1(t) = 0.$$

It means that the relation (22) is not fulfilled. Nevertheless the matrix

$$W(t_0, t_1) = W(0,1) = \begin{pmatrix} \int\limits_0^1 b_1^2(\tau) e^{2\alpha\tau} d\tau & 0 \\ 0 & \int\limits_0^1 b_2^2(\tau) e^{2\beta\tau} d\tau \end{pmatrix}$$

is nonsingular and consequently system (24) is completely controllable.

4. Observability of Linear Dynamic Systems

Consider a system

$$\dot{x}(t) = Ax(t), \quad t \in T = [t_0, t_1], \quad x \in R^n. \tag{25}$$

Suppose that due to unknown perturbation of the initial state $x(t_0)$ a transfer process $x(t)$, $t \geq t_0$, has began in this system, and it is inaccessible for measurement. We can observe only m-vector function $y(t)$, $t \in T$, which is connected with the state $x(t)$ of system (25) by the relation

$$y(t) = Cx(t), \quad t \in T, \quad y \in R^m, \tag{26}$$

where $C \in R^{m\times n}$, $m < n$ is the known matrix.

4.1. Relative G-observability of Linear Time-Invariant Systems

Let $x_0 \in R^n$ be an arbitrary vector, $G \in R^{q\times n}$ is a given constant matrix.

Problem 1. To restore vector Gx_0 by given matrices A, C, G and measured signal $y(t)$, $t \in T$.

In other words it is necessary to calculate q linear combinations Gx_0 of the coordinates x_0 or, equivalently, to find projections of the vector x_0 on q directions which are given by vector-rows of the matrix G.

Definition 4.1. *The observation system (25), (26) is said to be relatively G-observable on the interval T if for the known vector-function $y(t)$, $\in T$, it is possible uniquely to determine the value Gx_0 for any arbitrary initial state x_0.*

4.2. Conditional H-observability of Linear Time-Invariant Systems

Let $H \in R^{n\times s}$ be given matrix with constant elements, $z \in R^s$ is an arbitrary vector. Suppose that the observer is interested in initial states x_0 from the hyperplane

$$x_0 = Hz. \tag{27}$$

Problem 2. To restore the initial state x_0 of the form (27) by given matrices A, C, H and measured signal $y(t)$, $t \in T$, which is generated by unknown initial state (27).

Definition 4.2. *The observation system (25), (26) is said to be conditionally H-observable on the interval T if for the known vector-function $y(t)$, $t \in T$, it is possible uniquely to determine arbitrary initial state x_0 of the form* (27).

The combination of these definitions leads us to the following problem.

4.3. Conditionally-Relative $(H-G)$-observability of Linear Time-Invariant Systems

Problem 3. To restore vector Gx_0 by given matrices A, C, H, G and measured signal $y(t)$, $t \in T$, if the initial state x_0 has the form $x_0 = Hz$, $z \in R^s$.

Definition 4.3. *The system (25), (26) is said to be conditionally-relatively $(H - G)$-observable on the interval T if for the known vector-function $y(t)$, $t \in T$, it is possible uniquely to determine the vector Gx_0 for all initial states x_0, $x_0 = Hz$, $z \in R^s$.*

Theorem 4.1. *System (25) is $(H - G)$-conditionally-relatively observable on the segment $T = [t_0, t_1]$ with respect to the measurements (26) if and only if*

$$rank \left\{ \begin{array}{c} CA^kH \\ k = \overline{0, n-1} \end{array} \right\} = rank \left\{ \begin{array}{c} GH \\ CA^kH \\ k = \overline{0, n-1} \end{array} \right\}. \tag{28}$$

Example 4.1. Consider a material point of mass m which is under the influence of central force P, arising as a result of interaction with another material point of mass M $(m \ll M)$ (see Fig. 3).

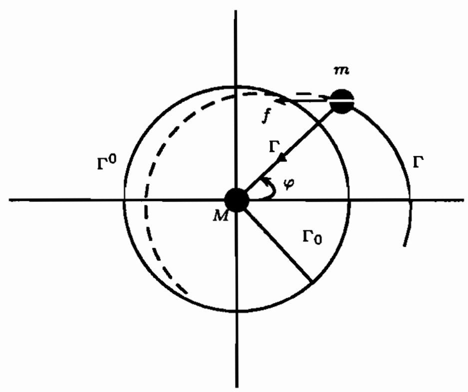

Fig. 3

A reactive force f is applied to point m as a control influence. Under the influence P and f the point m does a motion along the curve Γ which differs a little from some circular orbit Γ^0. If the vector of reactive force f is in the plane of the curve Γ^0 then the motion of the point m will be at the plane of this curve and it is determined completely by the change of its polar coordinates r, ψ. Let $r(t)$, $\dot{r}(t)$, $\psi(t)$, $\dot{\psi}(t)$ be phase coordinates. Then we can obtain differential equation which describes the change of r, ψ :

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = -\frac{\nu}{y_1^2} + y_1 y_4^2 - u_1,$$

$$\dot{y}_3 = y_4, \quad \dot{y}_4 = -2\frac{y_4 y_2}{y_1} - \frac{1}{y_1}u_2, \tag{29}$$

where

$$y_1 = r,\ y_2 = \dot{r},\ y_3 = \psi,\ y_4 = \dot{\psi},$$

$$u_1 = \frac{a_r \dot{m}}{m},\ u_2 = \frac{a_\psi \dot{m}}{m},\ \nu = \nu^0 M,$$

ν^0 is a gravitation constant, a_r, a_ψ are the projections on the radius and cross direction of the velocity vector of the leaving small part. Under some assumptions we obtain the following equations:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = a_{21}x_1 + a_{24}x_4 + u_1,$$

$$\dot{x}_3 = x_4, \quad \dot{x}_4 = a_{42}x_2 + \beta u_2, \tag{30}$$

where a_{ij}, β are some constants expressed through the parameters of the system. Suppose that u_1, u_2 are known. Write the system in the form

$$\dot{x} = Ax + Bu, \quad x \in R^4, \quad u \in R^2, \tag{31}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & 0 & 0 & a_{24} \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \beta \end{bmatrix},$$

with unknown initial states

$$x_1(t_0) = r(t_0) - r_0, \quad x_2(t_0) = \dot{r}(t_0),$$

$$x_3(t_0) = \psi(t_0) - \alpha t_0.$$

The observer knows only the value

$$x_4(t_0) = \dot{\psi}(t_0) - \alpha, \quad \alpha = \sqrt{\frac{\nu}{r_0^3}}.$$

P r o b l e m 4. To define $x_1(t_0)$ if the motion of the material point is described by the equation (30) and the vector-function $y(t) = x_4(t)$, $t \in T$, is accessible for the measurements.

It is necessary to restore vector Gx_0 at this problem where

$$x_0 = Hx + d,\ c' = (0,0,0,1),\ G = [1,0,0,0],$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{\psi}(t_0) - \alpha \end{bmatrix}.$$

So we have the problem of $(H - G)$-observability. Its criterion is

$$rank \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{42} & 0 & 0 \\ a_{42}a_{21} & 0 & 0 & 0 \\ 0 & a_{42}a_{21} + a_{42}^2 a_{24} & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a_{42} & 0 & 0 \\ a_{42}a_{21} & 0 & 0 & 0 \\ 0 & a_{42}a_{21} + a_{42}^2 a_{24} & 0 & 0 \end{bmatrix}$$

We can see that system (29) is $(H - G)$-observable with respect to output $y(t) = x_4(t)$. However this system is not completely observable with respect to the same output $y(t) = x_4(t)$ since the observability criterion

$$rank \left\{ \begin{matrix} c'A^k \\ k = \overline{0,3} \end{matrix} \right\} = 4$$

of this system with respect to the output $y(t) = x_4(t)$ is not fulfilled.

4.4. Observability of Linear Time-Varying Dynamic Systems

Consider the problem of finding the state vector $x(t)$ of linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x \in R^n, \tag{32}$$

on the basis of incomplete data of its components $x_i(t)$, $i = \overline{1,n}$, i.e. we have system (32) with the output

$$y(t) = C(t)x(t), \quad y \in R^m, \quad m < n. \tag{33}$$

As we suppose further that the input function $u(t)$ is fixed one, we can consider the system

$$\dot{x}(t) = A(t)x(t), \tag{34}$$

where $A(t) \in R^{n \times n}$ is continuous matrix on the open set $T \subset R$ and $C(t)$ is continuous $(m \times n)$-matrix function on T.

Suppose that $[t_0, t_1] \in T$. To each element $x_0 \in R^n$ we can put into the correspondence m-vector function

$$y(t) = y(t, x_0) = C(t)x(t, t_0, x_0), \quad t \in [t_0, t_1],$$

due to (34), (33) by the single way. Here $x(t, t_0, x_0)$ is the solution of (34) with the initial condition $x(t_0) = x_0$. Let us compose the set

$$Y_{[t_0,t_1]} = \{y(t, x_0)\, |t_0 \leq t \leq t_1, \, \forall x_0 \in R^n\}.$$

It means that all functions $y(t, x_0)$, $t_0 \leq t \leq t_1$, are the elements of this set for all $x_0 \in R^n$.

Definition 4.4. *System (34) is completely observable with respect to output (33) on the segment* $[t_0, t_1]$ *if the map* $x_0 \to y(t, x_0)$ *is injection of the space* R^n *to the set* $Y_{[t_0,t_1]}$.

This definition says that if the system (34) is completely observable then the knowledge of the output function $y(t, x_0)$ on the segment $[t_0, t_1]$ allow us to restore uniquely the initial state $x(t_0) = x_0$ which has generated this function in virtue of (34), (33).

For any $x_0 \in R^n$ the output $y(t) = y(t, x_0)$ is given by the relation

$$y(t) = C(t)F(t, t_0)x_0,$$

where $F(t, t_0)$ is a fundamental matrix of the system (34) with the property $F(t_0, t_0) = E_n$. Thus $Y_{[t_0,t_1]}$ is a vector space and the map $x_0 \to y(t, x_0)$ is a linear one. Therefore the system (34) is completely observable with respect to output (33) if and only if the equality $y(t, x_0) = 0$, $t \in [t_0, t_1]$, is fulfilled only for $x_0 = 0$.

Definition 4.5. *System (34) is differentially observable with respect to output (33) if it is completely observable with respect to this output on any segment* $[\tau_0, \tau_1] \subset [t_0, t_1]$.

It is clear that the differentially observable system is completely observable; the converse assertion is clearly not always true.

4.5. The Duality Principle and Some Observability Conditions

Theorem 4.2. *System (34) is completely observable with respect to output (33) on the segment* $[t_0, t_1]$ *if and only if the observability Grammian*

$$M(t_0, t_1) = \int_{t_0}^{t_1} F'(t, t_0)C'(t)C(t)F(t, t_0)dt \tag{35}$$

is nonsingular.

Notice that the observability Grammian is symmetric positive semidefinite and is the solution to the matrix differential equation

$$\frac{d}{dt}M(t,t_1) = -A'(t)M(t,t_1) - M(t,t_1)A(t) - C'(t)C(t)$$

with the initial condition

$$M(t_1,t_1) = 0_n.$$

Then the initial state $x(t_0)$ is given by the formula

$$x_0 = M^{-1}(t_0,t_1)\int_{t_0}^{t_1} F'(t,t_0)C'(t)y(t)dt,$$

where $A(t) \in C^{n-2}[t_0,t_1]$, $C(t) \in C^{n-1}[t_0,t_1]$.

Corollary 4.1. *System (34), (33) is completely observable on the set* $[t_0,t_1]$ *if and only if*

$$C(t)F(t,t_0)g \neq 0, \quad t \in [t_0,t_1], \tag{36}$$

for any unit n*-vector* $g, \|g\| = 1$.

Corollary 4.2. *System (34), (33) is differentially observable on the set* $[t_0,t_1]$ *if and only if for any segment* $[\tau_0,\tau_1] \subset [t_0,t_1]$ *the matrix* $M(\tau_0,\tau_1)$ *is nondegenerate or*

$$C(t)F(t,t_0)g \neq 0, \quad t \in [\tau_0,\tau_1]$$

for any unit n*-vector* $g, \|g\| = 1$.ç

Let us consider a system

$$\dot{z}(t) = -A'(t)z(t) + C'(t)u(t), \tag{37}$$

which is said to be *conjugate system* to the observation system (34), (33). It is clear that if $F(t,\tau)$ is a fundamental matrix of the system (34), then $Z(t,\tau) = (F^{-1}(t,\tau))'$ is a fundamental matrix of the conjugate system

$$\dot{z}(t) = -A'(t)z(t).$$

Therefore the matrix (35) can be rewritten in another form

$$M(t_0,t_1) = \int_{t_0}^{t_1} Z(t_0,\tau)C'(\tau)C(\tau)Z'(t_0,\tau)d\tau.$$

Theorem 4.3 (The duality principle). *System (34) is completely (differentially) observable with respect to the output (33) on the segment $[t_0, t_1]$ if and only if the conjugate system (37) is completely (differentially) state controllable on the same segment.*

Now we obtain some coefficient conditions of complete and differentiable observability which are analogous to the controllability conditions.

Definition 4.6. *We shall say that the observation system (34), (33) has the class p, $p \geq 0$, on the segment $\Delta \subset T$ if its any output $y(t)$ is p times continuously differentiable on Δ.*

Lemma 4.1. *The observation system (34), (33) has the class p on the set Δ if and only if the conjugate control system (37) belongs to the class p on the same set.*

The control system (37) belongs to the set Δ if and only if the matrix functions

$$Q_0(t) = C'(t), \quad Q_i(t) = -A'(t)Q_{i-1}(t) - \dot{Q}_{i-1}(t),$$
$$Q_i(t) \in R^{n \times m}, \quad i = \overline{1,p},$$

are defined and continuous on Δ. Consequently system (34), (33) has the class p on the set Δ if and only if there exist continuous matrices

$$S_0(t) = C(t), \quad S_i(t) = S_{i-1}(t)A(t) + \dot{S}_{i-1}(t),$$
$$S_i(t) \in R^{m \times n}, \quad i = \overline{1,p}. \tag{38}$$

for $t \in \Delta$.

Suppose that system (34), (33) has the class $n-1$ on the set Δ.

Definition 4.7. *Matrix function*

$$S(t) = \begin{pmatrix} S_0(t) \\ S_1(t) \\ \dots \\ S_{n-1}(t) \end{pmatrix}$$

is said to be the observability matrix for system (34), (33).

Theorem 4.4. *System (34), (33) of the class $n-1$ on the set Δ is*

— *completely observable on the segment $[t_0, t_1] \subset \Delta$ if*

$$rank\, S(t^*) = n$$

for some $t^* \in [t_0, t_1]$;

— *differentially observable on the set* $[t_0, t_1]$ *if and only if*

$$rank\, S(t) = n$$

for almost all $t \in [t_0, t_1]$.

Let the elements of matrices $A(t), B(t)$ be analytical functions on the set $T \in R$.

Theorem 4.5. *System (34) is completely observable with respect to the output (33) on the each segment* $[t_0, t_1] \subset T$ *if and only if*

$$rank\, S(t^*) = n$$

for some $t^* \in [t_0, t_1]$.

4.6. Examples

Example 4.2. Longitudinal oscillations of aircraft with respect to the center of masses are described by the equations

$$\begin{aligned} \dot{\theta} &= k_3 \alpha, \\ \ddot{\varphi} &= -k_1 \alpha - k_2 \delta, \\ \dot{\delta} + \beta_1 \delta &= \beta_2 u, \\ \alpha &= \varphi - \theta, \end{aligned} \tag{39}$$

where $\varphi, \theta, \alpha, \delta$ are some angles corresponding to the flight of the aircraft; $k_1, k_2, k_3, \beta_1, \beta_2$ are constants. Suppose $\xi_1 = \dot{\varphi}$, $\xi_2 = \varphi$, $\xi_3 = \delta$, $\xi_4 = \theta$; then we can write the system (39) in the form

$$\begin{aligned} \dot{\xi}_1 &= -k_1 \xi_2 - k_2 \xi_3 + k_1 \xi_4, \\ \dot{\xi}_2 &= \xi_1, \\ \dot{\xi}_3 &= -\beta_1 \xi_3 + \beta_2 u, \\ \dot{\xi}_4 &= k_3 \xi_2 - k_3 \xi_4. \end{aligned} \tag{40}$$

Suppose that the coordinates $\xi_3(t) = \delta(t), \xi_4(t) = \theta(t)$ are accessible for the observability. It means that the output $y(t)$ is defined by the relation

$$y(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x(t), \tag{41}$$

where the state vector $x(t)$ is $x(t) = \{\xi_1(t),\, \xi_2(t),\, \xi_3(t), \xi_4(t)\}$. Compose the observability matrix $S(t)$ for $u = 0$. We can see that

$$rank\, S = 4.$$

It means that system (40) is observable. However if we shall measure the coordinates $\xi_1(t) = \dot{\varphi}(t)$, $\xi_3(t) = \delta(t)$, i.e. we have the output

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t), \tag{42}$$

we can see that $rank\, S = 3$. This fact means that the system (40) is not observable.

