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Baire and σ-Borel Characterizations of Weakly Compact Sets in $M(T)$

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Abstract

Let T ba a locally compact Hausdorff space and let $M(T)$ be the Banach space of all bounded complex Radon measures on T. In this note we characterize weakly compact subsets A of $M(T)$ in terms of the Baire and σ-Borel restrictions of the members of A. These characterizations permit us to give a generalization of a theorem of Dieudonné which is stronger and more natural than that given by Grothendieck.

1 Introduction

For a locally compact Hausdorff space T, let $C_o(T)$ be the Banach space of all continuous complex functions vanishing at infinity in T, endowed with the supremum norm. If $\mathcal{B}(T)$ is the σ-algebra of Borel sets in T,the dual $M(T)$ of $C_o(T)$ is the Banach space of all bounded complex Radon measures μ on T, with $||\mu|| = |\mu|(T)$, where $|\mu|$ denotes the variation of μ in $\mathcal{B}(T)$.Let X be a quasicomplete locally convex Hausdorff space (briefly,a quasicomplete lcHs).

In the present note we characterize weakly compact sets A in $M(T)$ in terms of the Baire and σ-Borel restrictions of the members of A. As a consequence of the Baire characterizations, we obtain a generalization of Proposition 8 of Dieudonné [3], which is stronger and more natural than that of Grothendieck given on p. 150 of [8].

These characterizations are powerful enough to replace the use of Theorem 3 and Proposition 11 of [8] in the study of weakly compact operators in our succeeding work [13]. In fact, these results play a key role in [13] to provide a unified approach to the study of weakly compact operators $u : C_o(T) \to X$ and of regular Borel extension of X-valued σ-additive Baire measures on T. In this context we would like to point out that the study of weakly compact operators was carried out by Grothendieck [8] for complete lcHs-valued operators on

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$C_o(T)$, and by Bartle-Dunford-Schwartz [1] for Banach space valued operators on $C(\Omega)$, Ω compact, while the regular Borel extension problem for quasicomplete lcHs-valued Baire measures was studied by Dinculeanu and Kluvánek in [5,10] by vector measure methods. As far as we know, such a unification study has not been presented earlier in the literature.

2 Preliminaries

In this section we fix notation and terminology and also give some definitions and results which will be needed in the sequel.

Let T be a locally compact Hausdorff space and let $C_o(T)$ be the Banach space of all complex continuous functions vanishing at infinity in T, endowed with the supremum norm . Let $\mathcal{B}(T)$ be the σ-algebra of Borel sets in T, which is the σ-algebra generated by the class of all open sets in T. Then the dual of $C_o(T)$ is the Banach space $M(T)$ of all bounded complex Radon measures μ on T, with $||\mu|| = |\mu|(T)$,where $|\mu|$ denotes the variation of μ in $\mathcal{B}(T)$. $\mathcal{C}(T)$ (resp. $\mathcal{C}_o(T)$) is the class of all compact subsets (resp. compact G_δ subsets) of T. $\mathcal{B}_c(T)$ (resp. $\mathcal{B}_o(T)$) is the σ-ring generated by $\mathcal{C}(T)$ (resp. by $\mathcal{C}_o(T)$)) and $\mathcal{B}_c(T)$, (resp. $\mathcal{B}_o(T)$) is the σ-ring of the σ-Borel (resp. the Baire) sets in T.

We need the following lemma before we give some definitions.

Lemma 1. *For $\mu \in M(T)$, let $|\mu|(\cdot) = var(\mu, (\cdot))$ in $\mathcal{B}(T)$. Then*

$$|\mu||_{\mathcal{B}_o(T)}(\cdot) = var(\mu|_{\mathcal{B}_o(T)}, (\cdot)) \text{ and } |\mu||_{\mathcal{B}_c(T)}(\cdot) = var(\mu|_{\mathcal{B}_c(T)}, (\cdot)).$$

