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Dynamical Spectrum for Scalar Parabolic Equations.

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Notas de Matemática

Serie: Pre-Print

No. 172

Mérida - Venezuela

1998

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Abstract

In this paper we compute the dynamical spectrum for time-dependent scalar parabolic equations with both Neumann and Dirichlet boundary conditions. In order to do that, first, we put forward the concepts of negative continuation, exponential dichotomy and Dynamical Spectrum for linear skew-product semiflows. Second, we set the problem in the skew-product semiflow framework and compute explicitly the dynamical spectrum for this semiflow. Finally, we compute the dynamical spectrum for a time-dependent system of ordinary differential equations that is obtained by spatially discretizing of the parabolic equation.

1 Introduction

In this paper we shall characterize the dynamical spectrum for time dependent linear scale parabolic equations. In order to do that we will use the concepts of Skew-Product Semiflow and Exponential Dichotomy to define the dynamical spectrum. These concepts have been studied in [7], [1],[2] and [6].

More specifically we shall compute the dynamical spectrum for the following scalar parabolic equation with Neumann boundary conditions

$$u_t = a(t)u_{xx} + b(t)u \quad x \in (0, 1) \quad (1.1)$$

$$u_x(t, 0) = u_x(t, 1) = 0. \quad (NB) \quad (1.2)$$

Also, we shall compute the dynamical spectrum for this parabolic equation with Dirichlet boundary conditions:

$$u_t = a(t)u_{xx} + b(t)u \quad x \in (0, 1) \quad (1.3)$$

$$u(t, 0) = u(t, 1) = 0. \quad (DB) \quad (1.4)$$

One of the purpose of this work is to prove the following two Lemmas:

Lemma 1.1 *If $a(\cdot)$ and $b(\cdot)$ are continuous functions and*

$$\begin{aligned} \lim_{t \rightarrow \infty} b(t) &= \beta \geq \lim_{t \rightarrow -\infty} b(t) = \alpha \\ \lim_{t \rightarrow \infty} a(t) &= \gamma \geq \lim_{t \rightarrow -\infty} a(t) = \sigma \quad \text{and} \quad a(t) > 0. \end{aligned}$$

Then the dynamical spectrum for the equation (1.1) with Neumann boundary conditions is given by

$$\Sigma = \bigcup_{n=1}^{\infty} [-(n-1)^2\pi^2\sigma + \alpha, -(n-1)^2\pi^2\gamma + \beta],$$

and the dynamical spectrum for the equation (1.1) with Dirichlet boundary conditions is given by

$$\Sigma = \bigcup_{n=1}^{\infty} [-n^2\pi^2\sigma + \alpha, -n^2\pi^2\gamma + \beta].$$

Lemma 1.2 *If $a(\cdot), b(\cdot)$ are continuous τ -periodic functions.*

Then the dynamical spectrum for the equation (1.1) with Neumann boundary conditions is given by

$$\Sigma = \left\{ \frac{-(n-1)^2\pi^2}{\tau} \int_0^\tau a(s)ds + \frac{1}{\tau} \int_0^\tau b(s)ds : n = 1, 2, \dots \right\},$$

and the dynamical spectrum for the equation (1.1) with Dirichlet boundary conditions is given by

$$\Sigma = \left\{ \frac{-n^2\pi^2}{\tau} \int_0^\tau a(s)ds + \frac{1}{\tau} \int_0^\tau b(s)ds : n = 1, 2, \dots \right\}.$$

Finally, we consider the following more general time dependent scale parabolic equation

$$u_t = a(t, x)u_{xx} + b(t, x)u \quad x \in (0, 1) \quad (1.5)$$

$$u_x(t, 0) = u_x(t, 1) = 0. \quad (DB) \quad (1.6)$$

where $a, b \in C(\mathbb{R}^2, \mathbb{R})$ with $a(t, x) \geq \sigma > 0$ are continuous and bounded with the following property:

$$\lim_{t \rightarrow \infty} a(t, x) = \gamma(x) \geq \lim_{t \rightarrow -\infty} a(t, x) = \sigma(x) \quad (1.7)$$

$$\lim_{t \rightarrow \infty} b(t, x) = \beta(x) \geq \lim_{t \rightarrow -\infty} b(t, x) = \alpha(x). \quad (1.8)$$