Example 4.3. **Inverted pendulum**

Consider the inverted pendulum (see Fig. 4_1)

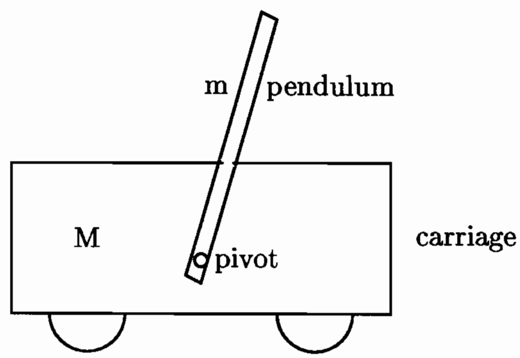

Fig. 4_1 An inverted pendulum positioning system.

whose behaviour is described by the the equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L'} & 0 & \frac{g}{L'} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{pmatrix} u(t), \tag{43}$$

where (see Fig. 4_2)

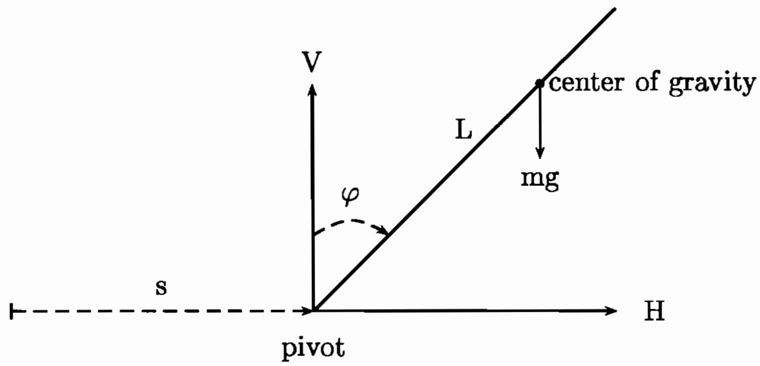

Fig. 4_2 Inverted pendulum: forces and displacements.

F represents the friction coefficient (friction is accounted for only in the motion of the carriadge and not at the pivot); g is the gravitational acceleration, M is the mass of carriage; L is the distance from the pivot to the center of gravity; $L' = L + \frac{J}{mL}$, J is the moment of inertia with respect to the center of gravity; $s(t)$ is the displacement of the pivot at time t. We choose the components of the state $x(t)$ as $x(t) = \{s(t), \dot{s}(t), s(t) + L'\varphi(t), \dot{s}(t) + L'\dot{\varphi}(t)\}$, where $\varphi(t)$ is the angular rotation at time t of the pendulum.

If we take as the output variable $y_1(t)$ the angle $\varphi(t)$, we have

$$y_1(t) = (-\frac{1}{L'}, 0, \frac{1}{L'}, 0)x(t). \tag{44}$$

The observability matrix has the form

$$Q = \begin{pmatrix} -\frac{1}{L'} & 0 & \frac{1}{L'} & 0 \\ 0 & -\frac{1}{L'} & 0 & \frac{1}{L'} \\ -\frac{g}{L'}\frac{1}{L'} & \frac{F}{M}\frac{1}{L'} & \frac{g}{L'}\frac{1}{L'} & 0 \\ 0 & -\frac{g}{L'}\frac{1}{L'} - (\frac{F}{M})^2\frac{1}{L'} & 0 & \frac{g}{L'}\frac{1}{L'} \end{pmatrix}. \tag{45}$$

We have

$$rank \quad Q = 3,$$

consequently system is not completely observable. If we add as a second component of the output variable the displacement $s(t)$ of the carriage, we have the output

$$y(t) = \begin{pmatrix} -\frac{1}{L'} & 0 & \frac{1}{L'} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} x(t).$$

With this output $rank\,Q = 4$ and the system (40) is completely observable.

Note that the canonical observability form for linear system (25) is

$$\dot{x}(t) = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} x(t),$$

$$y(t) = (C_1, 0)x(t). \tag{46}$$

5. Stabilization of Dynamic Systems

5.1. Statement of the Problem

Consider the following control system

$$\dot{x}(t) = f(t, x(t), u), \quad t \geq 0, \quad f(t, 0, 0) = 0,$$

$$x(0) = x_0, \quad x(t) \in R^n, \quad u \in R^r. \tag{47}$$

We shall seek a control function u of this system in the form of C-control $u = u(t, x(t))$, which depends on the time t and the values of running coordinates of $x(t)$. Besides a restriction $u = u(t, x(t)) \in U$ is valid where U is a given set from R^r. In addition we suppose that

$$u(t, 0) = 0.$$

It is possible to extract the following statements for control problems on the infinite time interval.

1^0. To find C-control $u(t, x)$ for which system (47) became asymptotically stable in some sense. The following problems may be considered in that case: the problems of *asymptotical stabilization, exponential stabilization, stabilization in the large* and so on, if the trivial solution of system (47) becomes asymptotically stable, exponentially stable, asymptotically stable in the large and so on for the control u, correspondingly. Such a control is called *asymptotically stabilizing, exponentially stabilizing, stabilizing in the large* and so on.

2^0. To find a control $u(t, x)$ which minimizes the cost functional

$$J(u) = \int_0^\infty F_0(t, x(t), u(t, x)dt \to \inf_{u \in U}. \tag{48}$$

Here F_0 is a given function

$$F_0 : R^1 \times R^n \times R^m \to R^1.$$

3^0. To find a control $u(t,x)$ which minimizes the cost functional (48) and transforms the system (47) into stable one simultaneously.

We have here *the problem of optimal stabilization* and the corresponding control is *the optimal stabilizing control.*

5.2. Stabilization of Linear System

The control problem of linear systems with quadratic cost functional is one of more well known and studied problem. Let us consider a linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u,\ t \geq 0, \tag{49}$$
$$x \in R^n,\ u \in R^r,\ x(0) = x_0$$

with the quadratic cost functional

$$J(u) = \int_0^\infty [x'(t)N_1(t)x(t) + u'N_0(t)u]dt \to \inf_u \tag{50}$$

which is minimized. The matrices $A(t)$, $B(t)$, $N_1(t)$, $N_0(t)$ are given and they have a bounded continuous elements. In addition the matrices $N_1(t), N_0(t)$ are uniformly positive definite:

$$N_1(t) \geq CE_n, \quad N_0(t) \geq CE_r, \tag{51}$$

where $C > 0$ is some constant, E_k is unite $(k \times k)$-matrix. Inequalities (51) have the following sense: for any $x \in R^n$, $u \in R^r$ and $t \geq 0$ the estimations

$$x'N_1(t)x \geq Cx'x, \quad u'N_0(t)u \geq Cu'u$$

take place. We shall seek the control u for the problem (49), (50) in the form $u = u(t, x(t))$. Any other restrictions on the control function u are absent.

The conditions of optimal stabilization in terms of scalar Lyapunov functions $V(t,x)$ with some properties are known. Now we remind them.

Let $\omega_i(r)$, $r \geq 0$, be a scalar continuous nondecreasing functions such that

$$\omega_i(r) > 0 \quad for \quad r > 0, \quad \omega_i(0) = 0,$$

and L_u is an operator of the form

$$L_u V(t,x) = \frac{\partial V(t,x)}{\partial t} + f'(t,x,u)\frac{\partial V(t,x)}{\partial x}.$$

Note that $L_u V(t,x)$ is a full derivative of the function $V(t,x)$ along the trajectories of system (47) for the control u.

Now we shall give the condition of the fact that some C-control $u(t,x)$ is stabilizing.

Theorem 5.1. *A control $u(t,x)$ is a stabilizing control for the system (47) if there exists such a continuously differentiable Lyapunov function $V(t,x)$ that*

$$\omega_1(|x|) \leq V(t,x) \leq \omega_2(|x|), \tag{52}$$

$$L_u V(t,x) \leq -\omega_3(|x|). \tag{53}$$

For such a control function the trivial solution of system (47) is uniformly asymptotically stable.

In accordance with Theorem 1 for the solving problem (49), (50) it is necessary to construct a Lyapunov function $V(t,x)$. It is naturally to seek this function in the form

$$V(t,x) = x'P(t)x,$$

where a symmetric function $P(t) > 0$ to be defined.

It is known from the theory of optimal control that optimal control u_0 and matrix $P(t)$ for the problem (49), (50) satisfy the relations

$$u_0(t,x) = -N_0^{-1}(t)B'(t)P(t)x, \quad t \geq 0, \tag{54}$$

$$\begin{gathered}\dot{P}(t) + A'(t)P(t) + P(t)A(t) - \\ -P(t)B(t)N_0^{-1}(t)B'(t)P(t) + N_1(t) = 0.\end{gathered} \tag{55}$$

Besides, if there exists such a solution $P(t)$ of the equation (55) that

$$P(t) \geq \alpha E_n, \quad \| P(t) \| < C,\ \alpha > 0, \quad C > 0, \tag{56}$$

then the system (49) is exponentially stable for the control (54).

Now we formulate a sufficient conditions for the existing and uniqueness of the solution $P(t)$, $t \geq 0$, which satisfies the condition (56). Suppose that the elements of the matrices $A(t), B(t)$, $t \geq 0$, have continuous bounded derivatives till $n-1$ order inclusive. Let us introduce n matrices $Q_0(t), \dots Q_{n-1}(t)$ of the dimension $n \times r$ and $(n \times nr)$-matrix $Q(t)$:

$$Q_0(t) = B(t), \quad Q_i(t) = -A(t)Q_{i-1} + \dot{Q}_{i-1}(t),$$

$$i = 1, 2, \dots, n-1,$$

$$Q(t) = \{Q_0(t),\ Q_1(t), \dots, Q_{n-1}(t)\}.$$

Theorem 5.2. *Suppose that all restrictions on the parameters of the problem (49), (50) are satisfied. Let, in addition, there exists such a number* $\Delta > 0$, *that for any segment* $[t, t+\Delta]$, $t \geq 0$, *there exists a point* $t^*(t)$ *.at which*

$$rank\, Q(t^*(t)) = n, \quad t \geq 0. \tag{57}$$

Then for $t \geq 0$ *there exists an unique solution* $P(t)$ *satisfying the conditions*

$$P(t) \geq \alpha E_n, \quad \|P(t)\| < C, \quad \alpha > 0, \quad C > 0.$$

The optimal control $u_0(t,x)$ *has the form*

$$u_0(t,x) = -N_0^{-1}(t)B'(t)P(t)x, \quad t \geq 0, \tag{58}$$

where

$$\dot{P}(t) + A'(t)P(t) + P(t)A(t) - \\ -P(t)B(t)N_0^{-1}(t)B'(t)P(t) + N_1(t) = 0, \tag{59}$$

system (49) is exponentially stable for this control and

$$J(u_0) = V(0, x_0).$$

5.3. Time-Invariant Linear-Quadratic Problem

Consider time-invariant linear-quadratic problem

$$\dot{x}(t) = Ax(t) + Bu, \quad t \geq 0,$$

$$x \in R^n, \quad u \in R^r, \quad x(0) = x_0 \tag{60}$$

with the quadratic cost functional

$$J(u) = \int_0^\infty [x'(t)N_1x(t) + u'N_0u]dt \to \inf_u,$$

$$N_1 > 0,\ N_0 > 0. \tag{61}$$

As the value of the quadratic cost functional is not changed when we change the initial time moment $t = 0$ into $t = t_0$ then the Lyapunov function V depends on only x and has the form

$$V(x) = x'Px, \tag{62}$$

optimal control is

$$u_0(t,x) = u_0(x) = -N_0^{-1}B'Px. \tag{63}$$

A constant symmetric matrix $P > 0$ from (62), (63) is the solution of the equation

$$A'P + PA - PBN_0^{-1}B'P + N_1 = 0. \tag{64}$$

The equation (64) is called *Riccati equation.*

Theorem 5.3. *Suppose that*

$$N_0 > 0, \quad N_1 > 0, \quad rank\,(B, AB, \ldots, A^{n-1}B) = n, \tag{65}$$

in the problem (60), (61).

Then equation (64) has the unique solution $P > 0$ and the relations

$$V(x) = x'Px, \quad u_0(t,x) = u_0(x) = -N_0^{-1}B'Px$$

are valid. System (60) is exponentially stable for the control (63) and

$$\min_u J(u) = J(u_0) = x_0'Px_0.$$

Thus for constant matrices A, B, N_0, N_1 the solution of the stabilization problem consists in the construction of positively defined solution for the Riccati equation (64).

5.4. Algebraic Riccati Equation

Consider so-called *establishment method* of constructing the solution of Riccati equation (64). It is based on the approximation of the problem (60), (61) on the infinite time interval by linear quadratic problem on the finite time interval $0 \le t \le T$ for linear time-invariant system (60)

$$\dot{x}(t) = Ax(t) + Bu \tag{66}$$

with the cost functional

$$J_1(u) = \int_0^T [x'(t)N_1x(t) + u'N_0u]dt. \tag{67}$$

If u_1 is an optimal control in the problem (66), (67) then

$$J_1(u_1) \le J_1(u_0),$$

where u_0 is an optimal control for the problem (60), (61). In addition it is clear that

$$J_1(u_1) \to J_1(u_0) \quad for \quad T \to \infty.$$

We obtain from this point and from the well known solution of the problem (66), (67) that, if there exists a positive definite solution P of Riccati equation (64), then this solution is a limit for $T \to \infty$ of the solutions $\alpha(t)$ for Cauchy problem

$$\dot{\alpha}(t) + \alpha(t)A + A'\alpha(t) - \alpha(t)BN_0^{-1}B'\alpha(t) + N_1 = 0,$$
$$0 \le t \le T, \quad \alpha(T) = 0.$$

It is convenient to suppose

$$\beta(t) = \alpha(T - t).$$

Then a function $\beta(t)$ is the solution of the equation

$$\dot{\beta} = \beta A + A'\beta - \beta BN_0^{-1}B'\beta + N_1, \quad \beta(0) = 0,$$

from which we have $\lim_{t\to\infty} \beta(t) = P$.

The ground of this fact you can find in **N.N.Krasovskii.** *Stabilizability problems for control motions. M. 1965 (in Russian).*

It is well known that for the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{68}$$

any initial state $x(0)$ can be present by the unique way as follows

$$x(0) = x_s(0) + x_u(0), \tag{69}$$

where $x_s(0)$, $x_u(0)$ belong to the subspace of stable and unstable states, correspondingly . It is evident that if we want to control correctly by the system it is necessary that unstable state should be controllable.

Definition 5.1. *Linear system (68) is said to be stabilizable if its subspace of unstable states is contained in its subspace of controllable states, that is any vector which belongs to the subspace of unstable states belongs the subspace of controllable states as well.*

Theorem 5.4. *Any asymptotically stable system with constant parameters is stabilizable. Any completely controllable system is stabilizable.*

The following result shows the convenience of applying the controllability canonical form.

Theorem 5.5. *Consider the system (68) and suppose that it is transformed into the controllability canonical form*

$$\dot{x}^*(t) = \begin{pmatrix} A_{11}^* & A_{12}^* \\ 0 & A_{22}^* \end{pmatrix} x(t) + \begin{pmatrix} B_1^* \\ 0 \end{pmatrix} u(t), \tag{70}$$

where the pair of matrices $\{A^*_{11}, B^*_1\}$ is completely controllable. System (68) is stabilizable if and only if the matrix A^*_{22} is asymptotically stable.

5.5. Examples

Example 5.1. **An inverted pendulum**

Consider the inverted pendulum (see Fig.4_1). Suppose that we wish to stabilize it. It is clear that if the pendulum starts falling to the right the carriadge must also move to the right. We therefore attempt a method of control whereby we apply a force $u(t)$ to the carriadge which is proportional to the angle $\varphi(t)$ (see Fig. 4_2). This angle can be measured by a potentiometer at the pivot, the force $u(t)$ is exerted through a small servomotor. Thus we have

$$u(t) = k\varphi(t), \tag{71}$$

where k is a constant. It may be proved (see, for example, H.Kwakernaak, R.Sivan *Linear Optimal Control Systems.* 1972), that the system (43) not be stabilized with the input function (71) for any value of the gain k. But it is possible, however, to stabilize the system (43) by feeding back the complete state $x(t) = col\{s(t), \dot{s}(t), s(t) + L'\varphi(t), \dot{s}(t) + L'\dot{\varphi}(t)\}$ as follows

$$u(t) = -k'x(t). \tag{72}$$

In (72) $k \in R^n$ is a constant row vector to be determined. We note that implementation of this controller requires measurement of all four state variables $s(t), \dot{s}(t), s(t) + L'\varphi(t), \dot{s}(t) + L'\dot{\varphi}(t)$, and not only of two as in (71) for $k = \{-\frac{1}{L'}, 0, \frac{1}{L'}, 0\}$. Since the behaviour of the inverted pendulum is described by the linear equation

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{73}$$

then substitution of (72) into (73) yields

$$\dot{x}(t) = (A - bk')x(t). \tag{74}$$

It is well known that the stability of this linear system is determined by the characteristic values of the matrix $A - bk'$.