Proof. Let $\mathcal{D}(\mathcal{C}_o(T))$ be the δ-ring generated by $\mathcal{C}_o(T)$. Then by Lemma 3.2 of [12], $|\mu||_{\mathcal{D}(\mathcal{C}_o(T))}(\cdot) = var(\mu|_{\mathcal{D}(\mathcal{C}_o(T))}, (\cdot))$. Given $E \in \mathcal{B}_o(T)$, there exists a disjoint sequence $(E_n)_{n=1}^{\infty} \subset \mathcal{D}(\mathcal{C}_o(T))$ such that $E = \bigcup_{n=1}^{\infty} E_n$. Then

$$\begin{aligned} |\mu|(E) &= \sum_{n=1}^{\infty} |\mu|(E_n) = \sum_{n=1}^{\infty} var(\mu|_{\mathcal{D}(\mathcal{C}_o(T))}, E_n) \\ &= \sum_{n=1}^{\infty} var(\mu|_{\mathcal{B}_o(T)}, E_n) = var(\mu|_{\mathcal{B}_o(T)}, E). \end{aligned}$$

Now, let $E \in \mathcal{B}_c(T)$. Since it is easy to check that each $F \subset E$ with $F \in \mathcal{B}(T)$ is σ-Borel it follows that $|\mu||_{\mathcal{B}_c(T)}(E) = var(\mu|_{\mathcal{B}_c(T)}, E)$.

Notation 1. For $\mu \in M(T)$, let $|\mu|(\cdot) = var(\mu, (\cdot))$ en $\mathcal{B}(T)$.

In the light of the above lemma, the variations used in the following definition are unambiguously defined. The first part is an adaptation of Definition

3.2 of [11].

Definition 1. Let $\mathcal{S}$ be a σ-ring of sets in T such that $\mathcal{C}(T) \subset \mathcal{S}$ or $\mathcal{C}_o(T) \subset \mathcal{S}$. A complex measure μ on $\mathcal{S}$ is said to be *$\mathcal{S}$-regular* if, given $E \in \mathcal{S}$ and $\varepsilon > 0$, there exists a compact set $K \in \mathcal{S}$ and an open set $U \in \mathcal{S}$ with $K \subset E \subset U$ such that $|\mu(B)| < \varepsilon$ for every $B \in \mathcal{S}$ with $B \subset U \backslash K$. When $\mathcal{S} = \mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$, $\mathcal{B}_o(T)$), we use the terminology *Borel (resp. σ-Borel, Baire) regularity* in place of $\mathcal{S}$-regularity. Let A be a subset of $M(T)$. We say that A is *uniformly Baire inner regular (resp. Baire regular) in a set $E \in \mathcal{B}_o(T)$* if, given $\varepsilon > 0$, there exists a compact $K \in \mathcal{B}_o(T)$ with $K \subset E$ (resp. and an open Baire set O in T with $K \subset E \subset O$) such that $\sup_{\mu \in A} |\mu|(E \backslash K) < \varepsilon$ (resp. such that $\sup_{\mu \in A} |\mu|(O \backslash K) < \varepsilon$). If A is uniformly Baire inner regular (resp. Baire regular) in each Baire set, then A is said to be *uniformly Baire inner regular (resp. Baire regular)*. Similarly, the *uniform Borel (resp. σ-Borel) regularity and inner regularity* of A and those of A in a Borel (resp. σ-Borel) set E are defined.

In virtue of Theorem 51.D of [9], we note that a compact $K \in \mathcal{B}_o(T)$ is necessarily a G_δ. It is well known that every complex Baire measure μ_o is Baire regular and that it has a unique extension μ to $\mathcal{B}(T)$(resp. μ_c to $\mathcal{B}_c(T)$) such that μ is a Borel (resp. μ_c is a σ-Borel) regular complex measure. Moreover, $\mu|_{\mathcal{B}_c(T)} = \mu_c$. (See, for example, Theorem 2.4 of [12].)

Definition 2. A faimly $\mathcal{F}$ of complex measures defined on a σ-ring Σ of sets is said to be *uniformly σ-additive*, if for each decreasing sequence (E_n) of members of Σ with $E_n \searrow \emptyset$, $\lim_n \mu(E_n) = 0$ uniformly in $\mu \in \mathcal{F}$.

Notation 2. Given a σ-ring Σ of sets, $ca(\Sigma)$ denotes the Banach space of all complex measures μ on Σ with $||\mu|| = \sup_{E \in \Sigma} var(\mu, E)$.

The following result is well known when Σ is a σ-algebra(see, for example, Theorem IV.9.1 of [6]).