Theorem 5.6. *A trivial solution of the linear system (73) is*

a) stable if and only if

1. all of the eigenvalues λ_i of the matrix $A - bk'$ have nonpositive real parts;

2. to any characteristic value on the imaginary axis with multiplicity m there correspond exactly m characteristic vectors of the matrix $A - bk'$.

b) asymptotically stable if and only if all of the characteristic values of $A - bk'$ have strictly negative real parts.

Example 5.2. **A Stirred Tank**
Consider a stirred tank (see Fig. 5)

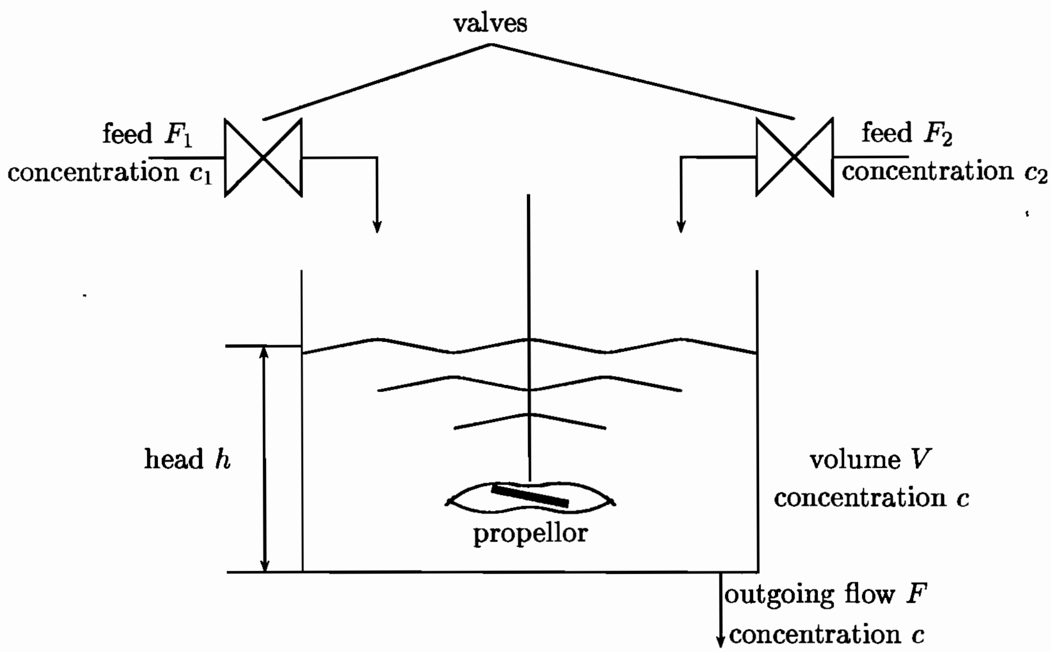

Fig. 5. A stirred tank

which is fed with two incoming flows with time-varying rates $F(t)$ and $F_2(t)$. Both feeds contain dissolved material with constant concentrations c_1 and c_2, respectively. The outgoing flow has a flow rate $F(t)$. It is assumed that the tank is stirred well so that the concentration of the outgoing flow equals the concentration $c(t)$ in the tank. Let $V(t)$ be the volume of the fluid in the tank.

Let us consider a steady-state situation where all quantities are constant, say F_{10}, F_{20}, F_0 for the flow rates, V_0 for the volume, and c_0 for the concentration in the bank. Under some conditions it is described by the differential equation

$$\dot{x}(t) = \begin{pmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{pmatrix} x(t) + \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} u(t), \qquad (75)$$

where $\theta = \frac{V_0}{F_0}$, θ is the time of completing the tank, $\theta > 0$, F_0- are expenditures, V_0-volume of a substance. This system is not completely controllable. Differential state equation (75) has the canonical controllability form (70). The matrix A_{22}^* has a characteristic number $-1/\theta$, so system (75) is stabilizable.

6. Linear Control Systems with Delay

6.1. Introduction to the Problem

A lot of control objects possess time delay either in control device or in a state. Let the motion of the controlled object be described by the equation

$$\dot{x}(t) = Ax(t) + A_1x(t-h) + Bu(t),$$

$$x \in R^n, \quad u \in R^r, \quad t \in T = [0, t_1], \tag{76}$$

where $h = const > 0$ is a number characterizing the delay, A, A_1, B are constant matrices of appropriate sizes. In order that the motion of the system be defined for $t \geq 0$, we must give an initial condition

$$x_0(\cdot) = \{x(t) = \varphi(t),\ -h \leq t < 0,\ x(0) = x_0\}, \tag{77}$$

where $\varphi(t)$ is a continuous function, and $x_0 \in R^n$. The state of the object at any moment $t \in T$ is not characterized by only a finite number of quantities, but by a function

$$\{x(t+\theta),\ \theta \in [-h, 0)\},$$

defined on an interval $[t-\theta, t)$. This peculiarity of delay systems essentially complicates solution of the control problem.

The solution of the problem (76), (77) has the form

$$x(t) = F(t,0)x_0 + \int_{-h}^{0} F(t,\tau+h)A_1\varphi(\tau)d\tau + \int_{0}^{t} F(t,\tau)Bu(\tau)d\tau, \quad t \geq 0, \tag{78}$$

where $F(t,\tau)$ is the Cauchy fundamental matrix, satisfying the following equations with respect to its first and second arguments:

$$\frac{dF(t,\tau)}{dt} = AF(t,\tau) + A_1F(t-h,\tau), \quad \tau < t, \tag{79}$$

$$\frac{dF(t,\tau)}{d\tau} = -F(t,\tau)A - F(t,\tau+h)A_1, quad\tau < t,$$

$$F(t,t-0)=E_n,\quad F(t,\tau)=0_n,\quad \tau\geq t. \tag{80}$$

It is possible to distinguish two forms of controllability. If for each state $x_0(\cdot)$ it is possible to find a time $t_1>0$ and a piecewise continuous control $u(t), t\in[0,t_1]$, for which $x(t_1)=0$, then we shall say that the state is *relatively controllable.* In many cases of such system it is not sufficient to "cut off" control at time $t=t_1$ $(u(t)\equiv 0,\ t\geq t_1)$ since the system may depart from the equilibrium $x(t)\equiv 0$ due to the action of the delay. Thus, we introduce the concept of a *completely controllable state* of (77) when the trajectory goes to the origin or to some prescribe function and remains there under an admissible control.

6.2. Relative Controllability of Linear Time-Invariant Systems with a Constant Delay

Definition 6.1. *The initial state $x_0(\cdot)$ of the system (76) is called* **relatively T-controllable** *if there exists a piecewise-continuous control $u(t)$, $t\in[0,t_1,]$ such that the trajectory $x(t)$ starting at x_0, generated by the control $u(t)$, satisfies the condition $x(t_1)=x_1$ for any prescribe state x_1. A system (76), all of whose initial states are relatively T-controllable, is called relatively T-controllable.*

Definition 6.2. *If the initial state $x_0(\cdot)$ of the system (76) is relatively T-controllable (T-controllable) for any t_1, then this state (system) is called relatively controllable (controllable).*

Consider the solution (78) at the moment t_1 :

$$x(t_1)=F(t_1,0)x_0+\int_{-h}^{0}F(t_1,\tau+h)A_1\varphi(\tau)d\tau+\int_{0}^{t_1}F(t,\tau)Bu(\tau)d\tau,\quad t\geq 0, \tag{81}$$

To find the control function, that transferes any initial state $x(-h+\theta),\theta\in[-h,0]$ to any prescribed final state $x_1\in R^n$ it is necessary to solve so-called *l-problem of moments:* to find a piecewise continuous control $u(t),t\in[0,t_1]$, for which

$$\int_{0}^{t_1}F(t_1,\tau)Bu(\tau)d\tau=y_0, \tag{82}$$

where y_0 is a known vector

$$y_0=x(t_1)-F(t_1,0)x_0-\int_{-h}^{0}F(t_1,\tau+h)A_1\varphi(\tau)d\tau. \tag{83}$$

It is well known from the problem of moments that the system (76) is controllable if and only if the rows of the matrix $F(t,\tau)B$ are linearly independend or, equivalently

$$\psi(g,\tau) = g'F(t_1,\tau)Bu(\tau) \not\equiv 0, \quad \tau \in [0,t_1], \tag{84}$$

for any g, $\| g \| = 1$.

The criterion (84) is implicit one for time-invariant system (76). Let us find an explicit criterion of relative controllability which is expressed through the matrices A, A_1, B of the system (76).

6.3. The Defining Equation

The solution of controllability and observability problems for time-invariant and time-varying systems with delay is based on the notion of *the defining equation* for the systems under investigation. This method gives us an effective and convenient for realization criteria of relative controllability for linear hereditary systems of various types. The defining equation is formed using the original kind of the control systems and allows us to verify if the system is controllable on structural schemes and experimental data or not. Note that the defining equation plays the same role in the theory of of controllability of the time-invariant systems as the characteristic equation in the stability theory.

For the system (76) let us introduce the $n\times n$-matrix function $X_k(s)$, depending on two arguments k, s, $(k = 1,2,\ldots; s = 0,h,2h,\ldots)$, and establish the following relations between the vector functions $x(t) \in R^n$, $u(t) \in R^r$ and matrix functions $X_k(s) \in R^{n\times r}$, $U_k(s) \in R^{r\times r}$:

$$x(t) \to X_k(s), \quad x(t-h) \to X_k(s-h), \quad \dot{x}(t) \to X_{k+1}(s), \quad u(t) \to U_k(s).$$

Then for the equation (76) we have

$$X_{k+1}(s) = AX_k(s) + A_1X_k(s-h) + BU_k(s), \quad s \geq 0, \quad k = 1,2,\ldots. \tag{85}$$

The equation (85) we shall call *the defining equation* of the system (76). We shall calculate the solutions $X_k(s)$, $k = 0,1,2,\ldots$ for the initial conditions

$$U_0(0) = E_r, \quad U_k(s) \equiv 0, \quad s \neq 0 \vee k \neq 0. \tag{86}$$

In subsequent discussions, this equation (86) will play very important role. It is clear that the solution of (86) is a sequence of matrices $X_k(s)$ defined for $k = 1,2,\ldots$, $s = 0,h,2h,\ldots$, where for fixed k we have $X_k(s) \equiv 0$ for $s = (k+1)h, (k+2)h, \ldots$. For each α we denote Π_α as the set

$$\Pi_\alpha = \{X_k(s),\ k = \overline{0,n-1}; \quad s \in [0,\alpha h]\}$$

the solution of the defining equation. The defining equation is called α *-nonsingular* if

$$rank\,\Pi_\alpha = n.$$

If this condition is satisfied for one $\alpha < \infty$, then we call the defining equation *nonsingular.*

Theorem 6.1. *The system (76) is relatively T-controllable if and only if the defining equation of this system is α-nonsingular, $\alpha = [T/h]$, where $[\cdot]$ is a whole part of the number.*

Remark 6.1. In the case $t_1 \leq h$, the number $\alpha = 0$ and the condition for relative controllability of a system with delay (76) coincides with the condition for complete controllability of a system without delay:

$$rank\{B,\, AB\, \ldots, A^{n-1}B\} = n.$$

6.4. Complete Controllability of Linear Time-Invariant Systems with a Constant Delay

We have introduced the notion of *completely controllable state* of the system (76) when the trajectory $x(t, x_0(\cdot), t_0, u(t))$ goes to the origin or to some prescribe function and remains there under an admissible control. Now let us introduce the accurate definitions.

Definition 6.3. *The initial state $x_0(\cdot)$ of the system (76) is called* **completely T-controllable** *if there exists a piecewise-continuous control $u(t)$, $t \in [0, t_1,]$ such that the trajectory $x(t)$, starting at $x_0(\cdot)$ and generated by the control $u(t)$, satisfies the condition $x(\theta) = \phi(\theta)$, $\quad \theta \in [t_1 - h, t_1]$ for any prescribe function $\phi(t_1+\theta)$, $\theta \in (-h, 0]$. A system (76), all of whose initial states are completely T-controllable, is called completely T-controllable.*

Definition 6.4. *If the initial state $x_0(\cdot)$ of the system (76) is completely T-controllable for any t_1, then this state (system) is called completely controllable.*

Theorem 6.3. *The system (76), (77) is completely controllable if and only if*

$$rank \quad \{\lambda E_n - A - A_1 e^{-\lambda h},\, B\} = n \tag{87}$$

for all complex numbers $\lambda \in C$.

Note. The delay system can have only a finite number of proper values λ with $Re\lambda \geq 0$ (see,for example, I.G.Malkin *The theory of stability of the motion.* Moscow, 1966). All proper numbers satisfy the equality

$$det(\lambda E_n - A - A_1 e^{-\lambda h}) = 0.$$

For nonproper numbers we have

$$det(\lambda E_n - A - A_1 e^{-\lambda h}) \neq 0;$$

in this case

$$rank \quad \{\lambda E_n - A - A_1 e^{-\lambda h},\, B\} = n$$

and the condition (87) holds automatically. Therefore it is necessary to check this condition for all λ, $Re\lambda \geq 0$.

6.5. Linear Systems with a Deviating Argument of Neutral Type. The defining Equation

Consider the behaviour of a system represented by the following equation with the deviating argument of neutral type

$$\dot{x}(t) = Ax(t) + A_1 x(t-h) + A_2 \dot{x}(t-h) + Bu(t), \tag{88}$$

$$x \in R^n, \ \ u \in R^r, \ \ t \in T = [0, t_1].$$

We assign the initial condition

$$x_0(\cdot) = \{x(\tau) = \varphi(\tau), \quad \tau \in [-h, 0);\, x(0) = x_0\}, \tag{89}$$

where $\varphi(\tau)$ is a continuously differentiable function and x_0 is an n-vector. Just as in the investigation of equation (76), here it is also possible to distinguish two forms of controllability: relative controllability and complete controllability. In this section we study the relative controllability of the time-invariant systems (88) in the sense of definitions of the previous section.

Let us find the solution of the equation (88). This solution is expressed through the fundamental matrix $F(t,\tau)$ from two arguments: with respect to its first argument t it satisfies the homogeneous part of (88):

$$\frac{dF(t,\tau)}{dt} = AF(t,\tau) + A_1 F(t-h,\tau) + A_2 \frac{dF(t-h,\tau)}{dt}, \quad \tau < t, \tag{90}$$

with the initial condition

$$F(t, t-0) = E_n.$$

It is necessary to know for the matrix $F(t,\tau)$ the equation with respect to the second argument τ. For this reason the equation (88) we multiply on the left side to some matrix $F(ttau)$ and take the integral $\int\limits_0^t$ from all adds in (88):

$$\int\limits_0^t F(t,\tau)\dot{x}(\tau)d\tau - \int\limits_0^t F(t,\tau)A_2\dot{x}(\tau-h)d\tau =$$

$$= \int\limits_0^t F(t,\tau)Ax(\tau)d\tau + \int\limits_0^t F(t,\tau)A_1x(\tau-h)d\tau + \int\limits_0^t F(t,\tau)Bu(\tau)d\tau. \quad (91)$$

For the first two integrals we suppose that matrix function $F(t,\tau)-F(t,\tau+h)A_2$ is continuous one for $\tau \in [0,t]$. Now we apply the integration in parts to the integral

$$(\int\limits_0^t F(t,\tau) - F(t,\tau+h)A_2)\dot{x}(\tau)d\tau$$

and after the change of variables get the formula

$$(F(t,\tau) - F(t,\tau+h)A_2x(\tau))|_0^t - \int\limits_0^t (\frac{dF(t,\tau)}{d\tau} - \frac{dF(t,\tau+h)A_2}{d\tau})x(\tau)d\tau -$$

$$- \int\limits_{-h}^0 F(t,\tau+h)A_2\varphi(\tau)d\tau = \int\limits_0^t F(t,\tau)Ax(\tau)d\tau + \int\limits_{-h}^0 F(t,\tau+h)A_1\varphi(\tau)d\tau +$$

$$+ \frac{dF(t,\tau+h)}{d\tau}A_1x(\tau)d\tau + \int\limits_0^t F(t,\tau)Bu(\tau)d\tau. \quad (92)$$

Let us choose the matrix $F(t,\tau)$ from the equation

$$\frac{dF(t,\tau)}{d\tau} = -F(t,\tau)A - F(t,\tau+h)A_1 + \frac{dF(t,\tau+h)}{d\tau}A_2, \quad \tau \le t, \quad (93)$$

and the conditions

$$F(t,t-0) = E_n, \quad F(t,\tau) = 0_n, \quad \tau \ge t,$$

$$F(t,\tau) - F(t,\tau+h)A_2 \quad (94)$$

is a continuous function for $\tau \in [0, t]$. Then the solution of the equation (88) will has the form

$$x(t) = (F(t,0) - F(t,h)A_2)x_0 + \int_{-h}^{0} F(t,\tau+h)A_1\varphi\tau)d\tau +$$

$$+ \int_{-h}^{0} F(t,\tau+h)A_2\dot{\varphi}(\tau)d\tau + \int_{0}^{t} F(t,\tau)Bu(\tau)d\tau, \quad t \geq 0, \tag{95}$$

where $F(t,\tau)$ is the Cauchy fundamental matrix, satisfying the equation (90) with respect to its first argument t and the equation (93) with respect to its second argument τ.

For the system (88) let us introduce $(n \times n)$-matrix function $X_k(s)$, depending on two arguments $k, s,\ k = 0,1,2,\ldots; s = 0, h, 2h, \ldots)$: and establish the following relations between the vector-functions $x(t) \in R^n$, $\dot{x}(t) \in R^n$, $u(t) \in R^n$, and matrix functions $X_k(s) \in R^{(n\times r)}$, $U_k(s) \in R^{(r\times r)}$:

$$x(t) \to X_k(s), \quad x(t-h) \to X_k(s-h), \quad \dot{x}(t) \to X_{k+1}(s), \quad \dot{u}(t) \to U_k(s).$$

Then instead of the equation (88) we have

$$X_{k+1}(s) = AX_k(s) + A_1X_k(s-h) + A_2X_{k+1}(s-h) + BU_k(s), \tag{96}$$

$$s \geq 0, \quad k = 1, 2, \ldots.$$

The equation (96) we shall call *the defining equation* of the system (88). For the unique solvability of (96) we shall calculate the solutions $X_k(s), k = 0,1,2,\ldots$: with the initial conditions

$$U_0(0) = E_r, \quad U_k(s) \equiv 0, s \neq \vee k \neq 0. \tag{97}$$

Then the following theorem may be proved.

bf Theorem 6.1.*The system (88) is relatively T-controllable if and only if*

$$rank\{X_k(s), \quad k = \overline{0}, n-1; s \in [0, \alpha h]\} = n,$$

where $X_k(s), = 0,1,2,\ldots$ are the solutions of the defining equation (96) with the initial conditions (97) for the system (88).

7. Controllability of Linear Time-Varying Delay Systems

Let us consider the system

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h) + B(t)u(t), \tag{98}$$

$$t \in T = [t_0, t_1], \quad x \in R^n, \quad u \in R^r.$$

with the initial state

$$x_0(\cdot) = \{x(\tau) = \varphi(\tau), \quad t_0 - h \le \tau \le t_0,\} \tag{99}$$

where $h = const > 0$ is a number characterizing the delay, $A(t), A_1(t), B(t)$ are time varying matrices of appropriate sizes. Suppose that $A(t), A_1(t), B(t)$ are analytical functions.

Definition 7.1. *The system (98) is said to be relatively controllable on $[t_0, t_1]$ if for any initial state $x_0(\cdot)$ and arbitrary n-vector x_1 there exists a piecewise continuous control $u(t)$, $t \in [t_0, t_1]$, such that the corresponding trajectory $x(t)$, $t \in [t_0, t_1]$, satisfies the condition $x(t_1) = x_1$.*

7.1. The Representation of Solution for Linear Time-Varying Delay Systems

Let us find the solution of the equation (98). As it is true for control linear differential equations the solution of (98), (99) is expressed through the fundamental matrix $F(t, \tau)$ from two arguments. This matrix with respect to its first argument t satisfied the homogeneous part of (98):

$$\frac{dF(t,\tau)}{dt} = A(t)F(t,\tau) + A_1(t)F(t-h,\tau), \quad t \ge t_0, \tag{100}$$

with the initial condition

$$F(t, t-0) = E_n.$$

Let us obtain the equation for the matrix $F(t, \tau)$ with respect to the second argument τ. For this reason the equation (98) we multiply on the left side to matrix $F(t, \tau)$ and take the integral $\int_{t_0}^{t}$ from all adds in (98):

$$\int_{t_0}^{t} F(t,\tau)\dot{x}(\tau)d\tau = \int_{t_0}^{t} F(t,\tau)A(\tau)x(\tau)d\tau + \int_{t_0}^{t} F(t,\tau)A_1(\tau)x(\tau-h)d\tau +$$

$$+ \int_{t_0}^{t} F(t,\tau)B(\tau)u(\tau)d\tau. \tag{101}$$

For the first integral we apply the integration in parts:

$$\int_{t_0}^{t} F(t,\tau)\dot{x}(\tau)d\tau = F(t,\tau)x(\tau)|_{t_0}^{t} - \int_{t_0}^{t} \frac{F(t,\tau)}{d\tau}x(\tau)d\tau. \tag{102}$$

For the third integral in (101) we introduce the change of variables $\tau - h = \theta$, define the matrix function $F(t,\tau) = 0_n$ for $\tau > t$ and obtain

$$\int_{t_0}^{t} F(t,\theta)A_1(\theta)x(\theta-h)d\theta = \int_{t_0-h}^{t-h} F(t,\theta+h)A_1(\theta)x(\theta)d\theta =$$

$$= \int_{t_0-h}^{t_0} F(t,\theta+h)A_1(\theta)(\theta+h)\varphi(\theta)d\theta +$$

$$+ \int_{t_0}^{t-h} F(t,\theta+h)A_1(\theta)x(\theta)d\theta + \int_{t-h}^{t} F(t,\theta+h)A_1(\theta)x(\theta)d\theta. \qquad (103)$$

Notice that the last integral is zero due to the property $F(t,\tau) = 0_n$ for $\tau > t$. Therefore due to (102), (103), (99) and the property $F(t,t-0) = E_n$ for the matrix $F(t,\tau)$ we have instead of (101)

$$x(t) - F(t,t_0)x_0 - \int_{t_0}^{t} \frac{F(t,\tau)}{d\tau}x(\tau)d\tau = \int_{t_0}^{t} F(t,\tau)A(\tau)x(\tau)d\tau +$$

$$+ \int_{t_0-h}^{t_0} F(t,\tau+h)A_1(\tau)\varphi(\tau)d\tau + \int_{t_0}^{t} F(t,\tau+h)A_1(\tau)x(\tau)d\tau + \int_{t_0}^{t} F(t,\tau)B(\tau)u(\tau)d\tau. \qquad (104)$$

Now we define the matrix $F(t,\tau)$ according to the equation

$$\frac{dF(t,\tau)}{d\tau} = -F(t,\tau)A(\tau) - F(t,\tau+h)A_1(\tau+h), \quad \tau < t,$$

$$F(t,t-0) = E_n, \quad F(t,\tau) = 0_n, \quad \tau \geq t. \qquad (105)$$

Then we obtain the solution of the problem (98), (99) from (104) in the form

$$x(t) = F(t,t_0)x_0 + \int_{t_0-h}^{t_0} F(t,\tau+h)A_1(\tau)\varphi(\tau)d\tau + \int_{t_0}^{t} F(t,\tau)B(\tau)u(\tau)d\tau, \; t \geq 0, \qquad (106)$$

where $F(t,\tau)$ is the Cauchy fundamental matrix, satisfying the equation (100) with respect to its first argument t and the equation (105) with respect to its second argument τ.

7.2. Implicit and Explicit Theorems of Relative Controllability

Theorem 7.1. *The system (98) is relatively T-controllable if and only if*

$$\psi(g,\tau) = g'F(t_1,\tau)Bu(\tau) \not\equiv 0, \quad \tau \in [0,t_1], \tag{107}$$

for any g, $\| g \| = 1$.

To obtain explicit conditions of relative T-controllability let us rewrite the equation (98) in the form

$$px(t) = A(t)x(t) + A_1(t)exp(-ph)x(t) + B(t)u(t), \tag{108}$$

where p is the differentiation operator: $px(t) \equiv \dot{x}(t);, exp(-ph)x(t) \equiv x(t-h)$, so that

$$p \cdot exp(-ph)x(t) \equiv \dot{x}(t-h).$$

Let us introduce the following correspondences between vector-functions $x(t) \in R^n$, $u(t) \in R^r$, the operator p and new matrix functions $X_k(t,s) \in R^{n\times r}$, $U_k(t,s) \in R^{r\times r}$ from two arguments t, s (here t reflects the nonstationarity of the system (98) and s reflects the presence of delay h in it), new operators Δ, D according to the rules:

$$x(t) \to X_k(t,s), \quad u(t) \to U_k(t,s), \quad p \to \Delta + D. \tag{109}$$

In (109) Δ is the shift operator for index of some matrix function, D is the differentiation operator with respect to t of some matrix function, so that, for example,

$$(\Delta + D)Z_k(t,s) \equiv Z_{k+1}(t,s) + \dot{Z}_k(t,s). \tag{110}$$

The index $k+j$, $j = 0,1$, at matrices $X_{k+j}(t,s)$ corresponds to j-derivative of vectors x from (98). The nonstationary of the system (98), in contrast to time-invariant case, is reflected due to (109) by the term D in the correspondences $p \to \Delta + D$ in (109). This correspondence transforms $\dot{x}(t) = px(t)$ to

$$px(t) \to (\Delta + D)X_k(t,s) \equiv X_{k+1}(t,s) + \dot{X}_k(t,s).$$

We have for time-invariant systems, obviously, from (109)

$$x(t) \to X_k(s), \quad u(t) \to U_k(s), \quad p \to \Delta$$

and we get instead of (110)

$$px(t) \to \Delta X_k(t,s) \equiv Z_{k+1}(s).$$

Due to (110) and the representation of the operator $A(t) + A_1(t)e^{-ph}$ of the right part of the system (108) we have that $e^{-\Delta h} \equiv exp{-}\Delta h$ is the shift operator of argument s for some matrix function $Z_k(t,s)$:

$$exp(-\Delta h)Z_k(t,s) = Z_k(t,s-h),$$

and $(exp{-}Dh)$ is the shift operator of argument t for some matrix function $Z_k(t,s)$, so that

$$exp(-Dh)Z_k(t,s) = Z_k(t-h,s).$$

Due to (110) we obtain for the system (108) the following algebraic reccurence on index k equations:

$$(\Delta+D)X_k(t,s) = A(t)X_k(t,s)+A_1(t)exp(-(\Delta+D)h)X_k(t,s)+B(t)U_k(t,s),$$
$$k = 0,1,2,\dots. \tag{111}$$

By virtue of properties of operators Δ, D this equation may be rewritten in the form:

$$X_{k+1}(t,s) + \dot{X}_k(t,s) = A(t)X_k(t,s) + A_1(t)X_k(t-h,s-h) + B(t)U_k(t,s),$$
$$k = 0,1,2,\dots. \tag{112}$$

To define uniquely the solution $X_k(t,s)$ of the equation (112) we introduce the initial conditions:

$$U_0(t,0) = E_r, \quad U_k(t,s) = 0_r,\, k \neq 0 \vee s \neq 0; \quad X_k(t,s) = 0_{n\times r}, \quad k \leq 0, \tag{113}$$

where $0_{l_1\times l_2}$ is a zero $l_1 \times l_2$-matrix, 0_l is a zero square $l \times l$-matrix, E_r is identity $l \times l$-matrix. Matrix algebraic reccurence on the index k equations (111) or, equivalently, (112) we shall call *the defining equation* of the time-varying system with delay (98). We shall say that a totality $X_k(t,s) \in R^{n\times r}$, $k = 0,1,2,\dots$ is *a solution of the defining equation* (112), (113).

Theorem 7.2. *System (98) is relatively controllable on* $T = [t_0, t_1]$ *if and only if*

$$rank \quad \{X_k(t,s), \quad s \in [0,\alpha h], \quad \alpha = [\frac{t_1 - t_0 - 0}{h}], \quad k = 0,1,\dots,N\} = n \tag{114}$$

holds for some integer N, *where* $X_k(t,s)$ *are the solutions of the defining equation* (112), (113).

We shall call that the system (98) satisfies *to maximum condition on* $T = [t_0, t_1]$ if the solution $X_k(t, s)$, $k \geq 0$, $t \in T$, $s \in [0, \alpha h]$ of the defining equation (112), (113) posesses the properties

$$rank\,\{X_k(t,s),\ k \geq 0, s \in [0,\alpha h]\} = max\ rank\,\{\bar{X}_k(t,s),\ k \geq 0, s \in [0,\alpha h]\} \tag{115}$$

for all $t \in T$ except, probably, nowhere dense subset. Here the function $\bar{X}_k(t,s)$ is the solution of the of the system (98) with the parameters $\bar{A}(t)$, $\bar{A}_1(t)$, $\bar{B}(t), \bar{h}$; maximum in (115) should be taken on various parameters $\bar{A}(t)$, $\bar{A}_1(t)$, $\bar{B}(t), \bar{h}$ of the system (98).

The results of this theorem are naturally generalized to the systems with general proper operator and influence operator of general type, to the time-varying systems with the deviating argument of neutral type:

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h) + A_2(t)\dot{x}(t-h) + B(t)u(t), \tag{116}$$

$$t \in T = [t_0, t_1], \quad x \in R^n, \quad u \in R^r,$$

with the initial state

$$x_0(\cdot) = \{x(\tau) = \varphi(\tau), \quad t_0 - h \leq \tau \leq t_0,\} \tag{117}$$

with a differentiable initial function $\varphi(\tau)$, $\quad t_0 - h \leq \tau \leq t_0$; to singularly perturbed dynamic systems (SPDS):

$$\dot{x}(t) = \mathcal{A}_1(t,p)x(t) + \mathcal{C}_1(t,p)y(t) + \mathcal{B}_1(t)u(t),$$

$$\mu\dot{y}(t) = \mathcal{A}_2(t,p)x(t) + \mathcal{C}_2(t,p)y(t) + \mathcal{B}_2(t)u(t), \tag{118}$$

where $t \in T = [t_0, t_1]$, $\quad x \in R^{n_1}$, $\quad y \in R^{n_2}$, $\quad u \in R^r$, μ is a small positive parameter, $\mu \in (0, \mu^0]$, $\mu^0 \ll 1$, $\mathcal{A}_i(t,p)$, $\mathcal{C}_i(t,p)$, $i = 1, 2$, are linear operators of the form

$$\mathcal{A}_i(t,p) = A_{i,0}(t) + A_{i,1}(t)exp(-ph) + A_{i,2}(t)exp(-ph),$$

$$\mathcal{C}_i(t,p) = C_{i,0}(t) + C_{i,1}(t)exp(-ph) + C_{i,2}(t)exp(-ph). \tag{119}$$

It is evident that (118), (119) is SPDS with the deviating argument of neutral type; for $A_{i,1} = A_{i,2} = 0$, $C_{i,1} = C_{i,2} = 0, i = 1, 2$, we obtain SPDS of ordinary differential equations; for $A_{i,2} = 0$, $C_{i,2} = 0, i = 1, 2$, we get SPDS with a constant delay h.

7.3. Output Controllability of Linear Time-Varying Delay Systems

Consider the behaviour of a system represented by the linear time-varying delay system

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h) + B(t)u(t), \tag{120}$$

$$t \in T = [t_0, t_1], \quad x \in R^n, \quad u \in R^r,$$

with the initial state

$$x_0(\cdot) = \{x(\tau) = \varphi(\tau), \quad t \in [t_0 - h, t_0], \tag{121}$$

with the output

$$y(t) = Cx(t), \quad y \in R^m. \tag{122}$$

Let us consider the problem of output controllability (see Definitions 3.1, 3.2). Using the formula for the solution of (120), (121) we can obtain the following representation for the output (122):

$$y(t) = C(t)[F(t,t_0) + \int_{t_0}^{t} F(t,\tau)B(\tau)u(\tau)d\tau], \quad t \geq t_0, \tag{123}$$

and for $t = t_1$ we have

$$\int_{t_0}^{t_1} C(t_1)F(t_1,\tau)B(\tau)u(\tau)d\tau = y_1 - C(t_1)F(t_1,t_0)x_0, \quad t \geq t_0. \tag{124}$$

So we have the implicit criterion of output controllability.

Theorem 7.3. *System (120), (121) is output controllable for analytical matrices functions $A(t)$, $A_1(t)$, $B(t)$ if and only if*

$$\psi(g,\tau) \equiv g'C(t_1)F(t_1,\tau)B \not\equiv 0, \quad \tau \in [t,t_1]. \tag{125}$$

Now we give the explicit criterion of the output controllability for the system (120), (121). Consider the defining equation for (120), (121):

$$X_{k+1}(t,s) = A(t)X_k(t,s) + A_1(t)X_k(t,s-h) + B(t)U_k(t,s) - \dot{X}_k(t,s),$$

$$Y_k(t,s) = C(t)X_k(t,s), \quad k = 0,1,2,\ldots. \tag{126}$$

with the initial conditions:

$$U_0(t,0) = E_r, \quad U_k(t,s) = 0_r, \, k \neq 0 \vee s \neq 0; \quad X_k(t,s) = 0_{n\times r}, \quad k \leq 0. \tag{127}$$

Matrix algebraic reccurence on the index k equations (126), (127) we shall call *the defining equations* of linear time-varying system (120) with the output (121). We shall say that the totality $\{X_k(t,s), Y_k(t,s)\} \in R^{(n+m)\times r}$, $k = 0,1,2,\dots$, is *a solution* of the defining equation (126), (127). Then matrices $X_k(t,s)\} \in R^{n\times r}$, $Y_k(t,s)\} \in R^{m\times r}$, calculated according to (126), (127) are called *the components of the solution* $\{X_k(t,s),\ Y_k(t,s)\} \in R^{(n+m)\times r}$ of these equations.