Proposition 1. *Let Σ be a σ-ring of subsets of a non empty set Ω. A subset A of $ca(\Sigma)$ is relatively weakly compact if and only if A is bounded and uniformly σ-additive.*

Proof. By the Eberlein- Šmulian theorem and by the fact that for each sequence $(\mu_n) \subset ca(\Sigma)$ there exists $E \in \Sigma$ such that $var(\mu_n, F) = 0$ for each $F \in \Sigma$ with $F \cap E = \emptyset$ and for each n, we can replace the space $ca(S, \Sigma, \lambda)$ in the proof of Theorem IV.9.1 of [6] by the space $ca(\Omega \cap E, \Sigma \cap E, \lambda)$ of all λ-continuous set functions in $ca(\Omega \cap E, \Sigma \cap E)$. Since the latter is a σ-algbera, the rest of the argument in the proof of Theorem IV.9.1 of [6] holds here to show that the conditions are necessary and sufficient.

3 Main Results

In the present section we obtain characteizations of bounded relatively weakly compact subsets of $M(T)$ in terms of the Baire and σ-Borel restrictions of the members of the set in question. These charcterizations are similar to those obtained by Grothendieck in Theorem 2 of [8] and those of Lemma VI.2.13 of Diestel and Uhl [2]. As mentioned in the introduction, these results are powerful enough to replace the use of Theorem 3 and Proposition 11 of [8] in our succeeding work [13] where we characterize quasicomplete lcHs-valued weakly compact operators on $C_o(T)$. Moreover, the isolated results of Dinculeanu and Kluvánek [5,10] on vector valued σ-additive Baire and Borel measures are deduced in [13] as corllaries of some of these characterizations. Finally, Theorem 1 below combined with the study of Grothendieck on p.150 of [8] provides a generalization of Proposition 8 of Dieudonné [3], which is stronger and more natural than that of Grothendieck [8] .See Corollary 1 below.

Theorem 1. *Let A be a bounded set in $M(T)$. Then the following statements are equivalent:*

(i) A is relatively weakly compact.

(ii) For each disjoint sequence (O_i) of open Baire sets in T, $\lim_i \mu(O_i) = 0$ uniformly in $\mu \in A$.

(iii) For each disjoint sequence (O_i) of open Baire sets in T, $\lim_i |\mu|(O_i) = 0$ uniformly in $\mu \in A$.

(iv) a) A is uniformly Baire inner regular in each open Baire set O in T.
b) For each $\varepsilon > 0$, there exists a $K \in C_0(T)$ such that

$$\sup_{\mu \in A} |\mu|(T \backslash K) < \varepsilon.$$

(v) A is uniformly Baire inner regular.

(vi) $A|_{\mathcal{B}_o(T)}$ is uniformly σ-additive on $\mathcal{B}_o(T)$.

(vii) A is uniformly Baire regular.

Proof. By Theorem 2 of Grothendieck [8] (which is the same as Theorem 4.22.1 of Edwards [7]), (i) implies (ii).

(ii) $\Rightarrow$ (iii). Since each $\mu|_{\mathcal{B}_o(T)}$ is Baire regular for $\mu \in A$, the argument in the proof of (a) $\Rightarrow$ (b) of Lemma VI.2.13 of Diestel and Uhl [2] can suitably be modified to show that (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv). Let O be an open Baire set in T or let $O = T$. Let $\varepsilon > 0$. If there exists no compact G_δ $K \subset O$ such that $\sup_{\mu \in A} |\mu|(O \backslash K) \leq \varepsilon$, then there is a $\mu_1 \in A$ such that $|\mu_1|(O) > \varepsilon$, for otherwise $K = \emptyset$ will provide a contradiction. If $O \in \mathcal{B}_o(T)$,then by the Baire regularity of $|\mu_1||_{\mathcal{B}_o(T)}$ there exists a compact G_δ $K_1 \subset O$ such that $|\mu_1|(K_1) > \varepsilon$.If $O = T$, then by the Borel regularity of $|\mu_1|$ there exists a compact K such that $|\mu_1|(K) > \varepsilon$. Then by Theorem 50.D of Halmos [9] there exists a compact G_δ K_1 such that $K \subset K_1$ and hence $|\mu_1|(K_1) > \varepsilon$. Since K_1 is a subset of O, again by Theorem 50.D of [9] there exists an open Baire set O_1 and a compact G_δ F_1 such that