Theorem 7.4. *Under the conditions* $A(t), A_1(t) \in C^{n-2}(T, R^{n\times n})$, $B(t) \in C^{n-1}(T, R^{n\times r})$, $C(t) \in C(T, R^{m\times n})$ *we have that*

1. the condition

$$rank\{C(t_1)X_k(t^*,s), s \in [0,\alpha h], \alpha = [\frac{t_1 - t_0 - 0}{h}], k = 0,\dots,N\} = m \quad (128)$$

is a sufficient one for the output controllability of the system (120), (121) for some $t^* \in T$,

2. if elements of matrices $A(t), A_1(t), B(t)$ *are analytical functions on* T *and* $C(t)$ *is a continuous function on* T, *then the condition (128) is a necessary one as well.*

8. Singularly Perturbed Dynamic Systems (SPDS)

8.1. Introduction to the Problem

The main goal of the study of different dynamical systems is to understand the long term behavior of states in these systems. In the original modelling process, various nature laws and simplifying assumptions are used to obtain a dynamical system which approximately describes physical phenomena. Therefore, we need to investigate not only the mathematical model but also different perturbations of this model.

We shall distinguish two kinds of the perturbations: regular and singular perturbations. In the last few years, there has been an increasing interest in the controllability and stabilizability problems for singularly perturbed dynamic systems (SPDS). SPDS are defined mathematically as the systems including the singular parameters whose small perturbations can change the order of the systems. The presence of such parameters often makes the analysis of the original system difficult due to the facts that the system has a singularity and the presence of singular parameters increases the dimension of the system. From the above reasons the original full $(n_1 + n_2)$-system is usually decomposed in two subsystems of smaller dimensions, i.e. *the reduced n_1-system* and *the boundary layer n_2-system,* which are analyzed instead of the full $(n_1 + n_2)$-system (see for example Chow J.H., Kokotovic P.V., 1976; Kokotovic P.V., 1984; Kokotovic P.V., Khalil H.K., O'Reilly J., 1986).

8.2. The Standard Singular Perturbation Model

The singular perturbation model of a dynamical system is a state-space model in which the derivatives of some of the states are multiplied by a small positive parameter μ; that is,

$$\dot{x} = f(t, x, z, \mu), \quad x \in R^{n_1},$$
$$\mu\dot{z} = g(t, x, z, \mu), \quad z \in R^{n_2}. \tag{129}$$

We assume that the functions f and g are continuously differentiable in their arguments for $(t, x, z, \mu) \in [0, t_1] \times D_1 \times D_2 \times [0, \mu^0]$, where $D_1 \subset R^{n_1}$, $D_2 \subset R^{n_2}$ are the open connected sets. When we set $\mu = 0$ in (129), the dimension of the state equation reduces from $n_1 + n_2$ to n_1 because the second differential equation in (129) degenerates into the equation

$$0 = g(t, x, z, 0). \tag{130}$$

We shall say that the model (129) is in *standard form* if and only if (130) has $k \geq 1$ isolated real roots

$$z = h_i(t, x), \quad i = \overline{1, k} \tag{131}$$

for each $(t, x) \in [0, t_1] \times D_1$. This assumption ensures that a well-defined n_1-dimensional reduced model will correspond to each root of (130). To obtain the i-th reduced model, we substitute (131) into (129), at $\mu = 0$, to obtain

$$\dot{x} = f(t, x, h(t, x), 0), \tag{132}$$

where we have dropped the subscript i from $h(t, x)$. It will be clear further from the context which root of (131) we are using. The model (129) is sometimes called a *quasi-steady-state model* because z, whose velocity $\dot{z} = \frac{g}{\mu}$ can be very large when $\mu \to 0$ and $g \neq 0$, may rapidly converge to a root of (130) which is the equilibrium of the second equation of (129). The model (132) is said to be *slow model* or *reduced model.*

Notice that modelling a physical system in the singularly perturbed form may not be easy, because it is not always clear how to pick the parameters to be considered as small. Fortunately, in many applications our knowledge of physical processes and components of the system sets us on the right road.

Singular perturbations cause a multitime-scale behavior of dynamic systems characterized by the presence of slow and fast transients in the system's response to externel disturbation. Loosely speaking, the slow response is approximated by the reduced model (132) while the discrepancy between the response of the reduced model (132) and that of the full model (129) is the fast transient. To see this point, let us consider the problem of solving the state equation

$$\dot{x} = f(t, x, z, \mu), \quad x(t_0) = \xi(\mu), \quad x \in R^{n_1},$$
$$\mu\dot{z} = g(t, x, z, \mu), \quad z(t_0) = \eta(\mu), \quad z \in R^{n_2}, \tag{133}$$

where $\xi(\mu)$, $\eta(\mu)$ depend smoothly on μ and $t_0 \in [0, t_1)$. Let $x(t, \epsilon)$ and $z(t, \epsilon)$ denote the solution of the full problem (133). When we define the corresponding problem for the reduced model (132), we can only specify n_1 initial conditions since the model is n_1-th order. Naturally, we retain the initial state for x to obtain *the reduced problem*

$$\dot{x} = f(t, x, h(t, x), 0), \quad x(t_0) = \xi_0 \equiv \xi(0), \quad x \in R^{n_1}. \tag{134}$$

Denote the solution of (134) by $\bar{x}(t)$. Since the variable z has been excluded from the reduced model and substituted by its "quasi-steady-state" $h(t, x)$, the only information we can obtain about z by solving (134) is to compute

$$\bar{z}(t) \equiv h(t, \bar{x}(t)),$$

which describes the quasi-steady-state behaviour of z when $x = \bar{x}$. By contrast to the original variable z starting at t_0 from a prescribed initial state $\eta(\epsilon)$, the quasi- steady-state $\bar{z}$ is not free to start from a prescribed value, and there may be a large discrepancy between its initial value

$$\bar{z}(t_0) \equiv h(t_0, \xi_0)$$

and the prescribed initial state $\eta(\epsilon)$. Thus, $\bar{z}(t)$ cannot be a uniform approximation of $z(t, \mu)$. The best we can expect is that the estimate

$$z(t, \mu) - \bar{z}(t) = O(\mu)$$

will hold on an interval excluding t_0, that is, for $t \in [t_b, t_1]$ where $t_b > t_0$. On the other hand, it is reasonable to expect the estimate

$$x(t, \mu) - \bar{x}(t) = O(\mu)$$

to hold uniformly for all $t \in [t_0, t_1]$ since

$$x(t_0, \mu) - \bar{x}(t_0) = \xi(\mu) - \xi(0) = O(\mu).$$

If the error $z(t, \mu) - \bar{z}(t)$ is indeed $O(\mu)$ over $[t_0, t_1]$, then it must be true that during the initial *("boundary-layer")* interval $[t_0, t_b]$ the variable z approaches $\bar{z}$. Let us remember that tha speed of z can be large since

$$\dot{z} = \frac{g}{\mu}.$$

In fact, having set $\mu = 0$ in the second equation of (133), we have made the transient of z instantaneous whenever $g \neq 0$. From the stability of equilibria, it should be clear that we cannot expect z to converge to its quasi-steady-state $\bar{z}$ unless certain stability conditions are satisfied. Such conditions will result from the forthcoming analysis.

It is more convenient in the analysis to perform the change of variables

$$y = z - h(t, x) \tag{135}$$

that shifts the quasi-steady-state of z to the origin. In the new variables (x, y), the full problem (129) is

$$\dot{x} = f(t, x, y + h(t, x), \mu), \quad x(t_0) = \xi(\mu),$$
$$\mu\dot{y} = g(t, x, y + h(t, x), \mu) - \mu\frac{\partial h}{\partial t} - \mu\frac{\partial h}{\partial x} f(t, x, y + h(t, x), \mu),$$
$$y(t_0) = \eta(\mu) - h(t_0, \xi(\mu)). \tag{136}$$

The quasi-steady-state of the second equation of (136) is now $y = 0$, which when substituted in the first equation of (136) results in the *reduced model* (134). To analyze the second equation of (136), let us note that $\mu\dot{y}(t)$ may remain finite even when μ tends to zero and $\dot{y}(t)$ tends to infiity. We set

$$\mu\frac{dy}{dt} = \frac{dy}{d\tau}; \quad hence \quad \frac{d\tau}{dt} = \frac{1}{\mu}$$

and we use $\tau = 0$ as the initial value at $t = t_0$. The new variable

$$\tau = \frac{t - t_0}{\mu}$$

is "stretched"; that is, if μ tends to zero ($\mu \to 0$) we have $\tau \to \infty$ even for finite t only slightly larger than t_0 by a fixed (independent of μ) difference. In the τ time scale, the second equation of (136) is represented by

$$\frac{dy}{d\tau} = g(t, x, y + h(t, x), \mu) - \mu\frac{\partial h}{\partial t} - \mu\frac{\partial h}{\partial x} f(t, x, y + h(t, x), \mu),$$

$$y(0) = \eta(\mu) - h(t_0, \xi(\mu)). \tag{137}$$

The variables t and x in the foregoing equation will be *slowly varying* since, in the τ time scale, they are given by the formulas

$$t = t_0 + \mu\tau,$$

$$x = x(t_0 + \mu\tau, \mu).$$

Setting $\mu = 0$ freezes these variables at $t = t_0$ and $x = \xi_0$, and reduces (137) to the time-invariant system

$$\frac{dy}{d\tau} = g(t_0, \xi_0, y + h(t_0, \xi_0), 0), \quad y(0) = \eta(0) - h(t_0, \xi_0) \tag{138}$$

which has equilibrium at $y = 0$. The frozen parameters (t_0, ξ_0) in (138) depend on the given initial time and initial state for the problem under consideration. In our investigation we should allow the frozen parameters to take any values in the region of slowly-varying parameters (t, x). Assume that the solution $\bar{x}(t)$ of the *reduced problem* (134) is defined for $t \in [0, t_1]$ and

$$\| \bar{x}(t) \| \leq r_1 \quad over \quad [0, t_1].$$

Define the set

$$B_r = \{x \in R_1^n \mid \quad \| x \| \leq r\},$$

where $r > r_1$. We rewrite (138) as

$$\frac{dy}{d\tau} = g(t, x, y + h(t, x), 0), \tag{139}$$

where $(t, x) \in [0, t_1] \times B_r$ are treated as *fixed parameters.* We shall refer to (139) as *the boundary-layer model* or *the boundary-layer system.* Sometimes, we shall also refer to (138) as *the boundary-layer model.* This should cause no confusion since (138) is an evaluation of (139) for a given initial time and initial state. The model (139) is more suitable when we study stability, controllability, observability, stabilizability properties of the the boundary-layer system.

Now we represent very important theorem *(Tikhonov's theorem)* for the singularly perturbed dynamic systems (SPDS) (133).

Theorem 9.1 (Tikhonov's theorem). *Consider SPDS (133) and let $z = h(t, x)$ be an isolated root of the equation*

$$0 = g(t, x, z, \mu).$$

Assume that the following conditions are satisfied for all

$$[t, x, z - h(t, x), \mu] \in [0, t_1] \times B_r \times B_\rho \times [0, \mu_0],$$

where

$$B_r \equiv \{x \in R_1^n \mid \quad \| x \| \leq r\},$$

$$B_\rho = \{y \in R^{n_2} \mid \quad \| y \| \leq \rho\} :$$

1. The functions f, g and their first partial derivatives with respect to (x, z, μ) are continuous. The function $h(t, x)$ and the Jacobian $\frac{\partial g(t,x,z,0)}{\partial z}$ have continuous first partial derivatives with respect to their arguments. The initial data $\xi(\mu)$ and $\eta(\mu)$ are smooth functions of μ.

2. The reduced problem (134) has a unique solution $\bar{x}(t)$, defined on $[t_0, t_1]$, and

$$\| \bar{x}(t) \| \leq r_1 < r$$

for all $t \in [t_0, t_1]$.

3. The origin of the boundary-layer model (139) is exponentially stable, uniformly in (t, x).

Then, there exist positive constants c and μ^* such that for all

$$\| \eta(0) - h(t_0, \xi(0)) \| < c$$

and

$$0 < \mu < \mu^*,$$

the singular perturbation problem (133) has a unique solution $x(t, \mu), z(t, \mu)$ on $[t_0, t_1]$, and

$$x(t, \mu) - \bar{x}(t) = O(\mu), \tag{140}$$

$$z(t, \mu) - h(t, \bar{x}(t)) - \tilde{y}(t/\mu) = O(\mu), \tag{141}$$

hold uniformly for $t \in [t_0, t_1]$, where $\tilde{y}(t/\mu)$ is the solution of the boundary-layer model (139). Moreover, given any $t_b > t_0$, there is $\mu^{**} \leq \mu^*$ such that

$$z(t, \mu) - h(t, \bar{x}(t)) = O(\mu), \tag{142}$$

holds uniformly for $t \in [t_b, t_1]$ whenever $\mu < \mu^{**}$.

There are other versions of this theorem which use slightly different technical assumptions (see, for example, **P.V.Kokotovic, h.K.Khalil, O'Reilly.** *Singular Perturbation Methods in Control: Analysis and Design.* Academic Press, New York, 1986.)

Let us consider example of SPDS.

8.3. Example of Armature-Controlled DC Motor

An armature-controlled DC motor can be modeled by the second-order state equation

$$J\frac{d\omega}{dt} = ki$$

$$L\frac{di}{dt} = -k\omega - Ri + u, \tag{143}$$

where i, u, R, L are the armature current, voltage, resistance, and inductance, J is the moment of inertia, ω is the angular speed, and $ki, k\omega$ are, correspondingly, the torque and the back electromotive force developed with constant excitation flux Φ (see Fig. 6). The first state equation in (143) is a mechanical torque equation, and the second one is an equation for the electric transient in the armature circuit. Typically, L is "small" and can play the role of the parameter μ. This means that, with

$$\omega = x, \quad i = y$$

the motor's model is *in standard form*

$$\dot{x} = f(t, x, y, \mu), \quad x \in R^{n_1},$$

$$\mu\dot{y} = g(t, x, y, \mu), \quad z \in R^{n_2}, \tag{144}$$

whenever $R \neq 0$. Neglecting L, we solve

$$0 = -k\omega - Ri + u$$

to obtain

$$i = \frac{u - k\omega}{R}$$

which is the only root, and substitute it in the torque equation. The resulting model

$$J\frac{d\omega}{dt} = -\frac{k^2}{R}\omega + \frac{k}{R}u$$

is the commonly used first-order model of the DC motor. In formulating perturbation models it is preferable to choose the perturbation parameter μ as a dimensionless ratio of two physical parameters. To that end, set us define the dimensionless variables

$$\omega_r = \frac{\omega}{\Omega}, i_r = \frac{iR}{k\Omega}, u_r = \frac{u}{k\Omega}$$

and rewrite the state equation as

$$T_m \frac{d\omega_r}{dt} = i_r$$

$$T_l \frac{di_r}{dt} = -\omega_r - i_r + u_r,$$

where $T_m = \frac{JR}{k^2}$ is the mechanical time constant and $T_l = \frac{L}{R}$ is the electrical time constant. Since

$$T_m \gg T_l,$$

we let T_m be the time unit; that is, we introduce the dimensionless time variable

$$t_r = \frac{t}{T_m}$$

and rewrite the state equation in the form

$$\frac{d\omega_r}{dt} = i_r$$

$$\frac{T_l}{T_m} \frac{di_r}{dt} = -\omega_r - i_r + u_r. \tag{145}$$

This scaling has brought the model (143) into *the standard form* (145) with a physically meaninful dimensionless parameter

$$\mu = \frac{T_l}{T_m} = \frac{k^2 L}{JR^2}.$$

9. Controllability and Observability Problems for Linear Time-Varying SPDS

9.1. Statement of the Problems

In the standard singular perturbation model of an ordinary differential equations (control systems) some of derivatives are multiplied by a scalar parameter, that is

$$\dot{x}(t) = A_1 x(t) + A_2 y(t) + B_1 u(t)$$

$$\mu \dot{y}(t) = A_3 x(t) + A_4 y(t) + B_2 u(t) \tag{146}$$

where $x(t) \in R^{n_1}$ is the slow variable, $y(t) \in \mathcal{R}^{n_2}$ is a fast variable, $m \leq n$ is the output, $u(t) \in \mathcal{R}^r$ is the input of (146). This time-invariant $(n_1 + n_2)$-SPDS is the partial case of the following control SPDS

$$\dot{x}(t) = \mathcal{A}_1(t,p)x(t) + \mathcal{C}_1(t,p)y(t) + B_1(t)u(t),$$

$$\mu\dot{y}(t) = \mathcal{A}_2(t,p)x(t) + \mathcal{C}_2(t,p)y(t) + B_2(t)u(t), \tag{147}$$

with the initial states

$$x_0(\cdot) = \{x(\vartheta) = \varphi(\vartheta), \vartheta \in [t_0 - h, t_0),\ x(t_0) = x_0\},$$

$$y_0(\cdot) = \{y(\vartheta) = \psi(\vartheta),\ \vartheta \in [t_0 - h, t_0),\ y(t_0) = y_0\},$$

$$x_0 \in \Re^{n_1},\ y_0 \in \Re^{n_2}, \tag{148}$$

$$\dot{x}_0(\cdot) = \{\dot{x}(\vartheta) = \dot{\varphi}(\vartheta),\ \vartheta \in [t_0 - h, t_0]\},$$

$$\dot{y}_0(\cdot) = \{\dot{y}(\vartheta) = \dot{\psi}(\vartheta),\ \vartheta \in [t_0 - h, t_0]\}, \tag{149}$$

where $x(\cdot) \in C([t_0 - h,\ t_1], \Re^{n_1})$, x is a slow variable, $y(\cdot) \in C([t_0 - h, t_1], \Re^{n_2})$, y is a fast variable, h is constant delay, $h = const > 0$, u is a control, $u(\cdot) \in C(T, \Re^r)$, $C([a,b], \Re^p)$ is a Banach space of continuous functions mapping $[a,b]$ in $\Re^p$ with the topology of uniform convergence, $t \in T = [t_0, t_1]$, μ is a small parameter, $\mu \in (0, \mu^0]$, $\mu^0 \ll 1$, p is a differentiation operator:

$$px(t) \equiv \dot{x}(t);$$

$exp(-ph)$ is a shift operator of function's argument:

$$exp(-ph)x(t) \equiv x(t-h),$$

so that

$$p \cdot exp(-ph)x(t) \equiv \dot{x}(t-h);$$

$\mathcal{A}_i(t,p)$, $\mathcal{C}_i(t,p)$, $i = 1,2$, are operators of the form

$$\mathcal{A}_i(t,p) = A_{i0}(t) + A_{i1}(t)exp(-ph) + A_{i2}(t)p \cdot exp(-ph)$$

$$\mathcal{C}_i(t,p) = C_{i0}(t) + C_{i1}(t)exp(-ph) + C_{i2}(t)p \cdot exp(-ph). \tag{150}$$

In (147), (150) $B_i(t)$, $A_{ij}(t)$, $C_{ij}(t)$, $i = 1,2$, $j = \overline{0,2}$, are sufficiently smooth on T matrix functions of the corresponding dimensions; in (148) $\varphi(t)$, $\psi(t)$ are continuously differentiable vector-functions; in (149) $\dot{\varphi}(t)$, $\dot{\psi}(t)$ are continuous n_1- and n_2-vector-functions correspondingly. System (147) – (150) is said to be the linear nonstationary control SPDS with the deviating argument of neutral type (LNCSPDSNT).

Definition 9.1.*LNCSPDSNT (147) – (150), $\mu \in (0, \mu^0]$, is called $\{x,y\}$-relatively controllable (x-relatively controllable, y-relatively controllable) on T, if for any $\{x_1, y_1\} \in \Re^{n_1+n_2}$ and any initial states (148), (149)* *there exists an admissible control $u(t)$ such, that the corresponding solution $\{x(t,\mu),y(t,\mu)\}$ satisfies the condition $\{x(t_1,\mu),y(t_1,\mu)\} = \{x_1,y_1\} \in \Re^{n_1+n_2}$, $(x(t_1,\mu) = x_1 \in \Re^{n_1};\ y(t_1,\mu) = y_1 \in \Re^{n_2})$.*

P r o b l e m 1 (*$\{x,y\}$-relative, x-relative, y-relative controllability problem*). Find conditions of $\{x,y\}-$, $x-$, y-relative controllability for LNCSPDSNT (147) – (150), $\mu \in (0,\mu^0]$, expressed through its parameters $A_{ij}(t)$, $C_{ij}(t), B_i(t), i = 1,2, j = \overline{0,2}$.

LNCSPDSNT (147) – (150), $\mu \in (0,\mu^0]$, is $\{x,y\}$-relatively (x-relatively, y-relatively) controllable on T if the corresponding controllability problem is solved for any initial states (148), (149).

Consider now the following linear nonstationary observation SPDS

$$\dot{x}(t) = \mathcal{A}_1(t,p)x(t) + \mathcal{C}_1(t,p)y(t),$$

$$\mu\dot{y}(t) = \mathcal{A}_2(t,p)x(t) + \mathcal{C}_2(t,p)y(t), \tag{151}$$

with the output

$$w(t) = D_1(t)x(t) + D_2(t)y(t), \tag{152}$$

where $x \in \Re^{n_1}$, $y \in \Re^{n_2}$, $w \in \Re^{n_3}$, $\quad n_3 \le n_1 + n_2$, $t \in T = [t_0,t_1]$, operators $\mathcal{A}_i(t,p), \mathcal{C}_i(t,p), i = 1,2$, have the form (151) and the initial state has the form (148), (149). Suppose that $A_{ij}(t)$, $C_{ij}(t)$, $D_i(t)$, $i = 1,2$, $j = \overline{0,2}$, are sufficiently smooth matrix functions on $T = [t_0,t_1]$ in (151), (152) with $t_1 \in (lh,(l+1)h]$, $l = const > 0$.

System (151) with the output (152) is said to be the linear nonstationary observation SPDS with the deviating argument of neutral type (LNOSPDSNT).

Definition 9.2.LNOSPDSNT (151), $\mu \in (0,\mu^0]$, (148), (149) is called $\{x,y\}$-relatively observable (x-relatively observable, y-relatively observable) on T with respect to the output $w(t), t \in T$, if for known initial functions $\varphi(\vartheta)$, $\psi(\vartheta)$, $\vartheta \in [t_0-h,t_0)$, it is possible uniquely to restore the vector $\{x(t_0),y(t_0)\} \in \Re^{n_1+n_2}$ (the component $x(t_0) \in \Re^{n_1}$, the component $y(t_0) \in \Re^{n_2}$) of the initial state (148) which generates the output (152).

P r o b l e m 2 (*$\{x,y\}$-relative, x-relative, y-relative observability problem.*) Find conditions of $\{x,y\}$-, x-, y-relative observability for LNOSPDSNT (151), $\mu \in (0,\mu^0]$, (148), (149) with respect to the output (152) expressed through its parameters $A_{ij}(t), C_{ij}(t), D_i(t), i = 1,2, j = \overline{0,2}$.

9.2. The Defining Equations for Control Systems

To find algebraic controllability conditions for LNCSPDSNT (147) we rewrite it in the form

$$px(t) = \mathcal{A}_1(t,p)x(t) + \mathcal{C}_1(t,p)y(t) + B_1(t)u(t),$$

$$\mu p y(t) = \mathcal{A}_2(t,p)x(t) + \mathcal{C}_2(t,p)y(t) + B_2(t)u(t), \qquad (153)$$

where $x \in \Re^{n_1}$, $y \in \Re^{n_2}$, $u \in \Re^r, t \in T = [t_0, t_1]$. Let us introduce some correspondences [2 – 4] between vector-functions $x(t)$, $y(t)$, $u(t)$, the operator p, small parameter μ of the system (153) and new matrix functions $X_k^i(t,s) \in \Re^{n_1 \times r}$, $Y_k^i(t,s) \in \Re^{n_2 \times r}$, $U_k^i(t,s) \in \Re^{r \times r}$ of two arguments t, s (t reflects the nonstationarity of system (153), s reflects the presence of delay h), new operators Δ_+, Δ^+, D according to the rules:

$$x(t) \longrightarrow X_k^i(t,s),\ y(t) \longrightarrow Y_k^i(t,s),\ u(t) \longrightarrow U_k^i(t,s),$$

$$p \longrightarrow \Delta_+ + D, \quad \mu \longrightarrow \Delta^+. \qquad (154)$$

In (154) $\Delta_+ \,(\Delta^+)$ is the shift operator to the right on 1 for lower (upper) index of some matrix function, D is the differentiation operator with respect to t of some matrix function, so that for example $(\Delta_+ + D)Z_k^i(t,s) \equiv Z_{k+1}^i(t,s) + \dot{Z}_k^i(t,s)$. The index $k + j\,(j = 0,1)$ at matrices X_{k+j}^i, Y_{k+j}^i corresponds to j-derivative of vectors x, y from (1), the index $i + m$ $(m = 0,1)$ at X_k^{i+m}, Y_k^{i+m} corresponds to power m of μ at derivatives $\dot{x}$, $\dot{y}$ of the system (147). The nonstationarity of system (136) in contrast to autonomous case is reflected due to (137) by the term D in the correspondence $p \longrightarrow \Delta_+ + D$. This correspondence transforms $\dot{x}(t)$, $\mu\dot{y}(t)$ from (147) in virtue of (153) to $px(t) \rightarrow (\Delta_+ + D)X_k^i(t,s) \equiv X_{k+1}^i(t,s) + \dot{X}_k^i(t,s)$, $\mu p y(t) \rightarrow \Delta^+(\Delta_+ + D)Y_k^i(t,s) \equiv Y_{k+1}^{i+1}(t,s) + \dot{Y}_k^{i+1}(t,s)$. Obviously for autonomous system (153) we have $x(t) \longrightarrow X_k^i(s)$, $y(t) \longrightarrow Y_k^i(s)$ instead of (154) and correspondingly we obtain $px(t) \rightarrow (\Delta_+ + D)X_k^i(s) \equiv X_{k+1}^i(s)$, $\mu p y(t) \rightarrow \Delta^+(\Delta_+ + D)Y_k^i(s) \equiv Y_{k+1}^{i+1}(s)$. Due to (154), (133) $exp(-\Delta_+ h)$ is a shift operator of argument t for some matrix function: $exp(-\Delta_+ h)X_k^i(t,s) \equiv X_k^i(t-h,s)$; $exp(-Dh)$ is a shift operator of argument s for some matrix function: $exp(-Dh)X_k^i(t,s) \equiv X_k^i(t,s-h)$, so that $exp(-(\Delta_+ + D)h)X_k^i(t,s) \equiv X_k^i(t-h,s-h)$. Due to (137) we have for system (153) the following matrix algebraic reccurence on k, i equations:

$$(\Delta_+ + D)X_k^i(t,s) = \mathcal{A}_1(t, \Delta_+ + D)X_k^i(t,s) +$$

$$+\mathcal{C}_1(t, \Delta_+ + D)Y_k^i(t,s) + B_1(t)U_k^i(t,s),$$

$$\Delta^+(\Delta_+ + D)Y_k^i(t,s) = \mathcal{A}_2(t, \Delta_+ + D)X_k^i(t,s) +$$

$$+\mathcal{C}_2(t, \Delta_+ + D)Y_k^i(t,s) + B_2(t)U_k^i(t,s), \qquad (155)$$

$$i = 0,1,2,...,\ k = 0,1,2,....$$

By virtue of properties of operators Δ^+, Δ_+, D these equations can be rewritten in the form

$$X^i_{k+1}(t,s)+\dot{X}^i_k(t,s)=\mathcal{A}_1(t,\Delta_++D)X^i_k(t,s)+$$
$$+\mathcal{C}_1(t,\Delta_++D)Y^i_k(t,s)+B_1(t)U^i_k(t,s),$$
$$Y^{i+1}_{k+1}(t,s)+\dot{Y}^{i+1}_k(t,s)=\mathcal{A}_2(t,\Delta_++D)X^i_k(t,s)+$$
$$+\mathcal{C}_2(t,\Delta_++D)Y^i_k(t,s)+B_2(t)U^i_k(t,s), \tag{156}$$
$$i=0,1,2,...,\ k=0,1,2,....$$

To define uniquely the solution $\{X^i_k(t,s),Y^i_k(t,s)\}$ of (139) we introduce the initial conditions:

$$U^0_0(t,0)=E_r,\ U^i_k(t,s)=O_r,\ k\neq 0\vee i\neq 0,$$
$$X^i_k(t,s)=O_{n_1\times r},\ Y^i_k(t,s)=O_{n_2\times r},\ k\leq 0, \tag{157}$$

where $O_{(l_1\times l_2)}$ is zero $(l_1\times l_2)$-matrix, O_l- is zero square $(l\times l)$-matrix, E_l is the identity $(l\times l)$-matrix. Matrix algebraic reccurence on k, i equations (155) or (156) we shall call *the defining equations* of LNCSPDSNT (147). A totality $\{X^i_k(t,s),\ Y^i_k(t,s)\}\in\Re^{(n_1+n_2)\times r}$ $(k=0,1,2,\ldots,i=0,1,2,\ldots)$ is said to be *a solution of the defining equation* (156), (157). Matrices $X^i_k(t,s)\in\Re^{n_1\times r}$, $Y^i_k(t,s)\in\Re^{n_2\times r}$ calculated according to (156), (157) we shall call *the components of the solutions* $\{X^i_k(t,s),\ Y^i_k(t,s)\}\in\Re^{(n_1+n_2)\times r}$ $(k=0,1,2,\ldots,i=0,1,2,\ldots)$ of these equations.

R e m a r k . Note that for control systems of linear ordinary differential equations the defining equations (156), (157) coincide to $B(t)U_k(t)$, $B(t)U_k(t,s)$ accuracy with the well known ones before [5 – 7]. For stationary systems with the deviating argument of neutral type these equations coincide with [8], but for nonstationary systems of such type they are new equations and differ from known before [9]. For LNCSPDSNT (147) and its partial cases of LNCSPDS described by ordinary differential equations, LNCSPDSD the defining equations (156) are introduced for the first time.

9.3. The Defining Equations for Observation Systems

Let us consider LNOSPDSNT (151) in the form

$$px(t)=\mathcal{A}_1(t,p)x(t)+\mathcal{C}_1(t,p)y(t),$$
$$\mu py(t)=\mathcal{A}_2(t,p)x(t)+\mathcal{C}_2(t,p)y(t), \tag{158}$$

with the output (152). Let us introduce the following correspondences [10 – 12] between vector-functions $x(t)$, $y(t)$, $w(t)$, operator p, small parameter μ, $(l_1 \times l_2)$-matrix functions $Q(t)$ $(l_1 = n_1 \vee n_2 \vee n_3,\ l_2 = n_1 \vee n_2)$, $Q(t) \in \mathcal{Q} = \{A_{ij}(t),\ C_{ij}(t),\ D_i(t),\ i = 1,2\ , j = \overline{0,2}\}$ which are contained in (151) and (152), and new matrix functions $X_i^k(s,t) \in \Re^{n_1\times(n_1+n_2)}$, $Y_i^k(s,t) \in \Re^{n_2\times(n_1+n_2)}$, $W_i^k(s,t) \in \Re^{n_3\times(n_1+n_2)}$, $n_3 \leq n_1 + n_2$, of two arguments s, t, operators Δ_+, Δ^+ and new operator p^i according to the rules:

$$x(t) \to X_i^k(s,t), \quad y(t) \to Y_i^k(s,t), \quad w(t) \to W_i^k(s,t),$$
$$p \to \Delta_+, \quad \mu \to \Delta^+,\ Q(t) \to (E_{l_1} + p^i E_{l_1})Q(t),$$
$$Q(t) \in \{A_{ij}(t),\ C_{ij}(t),\ D_i(t),\ i = 1,2,\ j = \overline{0,2}\}. \tag{159}$$

Thus we have for $\mathcal{A}_1(t,p)$ from (158) due to (159):

$$\mathcal{A}_1(t,p) \longrightarrow (E_{n_1} + p^i E_{n_1})(A_{10}(t) +$$
$$A_{11}(t)exp(-\Delta_+ h) + A_{12}(t)\Delta_+ \cdot exp(-\Delta_+ h)),$$

where $exp(-\Delta_+ h)$ is a shift operator of two arguments s, t to the right on h for some matrix functions: $exp(-\Delta_+ h)Z_i^k(s,t) = Z_i^k(s-h, t-h)$.