$$O \supset F_1 \supset O_1 \supset K_1.$$

Moreover, $|\mu_1|(O_1) \geq |\mu_1|(K_1) > \varepsilon$. Since F_1 is a compact G_δ subset of O, by our assumption there exists $\mu_2 \in A$ such that $|\mu_2|(O \backslash F_1) > \varepsilon$. If $O \neq T$,then using the Baire regularity of $|\mu_2|$ in $O \backslash F_1$ and if $O = T$, then using the Borel regularity of $|\mu_2|$ in $O \backslash F_1$ and then applying Theorem 50.D of [9], we can choose a compact G_δ $C_1 \subset O \backslash F_1$ such that $|\mu_2|(C_1) > \varepsilon$. Let $K_2 = F_1 \cup C_1$. Then K_2 is a compact G_δ, $O \supset K_2 \supset F_1$ and $|\mu_2|(K_2 \backslash F_1) = |\mu_2|(C_1) > \varepsilon$. Again by Theorem 50.D of [9] there exists an open Baire set O_2 and a compact G_δ F_2 such that

$$O \supset F_2 \supset O_2 \supset K_2 \supset F_1 \supset O_1 \supset K_1.$$

Accordingly, $|\mu_2|(O_2 \backslash F_1) \geq |\mu_2|(K_2 \backslash F_1) > \varepsilon$. Next by our assumption there exists $\mu_3 \in A$ such that $|\mu_3|(O \backslash F_2) > \varepsilon$. If $O \neq T$,then using the Baire regularity of $|\mu_3|$ in $O \backslash F_2$ and if $O = T$, then using the Borel regularity of $|\mu_3|$ in $O \backslash F_2$ and then applying Theorem 50.D of [9], we can choose a compact G_δ $C_2 \subset O \backslash F_2$ such that $|\mu_3|(C_2) > \varepsilon$. Let $K_3 = F_2 \cup C_2$. Then K_3 is a compact G_δ, $O \supset K_3 \supset F_2$ and $|\mu_3|(K_3 \backslash F_2) = |\mu_3|(C_2) > \varepsilon$. Again by Theorem 50.D of [9] there exists an open Baire set O_3 and a compact G_δ F_3 such that

$$O \supset F_3 \supset O_3 \supset K_3 \supset F_2 \supset O_2$$

and hence, $|\mu_3|(O_3 \backslash F_2) \geq |\mu_3|(K_3 \backslash F_2) > \varepsilon$.

Proceeding as in the proof of (b) $\Rightarrow$ (c) of Lemma VI.2.13 of [2], applying Theorem 50.D of [9] in each step and using the Baire-regularity of each $|\mu||_{\mathcal{B}_o(T)}$ for $\mu \in A$ or using the Borel regularuty of $\mu \in A$ and then applying Theorem 50.D of [9], as the case may be, we can produce an increasing sequence (O_n) of open Baire sets in T, another two increasing sequences (K_n) and (F_n) of compact G_δs in T and a sequence (μ_n) in A such that

$$O \supset .... \supset F_{n+1} \supset O_{n+1} \supset K_{n+1} \supset F_n \supset O_n \supset ...... \supset K_2 \supset F_1 \supset O_1 \supset K_1$$

and

$$|\mu_{n+1}|(O_{n+1} \backslash F_n) > \varepsilon$$

for all $n \geq 1$. Let $G_{n+1} = O_{n+1} \backslash F_n, \quad n \geq 1$. Then (G_{n+1}) is a disjoint sequence of open Baire sets in T and satisfies $|\mu_{n+1}|(G_{n+1}) > \varepsilon$ for $n \geq 1$. This contardicts (iii) and hence (iv) holds.
(iv) $\Rightarrow$ (v). Let $\varepsilon > 0$. By (iv)(b) there exists a compact G_δ Ω in T such that

$$\sup_{\mu \in A} |\mu|(T \backslash \Omega) < \frac{\varepsilon}{2}. \quad (3.1)$$

We shall now modify the proof of (c) $\Rightarrow$ (d) of Lemma VI.2.13 of [2] to show that (v) holds. Let $\mathcal{C}_o(\Omega) = \{K \subset \Omega : K$ compact G_δ in Ω with respect to the relative topology of $\Omega\}$. It is easy to check that

$$\mathcal{C}_o(\Omega) = \mathcal{C}_o(T) \cap \Omega = \{K \subset \Omega : K \in \mathcal{C}_o(T)\}. \quad (3.2)$$

Let

$\mathcal{S} \;=\; \{E \in \mathcal{B}_o(\Omega)$: for each $\varepsilon' > 0$,there exists $K \in \mathcal{C}_o(\Omega)$ such that $E \cap K$ is compact and $\sup_{\mu \in A} |\mu|(\Omega \backslash K) \leq \varepsilon'\}$.

If $E \in \mathcal{B}_o(\Omega)$ and $K \in \mathcal{C}_o(\Omega)$ are such that $E \cap K$ is compact, then by Theorem 51.D of Halmos [9], $E \cap K \in \mathcal{C}_o(\Omega)$. Clearly, $\mathcal{C}_o(\Omega) \subset \mathcal{S}$, since for $C \in \mathcal{C}_o(\Omega)$ we have $C \cap \Omega = C$ is compact and $|\mu|(\Omega \backslash \Omega) = 0$ for $\mu \in A$.

Claim 1. For each open Baire set O in T, $O \cap \Omega \in \mathcal{S}$.

In fact, $O \cap \Omega \in \mathcal{B}_o(T) \cap \Omega = \mathcal{S}(\mathcal{C}_o(T)) \cap \Omega = \mathcal{S}(\mathcal{C}_o(T) \cap \Omega) = \mathcal{S}(\mathcal{C}_o(\Omega)) = \mathcal{B}_o(\Omega)$ by (3.2) and by Theorem 5.E of Halmos [9], where $\mathcal{S}(\mathcal{E})$ denotes the σ-ring generated by the class $\mathcal{E}$. Given $\varepsilon' > 0$, by (iv)(a) there exists $K \in \mathcal{C}_o(T)$ with $K \subset O$ such that

$$\sup_{\mu \in A} |\mu|(O \backslash K) \leq \varepsilon'. \quad (3.3)$$

Let $K_o = K \cap \Omega$. Then $K_o \in \mathcal{C}_o(\Omega)$ by (3.2) and moreover, $O \cap \Omega \cap K_o = K_o$ is compact. Further, as $(O \cap \Omega) \backslash K_o \subset O \backslash K$, by (3.3) we have

$$\sup_{\mu \in A} |\mu|((O \cap \Omega) \backslash K_o) \leq \varepsilon'. \quad (3.4)$$

Let $K_1 = K_o \cup (\Omega \backslash O)$. Then by Theorem 51.D of Halmos [9] and by (3.2), $\Omega \backslash O \in \mathcal{C}_o(\Omega)$ and hence $K_1 \in \mathcal{C}_o(\Omega)$. Moreover, $O \cap \Omega \cap K_1 = K_o$ is compact and by (3.4)

$$\sup_{\mu \in A} |\mu|(\Omega \backslash K_1) = \sup_{\mu \in A} |\mu|((O \cap \Omega) \backslash K_o) \leq \varepsilon'.$$

Thus $O \cap \Omega \in \mathcal{S}$.

Claim 2. For $K \in \mathcal{C}_o(\Omega)$, $\Omega \backslash K \in \mathcal{S}$.

In fact, by (3.2) K is of the form $K = \bigcap_1^\infty V_n$, where the V_n are open Baire sets in T (see Proposition 14, §14 ,Chapter III of [4]). Then $\Omega\backslash K = \bigcup_1^\infty(\Omega\backslash V_n)$. Now, by Theorem 50.D of Halmos [9] there exists an open Baire set W_n in T such that $\Omega\backslash V_n \subset W_n$ for $n \geq 1$. Let $W = \bigcup_1^\infty W_n$. Then W is an open Baire set in T and $\Omega\backslash K = (\Omega\backslash K) \cap W = \Omega \cap (W\backslash K)$. Since $W\backslash K$ is an open Baire set in T, by Claim 1 we conclude that $\Omega\backslash K \in \mathcal{S}$.