Taking into consideration the nonstationarity of LNOSPDSNT (158) let us introduce the operator

$$p^i : C^i(T, R^{l_1\times in}) \times C(T^2, R^{in\times l_2}) \longrightarrow C(T^2, R^{l_1\times l_2})$$

with the domain $\mathcal{D}(p^i) = C^i(T, R^{l_1\times in}) \times C(T^2, R^{in\times l_2})$. This operator puts in the correspondence to the product $Q(t)Q_i^k(s,t)$ of an arbitrary pair $(Q(t), Q_i^k(s,t))$ of matrix functions $Q(\cdot) \in C^i(T)$ ($Q(t) \in \mathcal{Q}$, $Q_i^k(s,t) \in \mathcal{Z} = \{X_i^k(s,t),\ Y_i^k(s,t),\ W_i^k(s,t)\}$, $Q_i^k(s,t) \in R^{l_1\times(n_1+n_2)}$, $s, t \in T$, i, k — indexes) the element $p^i(Q(t)Q_i^k(s,t)) \in C(T^2, R^{l_1\times l_2})$ according to the rules:

$$p^i(Q(t)Q_i^k(s,t)) = [C_i^q Q^{(q)}(t),\ q = \overline{1,i},] \times \begin{bmatrix} Q_{i-q}^k(s,t) \\ \\ q = \overline{1,i} \end{bmatrix}, \tag{160}$$

where $C_i^k = i!/k!(i-k)!$. The index i at p^i points to maximum order of the derivative $Q^{(i)}(t)$ of the matrix $Q(t)$, as well the number of low index for all blocks in the matrix row and a quantity of block matrices in a column matrix (160). It is not difficult to note that p^i is a bilinear operator. Then for example for $\mathcal{A}_1(t,p)x(t)$ from (158), (159) we obtain

$$\mathcal{A}_1(t,\Delta_+)X_i^k(s,t) = A_1(t)X_i^k(s,t) + A_2(t)X_i^k(s-h, t-h)$$

$$+A_3(t)X_{i+1}^k(s-h,t-h)+p^i(A_1(t)X_i^k(s,t))+$$
$$p^i(A_2(t)X_i^k(s-h,t-h))+p^i(A_3(t)X_{i+1}^k(s-h,t-h)).$$

In virtue of (151), (152), (142) we obtain from (158) the following reccurence on i, k relations

$$\Delta_+X_i^k(s,t)=(E_{n_1}+p^iE_{n_1})[\mathcal{A}_1(t,\Delta_+)X_i^k(s,t)+\mathcal{C}_1(t,\Delta_+)Y_i^k(s,t)],$$
$$\Delta^+\Delta_+Y_i^k(s,t)=(E_{n_2}+p^iE_{n_2})[\mathcal{A}_2(t,\Delta_+)X_i^k(s,t)+\mathcal{C}_2(t,\Delta_+)Y_i^k(s,t)],$$
$$W_i^k(s,t)=(E_{n_3}+p^iE_{n_3})[D_1(t)X_i^k(s,t)+D_2(t)Y_i^k(s,t)],$$
$$i=0,1,2,...,\ k=0,1,2,..., \tag{161}$$

which are constructed according to the type of LNOSPDSNT (151), (152) by the natural way. Equations (161) are said to be *the defining equations* of LNOSPDSNT (??), (152). They are equivalent to

$$X_{i+1}^k(s,t)=\mathcal{A}_1(t,\Delta_+)X_i^k(s,t)+\mathcal{C}_1(t,\Delta)Y_i^k(s,t)+$$
$$+p^i(\mathcal{A}_1(t,\Delta_+)X_i^k(s,t))+p^i(\mathcal{C}_1(t,\Delta_+)Y_i^k(s,t)),$$
$$Y_{i+1}^{k+1}(s,t)=\mathcal{A}_2(t,\Delta_+)X_i^k(s,t)+\mathcal{C}_2(t,\Delta_+)Y_i^k(s,t)+$$
$$+p^i(\mathcal{A}_2(t,\Delta_+)X_i^k(s,t))+p^i(\mathcal{C}_2(t,\Delta_+)Y_i^k(s,t)), \tag{162}$$
$$W_i^k(s,t)=D_1(t)X_i^k(s,t)+D_2(t)Y_i^k(s,t)+$$
$$+p^i(D_1(t)X_i^k(s,t)+p^i(D_2(t)Y_i^k(s,t)),$$
$$i=0,1,2,...,\ k=0,1,2,.... \tag{163}$$

Thus we obtained that to hybrid system consisting of SPDS (151) and algebraic system (152) in virtue of (159) corresponds the system of matrix algebraic reccurence on i,k equations (162), (163) (or the same equation (161)). We shall call the reccurence equations (162), (163) *the defining equations* for (151), (152) as well. To define uniquely the solution $\{X_i^k(s,t),Y_i^k(s,t),W_i^k(s,t)\}$ of equations (162), (163) we introduce the initial states

$$X_0^0(0,t)=[E_{n_1},\ O_{n_1\times n_2}],\ X_i^k(s,t)=O_{n_1\times(n_1+n_2)},$$
$$s\neq jh,\ j=\overline{0,l}\vee s=jh,\ k\geq j+i\vee$$
$$\vee i<0\vee k<0\vee j<0,$$
$$Y_0^0(0,t)=[O_{n_2\times n_1},\ E_{n_2}],\ Y_i^k(s,t)=O_{n_2\times(n_1+n_2)},$$

$$s \neq jh,\ j = \overline{0,l} \vee s = jh,\ \ k \geq j + i + 1 \vee$$
$$\vee i < 0 \vee k \leq 0 \vee j < 0. \qquad (164)$$

Triple of matrices $\{X_i^k(s,t),\ Y_i^k(s,t),\ W_i^k(s,t)\}$ will be called *the solution* of the defining equations (162), (163) with the initial states (164). Matrices $X_i^k(s,t),\ Y_i^k(s,t),\ W_i^k(s,t)$, calculated according to (162) – (164) we shall call *the components of solutions* $\{X_i^k(t), Y_i^k(t), W_i^k(t)\}$ of these equations. Note that for any arbitrary indexes i, k each component $Q_i^k(s,t) \in \mathcal{Z}$ consists of two blocks $Q_{ij}^k(s,t),\ j = 1,2,\ Q_{ij}^k \in R^{l_1 \times n_j},\ (l_1 = n_1 \vee n_2 \vee n_3)$. This fact reflects distinct scales of variables x, y in system (151) and the presence of motions with two essentially different velocities $\dot{x}(t), \mu\dot{y}(t)$.

The following proposition explains the sense of all introduced matrix elements from (159).

Lemma 1 [13]. *The following formula*

$$Q_i^k(s,t) = \Big(\frac{\partial}{\partial(\mu^{-k})}\frac{\partial}{\partial x},\ \frac{\partial}{\partial(\mu^{-k})}\frac{\partial}{\partial y}\Big)q^{(i)}(t),$$

takes place for any $i, k \quad (i = 0, 1,\ 2, \ldots,\ k = 0,\ \pm 1\ , \pm 2, \ldots,)$ *for any vector-function* $q(t) \in \{x(t),\ y(t),\ w(t)\},\ q \in R^{n_l}\ (l = \overline{1,3})$ *and for matrix function* $Q_i^k(s,t) \in \mathcal{Z}$, $(Q_i^k \in R^{l_1 \times (n_1+n_2)}, l_1 = n_1 \vee n_2 \vee n_3)$, *corresponding to it in virtue of (13), where* $\partial/\partial z$ *is the operator of partial differentiation with respect to* z, $z \in \{x, y\}$ and $\partial/\partial(\mu^{-k})$ *is the operator of partial differentiation of matrix function with respect to* μ^{-k}.

9.4. Main results

Now we shall find some conditions of relative controllability for NC-SPDSNT (147) – (150) and some conditions of relative observability for LNOSPDSNT (151) with the output (152) expressed through the components $X_k^i(t,s), Y_k^i(t,s),\ W_i^k(s,t)$ of the solutions of the appropriate defining equations (156), (157) and (152) – (154). First we shall demonstrate the main idea of the paper by investigating the controllability problem for linear nonstationary SPDS of ordinary differential equations (LNSPDSOD) (147) – (150). Let us represent this system in an extended space of dimension $n_1 + n_2$ in the form

$$\dot{z}(t) = \mathcal{A}(t,\mu)z(t) + \mathcal{B}(t)u(t),$$
$$z \in \Re^{n_1+n_2}, \quad t \in T = [t_0, t_1], \quad z(t_0) = z_0, \qquad (165)$$

where $\mathcal{A}(t,\mu) \in \Re^{(n_1+n_2)\times(n_1+n_2)}$, $z(t) = col(x(t), y(t))$,

$$\mathcal{A}(t,\mu) = \begin{bmatrix} \mathcal{A}_1(t) & \mathcal{C}_1(t) \\ \mathcal{A}_2(t)/\mu & \mathcal{C}_2(t)/\mu \end{bmatrix} =$$

$$= \begin{bmatrix} A_{10}(t) & C_{10}(t) \\ A_{20}(t)/\mu & C_{20}(t)/\mu \end{bmatrix}, \mathcal{B}(t) = \begin{bmatrix} B_1(t) \\ B_2(t)/\mu \end{bmatrix}. \tag{166}$$

Note that for $\mu \to 0$ system (165) singularly depends on μ.

Definition 9.3. *LNSPDSOD (165) , $\mu \in (0, \mu^0]$, is completely controllable on T if for any $z_1 \in \Re^{n_1+n_2}$ and for any initial state $z_0 \in \Re^{n_1+n_2}$ there exists control $u(t) \in U$ such that the corresponding solution $z(t, \mu)$, $t \in T$, satisfies the condition $z(t_1, \mu) = z_1$.*

For each fixed μ, $\mu \in (0, \mu^0]$, system (165) is a linear nonstationary one of ordinary differential equations. Sufficient conditions of complete controllability for this system are well known [14, Theorem 20.1]. In terms of solutions $Z_k(t)$ of the defining equations (156) , (157) these conditions have the form:

$$rank\ \{Z_k(t),\ k = \overline{1, n}\} = n, \quad \exists \quad t \in T,$$

where

$$Z_{k+1}(t) + \dot{Z}_k(t) = \mathcal{A}(t)Z_k(t) + \mathcal{B}(t)U_k(t), \tag{167}$$

$$U_0(t) = E_r,\ U_k(t) = O_r,\ k \neq 0, Z_k(t) = O_{n_1 \times r},\ k \leq 0. \tag{168}$$

Note that solutions $X_k(t), Y_k(t)$ of the defining equations (156) , (157) for system (147), (150) and solutions $Z_k(t)$ of the defining equations (167) with the initial states (168) are connected by the relation :

$$rank\ \{Z_k(t),\ k = \overline{1, n_1 + n_2}\} = rank\ Z(t, \mu),$$

which is analogous to [4, p.42]. Here $\mu \in (0, \mu^0]$,

$$Z(t, \mu) = \begin{bmatrix} \sum\limits_{m=0}^{i-1} \mu^m X_i^{i-m-1}(t, s) & \\ & i = \overline{1, n_1 + n_2} \\ \sum\limits_{m=0}^{i-1} \mu^m Y_i^{i-m}(t, s) & \end{bmatrix}.$$

The proof of this and consequent propositions are analogous to proofs of Lemma 9.1, Theorem 9.1, Corollaries 9.1 – 9.3 of [4] and we drop their.

Theorem 9.1. *Let us assume that $A_{i0}(t) \in C^{n_1+n_2-2}(T, \Re^{n_i \times n_1})$, $C_{i0}(t) \in C^{n_1+n_2-2}(T, \Re^{n_i \times n_2})$, $B_i(t) \in C^{n_1+n_2-1}(T, \Re^{n_i \times r})$, $i = 1, 2$. Then*

1. the condition

$$rank\ Z(t, \mu) = n_1 + n_2 \tag{169}$$

is a sufficient one of $\{x, y\}$-complete controllability for LNSPDSOD, $\mu \in (0, \mu^0]$, (165), (166) for some $t \in T$;

2. if elements of matrices $A_{i0}(t),\ C_{i0}(t),\ B_i(t),\ i=1,2$ are analytical functions on T then the condition (23) is necessary one as well.

Corollaries 1, 2 give sufficient conditions and corollary 3 gives necessary condition of $\{x,y\}$−complete controllability for LNSPDSOD (147)— (150).

Corollary 9.1. *Suppose that conditions of Theorem 9.1 are fulfilled. Then if for some set of integers $l_i,\ i=\overline{1,n_1+n_2},\ l_i=\overline{1,i}$, there exists $m,\ 1\le m\le n_1+n_2$, for which*

$$rank\begin{bmatrix} X_i^{i-m-l_i-1}(t) & \\ & i=\overline{1,n_1+n_2} \\ Y_i^{i-m-l_i}(t) & \end{bmatrix}=n_1+n_2,$$

then there exists $\mu^*>0$ such that LNSPDSOD (147)— (150) is completly controllable on T for all $\mu\in(0,\mu^*]$.

Corollary 9.2. *Suppose that conditions of Theorem 1 are fulfilled. Then if for some set of integers $l_1,\ l_2,\ l_i=\overline{1,n_1+n_2};\ i=1,2$, there exists $m,\ 1\le m\le n_1+n_2-min(l_1+l_2)$, for which*

$$rank\begin{bmatrix} X_i^{i-m-l_1-1}(t) & \\ & i=\overline{1,n_1+n_2} \\ Y_i^{i-m-l_2}(t) & \end{bmatrix}=n_1+n_2,$$

then there exists $\mu^*>0$ such that LNSPDSOD (147)— (150) is completly controllable on T for all $\mu\in(0,\mu^*]$.

Corollary 9.3. *Let us assume that elements of matrices (166) are analytical functions on T. If LNCSPDSOD (165) is completely controllable on T, then for $\mu\in(0,\mu^0]$*

$$rank\,[\sum_{m=0}^{i-1}\mu^m X_i^{i-m-1}(t),\quad i=\overline{1,n_1+n_2}\,]=n_1,$$

$$rank\,[\sum_{m=0}^{i-1}\mu^m Y_i^{i-m}(t),\quad i=\overline{1,n_1+n_2}\,]=n_2.$$

For LNCSPDSNT (147)— (150) let us create the matrix

$$Z(t,\mu)=\begin{bmatrix} \sum\limits_{m=0}^{i-1}\mu^m X_i^{i-m-1} & \\ & i=\overline{1,n_1+n_2},\ s=\overline{0,lh} \\ \sum\limits_{m=0}^{i-1}\mu^m Y_i^{i-m} & \end{bmatrix}$$

where $\mu \in (0, \mu^0]$, $X_k^i(t,s), Y_k^i(t,s)$ are the components of solutions of the defining equations (156), (157).

Theorem 9.2. *Let*

$$A_{ij}(t) \in C^{n_1+n_2-2}(T, \Re^{n_i \times n_1}), C_{ij}(t) \in C^{n_1+n_2-2}(T, \Re^{n_i \times n_2}),$$

$$B_i(t) \in C^{n_1+n_2-1}(T, \Re^{n_i \times r}), i = 1,2;\ j = 0,1,2.$$

Then

1. if for some $t \in T$, $\mu \in (0, \mu^0]$

$$rank\ Z(t,\mu) = n_1 + n_2 \tag{170}$$

then LNCSPDSNT (147)— (150) is $\{x,y\}$-relatively controllable on T.

2. if elements of matrices $A_{ij}(t)$, $C_{ij}(t)$, $B_i(t)$, $i = 1,2;\ j = 0,1,2$ are analytical functions on T, then (170) is necessary condition of $\{x,y\}$-relative controllability for (147)— (150) as well.

Now we shall formulate some algebraic conditions of relative observability for LNOSPDSNT (151), (152) in terms of the components $W_i^k(s,t)$ of the defining equations (162) – (163).

To find conditions of relative observability for LNOSPDSD (151) with the output (152) we create $(n_3(n_1+n_2)(l+1) \times (n_1+n_2))-$matrix

$$L(t,\mu) = \begin{bmatrix} \sum\limits_{m=0}^{i} \mu^{-m} W_i^m(jh, t+jh) \\ i = \overline{0, n_1+n_2-1} \\ j = \overline{0,l} \end{bmatrix} \tag{171}$$

from the components $W_i^k(s,t)$ of solutions of the defining equations (162) – (163) where $W_i^m(s,t)$ are $(n_3 \times (n_1+n_2))-$matrix so that in (171)

$$W_i^m(jh, t+jh) \doteq [\, W_{i1}^m(jh, t+jh), \quad W_{i2}^m(jh, t+jh) \,],$$

$$W_{i1}^m \in \Re^{n_3 \times n_1}, \quad W_{i2} \in \Re^{n_3 \times n_2}.$$

Theorem 9.3. *Suppose that matrices $A_{ij}(t)$, $C_{ij}(t)$ $(i = 1,2, j = 0,1)$ of system (151) are continuously differentiable (n_1+n_2-2) times and $D_i(t)$, $i = 1,2$, — (n_1+n_2-1) times on T. Then for $\{x,y\}-$relative observability of LNOSPDSD (151) with respect to the output (152) it is sufficiently that*

$$rank\ L(t_0,\mu) = n_1 + n_2, \quad \mu \in (0, \mu^0].$$

If $A_{ij}(t)$, $C_{ij}(t)$ $D_i(t)$, $(i = 1,2, j = 0,1), j = 1,2$ are analytical functions then for $\{x,y\}$-relative observability of (151), (152) it is necessary the existence of such time moment $t^* \in T$ that

$$rank\ L(t^*,\mu) = n_1 + n_2, \quad \mu \in (0,\mu^0].$$

Corollary 9.4. *Suppose that all elements in LNOSPDSD (151), (152) are analytical matrices. Then if (151) is $\{x,y\}$-relatively observable with respect to the output (152) then there exists a such time moment $t^* \in T$ that*

$$rank \left[\begin{array}{c} \sum\limits_{m=0}^{i} \mu^{-m} W_{i1}^{m}(jh, t^* + jh) \\ i = \overline{0, n_1 + n_2 - 1} \\ j = \overline{0, l} \end{array} \right] = n_1,$$

$$rank \left[\begin{array}{c} \sum\limits_{m=0}^{i} \mu^{-m} W_{i2}^{m}(jh, t^* + jh) \\ i = \overline{0, n_1 + n_2 - 1} \\ j = \overline{0, l} \end{array} \right] = n_2$$

for all $\mu \in (0,\mu^0]$.

To find some conditions of relative observability for LNOSPDSNT (151), (152) let us write it in $(n_1 + n_2)$-space:

$$\dot{z}(t) = \mathcal{A}(t,p,\mu)z(t), \quad z \in R^{n_1+n_2}, \tag{172}$$

$$w(t) = D(t)z(t), \quad w \in R^{n_3}, \quad t \in T,\ \mu \in (0,\mu^0], \tag{173}$$

with the initial conditions

$$z_0(\cdot) = \{z(\vartheta) = \zeta(\vartheta),\ \vartheta \in [t_0 - h, t_0),\ z(t_0) = z_0\},$$

$$\dot{z}_0(\cdot) = \{\dot{z}(\vartheta) = \dot{\zeta}(\vartheta),\ \vartheta \in [t_0 - h, t_0]\}, \tag{174}$$

where operator $\mathcal{A}(t,p,\mu)$ has the form $\mathcal{A}(t,p,\mu) = \bar{A}(t,\mu) + \bar{A}_1(t,\mu)exp(-ph) + \bar{A}_2(t,\mu)p \cdot exp(-ph)$,

$$z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},\ \zeta(\vartheta) = \begin{bmatrix} \varphi(\vartheta) \\ \psi(\vartheta) \end{bmatrix},$$

$$\bar{A}_0(t,\mu) = \left[\begin{array}{cc} A_{10}(t) & C_{10}(t) \\ A_{20}(t)/\mu & C_{20}(t)/\mu \end{array} \right],$$

$$\bar{A}_1(t,\mu) = \begin{bmatrix} A_{11}(t) & C_{11}(t) \\ A_{21}(t)/\mu & C_{21}(t)/\mu \end{bmatrix},$$

$$\bar{A}_2(t,\mu) = \begin{bmatrix} A_{12}(t) & C_{12}(t) \\ A_{22}(t)/\mu & C_{22}(t)/\mu \end{bmatrix},$$

$$D(t) = [D_1(t), D_2(t)]. \tag{175}$$

Note that for $\mu \longrightarrow 0$ system (172) in a singular way depends on μ. Introduce $(n_1 \times (n_1+n_2))$-matrix $H_x = [E_{n_1}, 0_{n_1\times n_2}]$ and $(n_2 \times (n_1+n_2))$-matrix $H_y = [0_{n_2\times n_1}, E_{n_2}]$.

Definition 9.4. *LNOSPDSNT (172) is relatively (H_x-relatively; H_y-relatively) observable with respect to output (173) on T, if from measurements of output $w(t)$ and known initial functions $\zeta(\vartheta)$, $\dot{\zeta}(\vartheta), \vartheta \in [t_0 - h, t_0)$ we can uniquely to restore vector z_0 $(H_x z_0;\ H_y z_0)$ of the initial state (174) which generated this output $w(t)$.*

For each fixed μ, $\mu \in (0,\mu^0]$, system (172) is a linear nonstationary system with the deviating argument of neutral type. Relative observability conditions of such type systems are well known [15]. Then the problem of (x-relative, y-relative) observability for LNOSPDSNT (151), (152), (149), (150) due to (175) is equivalent to the problem of relative (H_x-relative; H_y-relative) observability of (172), (173).

To obtain some conditions of relative observability for LNSPDSNT (172) – (173) let us establish the connection between the components $X_i^k(s,t)$, $Y_k^i(s,t)$ of the defining equations (162) – (164) for system (151), (152) and the components $Z_i(s,t,\mu)$ of solutions of the defining equations for system (172), (173), which can be constructed according to the rules describing in [13, 16]. The defining equations for system (172), (173) have the form

$$Z_{i+1}(s,t,\mu) = \mathcal{A}(t,\Delta_+,\mu)Z_i(s,t,\mu) + p^i(\mathcal{A}(t,\Delta_+,\mu)Z_i(s,t,\mu)), \tag{176}$$

$$W_i(s,t,\mu) = D(t)Z_i(s,t,\mu) + p^i(D(t)Z_i(s,t,\mu)),\ i = 0,1,2,\dots, \tag{177}$$

$t \in T$, $\mu \in (0,\mu^0]$, $Z_i \in R^{(n_1+n_2)\times(n_1+n_2)}$, $W_i \in R^{n_3\times(n_1+n_2)}$. Solutions $\{Z_i(s,t,\mu), W_i(s,t,\mu)\}$ $(i = 0,1,2\dots)$ of equations (176), (177) can be calculated with the initial states

$$Z_0(0,t,\mu) = E_{n_1+n_2}, \quad Z_i(s,t,\mu) = 0_{n_1+n_2},$$

$$s \neq jh,\ j = \overline{0,l}\ \vee\ i < 0\ \vee\ s = jh. \tag{178}$$

Lemma 9.2. *Components $Z_i(jh,t,\mu), \mu \in (0,\mu^0]$, of solutions of the defining equations (176) – (178) for each i,j $(i = 0,1,2,\dots,\ j = \overline{0,l})$ are* *connected with the components* $X_i^k(jh,t)$, $Y_i^k(jh,t)$ *of the defining equations (162)- (164) by the relation*

$$Z_i(jh,t,\mu) = \begin{bmatrix} \sum\limits_{m=0}^{i+j} \mu^{-m} X_i^m(jh,t) \\ \sum\limits_{m=0}^{i+j} \mu^{-m} Y_i^m(jh,t) \end{bmatrix}.$$

Proof of this and next proposition can be done by induction analogously to the proof of Lemma 1.2 from [13].

Lemma 9.3. *For each* i, j $(i = 0,1,2,\dots,\ j = \overline{0,l})$ *components* $W_i(jh,t,\mu)$ *of the defining equations (30), (31) and components* $W_i^k(jh,t)$ *of the solution* $\{X_i^k(s,t), Y_i^k(s,t), W_i^k(s,t)\}$ *of the defining equations (162)- (164) are connected by the relation*

$$W_i(jh,t,\mu) = \sum_{m=0}^{i+j} \mu^{-m} W_i^m(jh,t),\ \mu \in (0,\mu^0],\ j = \overline{0,l}.$$

To formulate conditions of relative observability for LNOSPDSNT (151), (152) we define now $(n_3 \times n_1)$- $(n_3 \times n_2)$-matrix functions $\bar{W}_{i1}^m(s,t), \bar{W}_{i2}^m(s,t)$, connected with the components $W_{ij}^m(s,t)$, $(j = 1{,}2)$ of the defining equations (162)–(164) by the relation

$$\bar{W}_{i1}^m(s,t) = W_{i1}^m(s,t+s) - W_{i1}^m(s-h,t+s)A_{11}(t+h) -$$
$$-W_{i2}^{m-1}(s-h,t+s)A_{12}(t+h),$$
$$\bar{W}_{i2}^m(s,t) = W_{i2}^m(s,t+s) - W_{i1}^m(s-h,t+s)C_{21}(t+h) -$$
$$-W_{i2}^{m-1}(s-h,t+s)C_{22}(t+h).$$

Now we create the $n_3(n_1+n_2)(l+1) \times (n_1+n_2)$-matrix

$$L_1(t,\mu) = \begin{bmatrix} \sum\limits_{m=0}^{i+j} \mu^{i+j-m} \bar{W}_i^m(jh,t), \\ i = \overline{0, n_1+n_2-1}, \\ j = \overline{0,l} \end{bmatrix}. \tag{179}$$

Theorem 9.4. *Suppose that matrices* $A_{ij}(t), C_{ij}(t), (i = 1,2, j = 0,1,2)$ *are* (n_1+n_2-2)*-times and* $D_j(t)$, $(j = 1,2)$ - (n_1+n_2-1)*-times continuously differentiable on* T. *Then for* $\{x,y\}$*-relative observability of LNOSPDSNT (151), (152) sufficiently that rank* $L(t_0,\mu) = n_1+n_2$, $\mu \in (0,\mu^0]$. *If* $A_{ij}(t)$, $C_{ij}(t)$, $(i = 1,2, j = 0,1,2)$, $D_i(t) (i = 1,2)$, *are matrix* *analytical functions then for $\{x,y\}$-relative observability of system (151), (152) it is necessary the existence of moment $t^* \in T$, such that*

$$rank L_1(t^*,\mu) = n_1 + n_2, \quad \mu \in (0,\mu^0]$$

Corollary 9.5. *Let us assume that elements of system (151), (152) are analytical functions. Then if LNOSPDSNT (151), (152), $\mu \in (0,\mu^0]$, is $\{x,y\}$-relatively observable on T then for some $t^* \in T$*

$$rank \left[\begin{array}{c} \sum\limits_{m=0}^{i+j} \mu^{-m}\bar{W}_{i1}^{m}(jh,t^*) \\ \overline{i = 0, n_1+n_2-1}, \\ j = \overline{0,l} \end{array} \right] = n_1$$

$$rank \left[\begin{array}{c} \sum\limits_{m=0}^{i+j} \mu^{-m}\bar{W}_{i2}^{m}(jh,t^*) \\ \overline{i = 0, n_1+n_2-1}, \\ j = \overline{0,l} \end{array} \right] = n_2.$$

9.5. Example

Consider LNOSPDSNT with the observable output

$$\dot{x}_1(t) = x_1(t) + t\dot{y}(t-1)$$

$$\dot{x}_2(t) = x_2(t) + ty(t-1)$$

$$\mu\dot{y}(t) = x_1(t) + y(t) \tag{180}$$

$$w(t) = x_1(t) + x_2(t), \quad t \in [1,2]. \tag{181}$$

This system has the following parameters $n_1 = 2,\ n_2 = 1,\ n_3 = 1,\ h = 1,\ l = 1, x(t) = col\{x_1(t), x_2(t)\}$,

$$A_{10}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_{11}(t) = A_{12}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_{10}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ C_{11}(t) = \begin{bmatrix} 0 \\ t \end{bmatrix},\ C_{12}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix},$$

$$A_{20}(t) = [1,0],\ A_{21}(t) = A_{22}(t) = [0,0],\ C_{20}(t) = 1,$$

$$C_{21}(t) = C_{22}(t) = D_2(t) = 0, \quad D_1(t) = [1,1]. \tag{182}$$

Investigate $\{x, y\}$-relative observability of (34), (35), using (16)–(18)

$$X_{i+1}^k(s,t) = A_{10}(t)X_i^k(s,t) + C_{11}(t)Y_i^k(s-1,t-1) +$$
$$+C_{12}(t)Y_{i+1}^k(s-1,t-1) + p^i(C_{11}(t)Y_i^k(s-1,t-1)) +$$
$$+p^i(C_{12}(t)Y_{i+1}^k(s-1,t-1)),$$
$$Y_{i+1}^{k+1}(s,t) = A_{20}(t)X_i^k(s,t) + C_{20}(t)Y_i^k(s,t), \tag{183}$$
$$W_i^k(s,t) = D_1(t)X_i^k(s,t), \tag{184}$$

with the initial states

$$X_0^0(0,t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad X_i^k(s,t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$i = k = 0,\ s = j = 1 \ \vee\ s \neq j,\ j = 0,1$$
$$\vee s = j,\ k \geq i + j \ \vee\ k < 0 \ \vee\ i < 0,\ j < 0;$$
$$Y_0^0(0,t) = [0\ 0\ 1];\ Y_i^k(s,t) = [0\ 0\ 0],\ i = k = 0,$$
$$s = j = 1 \ \vee\ s \neq j,\ j = 0,1, \quad \vee\ s = j,$$
$$k \geq i + j + 1 \ \vee\ k \leq 0 \ \vee\ i < 0,\ j < 0. \tag{185}$$

Note that in virtue to (160) and (182) for parameters in (183), (184) members

$$p^i(A_{10}(t)X_i^k(s,t)), p^i(A_{20}(t)X_i^k(s,t)), p^i(C_{20}(t)Y_i^k(s,t)), p^i(D_1(t)X_i^k(s,t))$$

are absent. Calculate components $X_i^k(s,t)$, $Y_i^k(s,t)$, $W_i^k(s,t)$ of solutions of the defining equations (183) – (185):

$$X_0^0(0,t) = X_1^0(0,t) = X_2^0(0,t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$Y_0^0(0,t) = [0\ 0\ 1],\ W_0^0(0,t) = W_1^0(0,t) = W_2^0(0,t) =$$
$$= [1\ 1\ 0];\ Y_1^0(0,t) = W_1^1(0,t) = Y_2^0(0,t) = W_2^1(0,t) =$$
$$= W_2^2(0,t) = Y_1^0(1,t) = Y_1^1(1,t) = Y_2^0(1,t) = [0\ 0\ 0],$$
$$X_1^1(0,t) = X_2^1(0,t) = X_2^2(0,t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$Y_1^1(0,t) = Y_2^2(0,t) = [1\ 0\ 1], Y_2^1(0,t) = Y_2^1(1,t) =$$

$$= [1\ 0\ 0],\ X_1^0(1,t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & t \end{bmatrix}, \quad W_1^0(1,t) =$$

$$= [0\ 0\ t]; X_1^1(1,t) = \begin{bmatrix} t & 0 & t \\ 0 & 0 & 0 \end{bmatrix},\ W_1^1(1,t) =$$

$$= Y_2^2(1,t) = W_2^2(1,t) = [t\ 0\ t],\ X_2^0(1,t) =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & t+1 \end{bmatrix}, W_2^0(1,t) = [0\ 0\ t+1];$$

$$X_2^1(1,t) = \begin{bmatrix} 2t+1 & 0 & t+1 \\ t & 0 & t \end{bmatrix},\ W_2^1(1,t) =$$

$$= [3t+1\ 0\ 2t+1];\ X_2^2(1,t) = \begin{bmatrix} t & 0 & t \\ 0 & 1 & 0 \end{bmatrix}. \qquad (186)$$

Using (186), (182) let us calculate the matrix $L_1(t,\mu)$ from (179) and according to Theorem 9.4 we obtain for $t_0 = 1$

$$rank L_1(1,\mu) =$$

$$rank \begin{bmatrix} 1 & 1 & 0 \\ \mu & \mu & 0 \\ \mu^2 & \mu^2 & 0 \\ 0 & 0 & 2\mu \\ 2\mu & 0 & \mu^2+\mu \\ 7\mu^2+2\mu & 0 & \mu^3+5\mu^2+2\mu \end{bmatrix} = 3$$

$$\forall \mu \in (0,\mu^0].$$

Consequently LNOSPDSNT (180), (181) is $\{x,y\}$-relatively observable. Note that system is also x-relatively observable because for the matrix $H_x = [E_2, 0_{2\times 1}]$ the condition

$$rank L_1(t_0,\mu) = rank \begin{bmatrix} H_x \\ L(t_0,\mu) \end{bmatrix}, \quad \mu \in (0,\mu^0]$$

is fulfilled.

9.6. Conclusion

The unified method of investigating controllability and observability problems for various classes of dynamical systems is suggested. It combines the state space method and the method of defining equations [1, 6, 7] and allows to obtain conditions of controllability and observability in an explicit form. It may be safely suggested that this approach can be useful in studying another more complicated mathematical objects (both time-invariant and time-varying) but another problems (stabilizability, decoupling problem etc.).

9.7. Some Open Problems

There are some open problems in the frames of the theme "The Qualitative Theory of Control Processes":

1. To find some controllability and observability conditions for linear time-variable systems with *continuous matrices* $A(t), B(t), C(t)$.

2. To study controllability and observability problems for another class of linear systems - discrete event systems (statement of the problems, conditions of controllability and observability in (max, +)-algebra).

3. To study controllability and observability problems for linear time-delay systems, for the systems with the deviating argument of neutral type using the approach of approximation for such kinds of systems by the systems of ordinary differential equations.

4. To construct an inversion systems for:

a). linear time-invariant and time-varying systems with the measurement output;

b). linear singularly perturbed dynamic systems of the general form (without delay, with delay, with the deviating argument of neutral type) with linear output.

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