To show that $\mathcal{S}$ is closed under countable intersections, let (E_n) be a sequence in $\mathcal{S}$ and let $\varepsilon' > 0$. Then, proceeding as on p.158 of [2], there exists a sequence (K_n) in $\mathcal{C}_o(\Omega)$ such that $E_n \cap K_n$ is compact and $\sup_{\mu\in A} |\mu|(\Omega\backslash K_n) \leq \frac{\varepsilon'}{2^n}$ for each $n \geq 1$. Then the set

$$(\bigcap_{n=1}^\infty E_n) \cap (\bigcap_{n=1}^\infty K_n) = \bigcap_{n=1}^\infty (E_n \cap K_n)$$

is compact and

$$\sup_{\mu\in A} |\mu|(\Omega\backslash\bigcap_1^\infty K_n) \leq \sum_{n=1}^\infty \sup_{\mu\in A} |\mu|(\Omega\backslash K_n) \leq \varepsilon'.$$

Thus $\bigcap_1^\infty E_n \in \mathcal{S}$.

To verify that $\mathcal{S}$ is also closed under complements in Ω, let $E \in \mathcal{S}$ and let $\varepsilon' > 0$. Then there exists $K_1 \in \mathcal{C}_o(\Omega)$ such that $E \cap K_1$ is compact and $\sup_{\mu\in A} |\mu|(\Omega\backslash K_1) \leq \frac{\varepsilon'}{2}$. Now, by Claim 2 and by Theorem 51.D of [9], we have $\Omega\backslash(E\cap K_1) \in \mathcal{S}$. Therefore there exists $K_2 \in \mathcal{C}_o(\Omega)$ such that $(\Omega\backslash(E\cap K_1))\cap K_2$ is compact and $\sup_{\mu\in A} |\mu|(\Omega\backslash K_2) \leq \frac{\varepsilon'}{2}$. Then $(K_1 \cap K_2) \cap (\Omega\backslash E) = K_1 \cap K_2 \cap (\Omega\backslash(E \cap K_1))$ is compact and

$$\sup_{\mu\in A} |\mu|(\Omega\backslash(K_1 \cap K_2)) \leq \sup_{\mu\in A} |\mu|(\Omega\backslash K_1) + \sup_{\mu\in A} |\mu|(\Omega\backslash K_2) < \varepsilon'.$$

Thus $\Omega\backslash E \in \mathcal{S}$. Consequently, $\mathcal{S}$ is a σ-algebra in Ω.

Since $\mathcal{C}_o(\Omega) \subset \mathcal{S} \subset \mathcal{B}_o(\Omega)$, it follows that $\mathcal{S} = \mathcal{B}_o(\Omega)$. Thus, for each $E \in \mathcal{B}_o(\Omega)$ and $\varepsilon' > 0$, there exists $K \in \mathcal{C}_o(\Omega)$ such that $E \cap K$ is compact and $\sup_{\mu\in A} |\mu|(\Omega\backslash K) \leq \varepsilon'$. Then

$$\sup_{\mu\in A} |\mu|(E\backslash E \cap K) \leq \sup_{\mu\in A} |\mu|(\Omega\backslash K) \leq \varepsilon'. \qquad (3.5)$$

Now let $E \in \mathcal{B}_o(T)$ and $\varepsilon' = \frac{\varepsilon}{2}$. Then $E \cap \Omega \in \mathcal{B}_o(\Omega)$ by (3.2) and by Theorem 5.E of [9]. Consequently, using $E \cap \Omega \in \mathcal{B}_o(\Omega)$ in place of E above, as in (3.5) there exists $K \in \mathcal{C}_o(\Omega)$ such that $(E \cap \Omega) \cap K = K_o$ (say) is compact and

$$\sup_{\mu\in A} |\mu|((E \cap \Omega)\backslash K_o) \leq \frac{\varepsilon}{2}. \qquad (3.6)$$

Thus $K_o \in \mathcal{C}_o(T)$, $K_o \subset E$ and

$$\sup_{\mu \in A} |\mu|(E \backslash K_o) \leq \sup_{\mu \in A} |\mu|((E \cap \Omega) \backslash K_o) + \sup_{\mu \in A} |\mu|(T \backslash \Omega) < \varepsilon$$

by (3.6) and (3.1).Thus (v) holds.

Replacing in the proof of (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (a) of Lemma VI.2.13 of [2] compact sets, Borel sets and open sets in Ω respectively by compact G_δ sets in T, Baire sets in T and open Baires sets in T , one can easily show that (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (ii).

Finally, to show that (vi) $\Rightarrow$ (i), let $\Phi(\mu) = \mu|_{\mathcal{B}_o(T)}$ for $\mu \in M(T)$ and let $\mathcal{M}_o(T) = \{\mu : \mathcal{B}_o(T) \to \mathbb{C}, \mu\ \sigma\text{-additive}\}$ with $||\mu||_o = \sup_{E \in \mathcal{B}_o(T)} |\mu|(E)$ for $\mu \in \mathcal{M}_o(T)$. Then by Lemma 1 and by Theorem 5.3 of [12], Φ is an isometric isomorphism of $M(T)$ onto $\mathcal{M}_o(T)$. By Proposition 1, and by (vi), $\Phi(A)$ is relatively weakly compact in $\mathcal{M}_o(T)$. Consequently, A is relatively weakly compact in $M(T)$.

This completes the proof of the theorem.

Corollary 1 (Generalization of Proposition 8 of Dieudonné [3]). *A bounded sequence (μ_i) in $M(T)$ is weakly convergent if and only if, for each open Baire set O in T, $\lim_i \mu_i(O)$ exists in $\mathbb{C}$.*

Proof. We only have to show that the condition is sufficient. By regularity, each complex Baire measure μ in T is determined by its restriction on the lattice of all open Baire sets. Moreover, each complex Baire measure has a unique regular Borel extension. These facts and the Eberlein-Šmulian theorem ensure that it suffices to show that (μ_i) is relatively weakly compact in $M(T)$. Arguing as in the proof of Corollary 4.22.2 of Edwards [7], one can show that $\lim_i \mu_n(O_i) = 0$ uniformly in $n \in I\!N$ for each disjoint sequence (O_i) of open Baire sets in T. Then by the equivalence of (i) and (ii) of Theorem 1 it follows that (μ_i) is relatively weakly compact in $M(T)$.

Remark 1. When T is compact, the proof of Proposition 9 in [3] holds verbatim to show that the hypothesis that $\lim_i \mu_i(U)$ exists in $\mathbb{C}$ for each open set U in T implies that (μ_i) is bounded. When T is locally compact, one can argue with its one-point compactification as on p.177 of Thomas [14] to show that the above hypothesis also ensures the boundedness of (μ_i) in $M(T)$. Again, when T is compact, using Theorem 50.D of [9] and the Baire regularity of the $\mu_i|_{\mathcal{B}_o(T)}$ we can modify the proof of Proposition 9 in [3] to show that (μ_i) is bounded when $\lim_i \mu_i(O)$ exists in $\mathbb{C}$ for each open Baire set O in T. However, when T is locally compact and not compact,we do not know whether the boundedness condition can be dispensed with in the above corollary. When T is metrizable and compact, $\mathcal{B}(T) = \mathcal{B}_o(T)$ and hence the above corollary reduces to Proposition 8 of Dieudonné [3]. Thus the present generalization is more natural and is further stronger than that of Grothendieck on p.150 of [8].

Theorem 2. *Let A be a bounded set in $M(T)$. Then the following statements are equivalent:*

(i) A is relatively weakly compact.

(ii) For each disjoint sequence (O_i) of σ-Borel open sets (resp. (ii)'open sets) in T,

$$\lim_i \mu(O_i) = 0$$

uniformly in $\mu \in A$.

(iii) For each disjoint sequence (O_i) of σ-Borel open sets (resp. (iii)' open sets) in T,

$$\lim_i |\mu|(O_i) = 0$$

uniformly in $\mu \in A$.

(iv) (a) A is uniformly σ-Borel inner regular in each σ-Borel open set O in T.

(b) For each $\varepsilon > 0$, there exists a compact K in T such that

$$\sup_{\mu \in A} |\mu|(T \backslash K)) < \varepsilon.$$

(resp.(iv)' A is uniformly Borel inner regular in each open set O in T).

(v) A (resp. (v)' A) is uniformly σ-Borel (resp. Borel) inner regular.

(vi) $A|_{\mathcal{B}_c(T)}$ (resp. (vi)' A) is uniformly σ-additive on $\mathcal{B}_c(T)$ (resp. on $\mathcal{B}(T)$).

(vii) A (resp. (vii)' A) is uniformly σ-Borel (resp. Borel) regular.

Proof. Let $\mathcal{M}_c(T) = \{\mu : \mathcal{B}_c(T) \to \mathbb{C},\ \mu\ \sigma\text{-additive and } \sigma\text{-Borel regular }\}$ with $||\mu||_c = \sup_{E \in \mathcal{B}_c(T)} |\mu|(E)$, and let $\Psi : M(T) \to \mathcal{M}_c(T)$ be given by $\Psi(\mu) = \mu|_{\mathcal{B}_c(T)}$. Then by Lemma 1 and by Theorem 5.3 of [12], Ψ is an isometric isomorphism of $M(T)$ onto $\mathcal{M}_c(T)$. This fact and an argument similar to that in the proof of Theorem 1 can be used to show that (i) $\Rightarrow$ (ii) (resp. (ii)') $\Rightarrow$ (iii) (resp. (iii)') $\Rightarrow$ (iv) (resp.(iv)'); (v) (resp. (v)') $\Rightarrow$ (vi) (resp. (vi)') $\Rightarrow$ (vii) (resp. (vii)') $\Rightarrow$ (ii) (resp. (ii)') and (vi) (resp.(vi)') $\Rightarrow$ (i).

Now we shall prove (iv) (resp. (iv)') $\Rightarrow$ (v) (resp. (v)'). Given $\varepsilon > 0$, by (iv)(b) (resp. by (iv)') there exists a compact set Ω in T such that

$$\sup_{\mu \in A} |\mu|(T \backslash \Omega) < \frac{\varepsilon}{2}. \qquad (3.7)$$

Let

$$\Sigma = \{E \in \mathcal{B}(\Omega) : \text{for each } \varepsilon' > 0, \text{ there exists a compact } K \subset \Omega \text{ such that } E \cap K \text{ is compact and } \sup_{\mu \in A} |\mu|(\Omega \backslash K) < \varepsilon'\}.$$

Clearly, $\mathcal{C}(\Omega) = \{K \subset \Omega : K \text{ compact}\} \subset \Sigma$.

We claim that $O \cap \Omega \in \Sigma$ for each σ-Borel open set (resp. open set) O in T. In fact, given $\varepsilon' > 0$, by (iv)(a) (resp.(iv)') there exists a compact K in T with $K \subset O$ such that $\sup_{\mu \in A} |\mu|(O \backslash K) \leq \varepsilon'$. Let $K_o = K \cap \Omega$. Then $O \cap \Omega \cap K_o = K_o$ is compact and clearly, $O \cap \Omega \in \mathcal{B}(\Omega)$. Moreover,

$$\sup_{\mu \in A} |\mu|((O \cap \Omega) \backslash K_o) \leq \sup_{\mu \in A} |\mu|(O \backslash K) \leq \varepsilon'.$$

Setting $K_1 = K_o \cup (\Omega \backslash O)$, we note that K_1 is compact , $K_1 \subset \Omega$, $(\Omega \cap O) \cap K_1 = K_o \in \mathcal{C}(T)$ and $\sup_{\mu \in A} |\mu|(\Omega \backslash K_1) \leq \varepsilon'$. Thus $O \cap \Omega \in \Sigma$.

We also claim that $\Omega \backslash K \in \Sigma$ for each compact $K \subset \Omega$. In fact, by Theorem 50.D of [9] there exists a relatively compact open set U in T such that $\Omega \subset U$. Clearly U is a σ-Borel open set in T and $\Omega \backslash K = (\Omega \backslash K) \cap U = \Omega \cap (U \backslash K)$ with $U \backslash K$ a σ-Borel open set in T. Then by the foregoing claim it follows that $\Omega \backslash K \in \Sigma$.

Proceeding as on p.158 of [2], one can show that Σ is closed under countable intersections. The argument used in the proof of (iv) $\Rightarrow$ (v) of Theorem 1 to show that $\mathcal{S}$ is closed under complements can be modified here to prove that Σ is also closed under complements in Ω. Thus Σ is a σ-algebra in Ω. As $\mathcal{C}(\Omega) \subset \Sigma \subset \mathcal{B}(\Omega)$, it follows that $\Sigma = \mathcal{B}(\Omega)$. Then arguing as in the last part of (iv) $\Rightarrow$ (v) of Theorem 1 by using (3.7) in place of (3.1), we conclude that (v) (resp. (v)') holds.

This completes the proof of the theorem.

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