Instead of studying the dynamical spectrum of the equation (1.5) with (NB), we shall characterize the dynamical spectrum of the time-dependent system of ordinary differential that are obtained by spatially discretizing (1.5) in $[0, 1]$.

$$\dot{u}_i = a_i(t)n^2(u_{i-1} - 2u_i + u_{i+1}) + b_i(t)u_i \quad (1.9)$$

$$u_{-1} = u_0, \quad u_n = u_{n+1}. \quad (1.10)$$

Where each u_i is a function of the time t and $\frac{1}{n}$ is the spacing between mesh-points. This discrete version of the equation (1.5) can be studied for several reasons. First, they represent a simple scheme that might be used to simulate equation (1.5) numerically. Second, the partial differential equations are usually derived as a (or simply the dynamical spectrum)continuous approximation of discrete systems. Another reason, could be purely mathematical. Then for the discrete system (1.9) we prove the following spectral Theorem

Lemma 1.3 *The dynamical spectrum for the system of ODEs (1.9) is given by:*

$$\Sigma = \bigcup_{i=0}^n \left[-2n^2 \left(1 - \cos \frac{i\pi}{n+1} \right) \sigma + \alpha, -2n^2 \left(1 - \cos \frac{i\pi}{n+1} \right) \gamma + \beta \right],$$

where

$$\begin{aligned} \lim_{t \rightarrow \infty} a(t) &= \gamma \geq \lim_{t \rightarrow -\infty} a(t) = \sigma \\ \lim_{t \rightarrow \infty} b(t) &= \beta \geq \lim_{t \rightarrow -\infty} b(t) = \alpha. \end{aligned}$$

Our last Theorem is the following:

Lemma 1.4 *If the coefficients $a(t, x) = a(t)$ and $b(t, x) = b(t)$ are assumed to be τ -periodic,*

$$a(t + \tau) = a(t), \quad b(t + \tau) = b(t),$$

then the dynamical spectrum for the discrete system (1.9) is

$$\Sigma = \left\{ -2n^2 \left(1 - \cos \frac{i\pi}{n+1} \right) \frac{1}{\tau} \int_0^\tau a(s) ds + \frac{1}{\tau} \int_0^\tau b(s) ds : i = 0, 1, 2, \dots, n \right\}.$$

2 Preliminaries

In this section we shall present some definitions, notations and results about skew-product semiflow in infinite dimensional Banach spaces.

2.1 Skew-Product Semiflow

We begin with the notion of skew-product semiflow on the trivial Banach bundle $\mathcal{E} = X \times \Theta$ where X is a fixed Banach space (the state space) and Θ is a compact Hausdorff space.

Definition 2.1 Suppose that $\sigma(\theta, t) = \theta_t$ is a flow on Θ , i.e., the mapping $(\theta, t) \rightarrow \theta_t$ is continuous, $\theta_0 = \theta$ and $\theta_{s+t} = \theta_s \circ \theta_t$, for all $s, t \in \mathbb{R}$.

A semiflow π on $\mathcal{E} = X \times \Theta$ is said to be **Linear Skew-Product Semiflow**, if it can be written as follows

$$\pi(x, \theta, t) = (\Phi(\theta, t)x, \theta_t), \quad t \geq 0,$$

where $\Phi(\theta, t) \in L(X)$ has the following properties:

- (1) $\Phi(\theta, 0) = I$, the identity operator on X , for all $\theta \in \Theta$
- (2) $\lim_{t \rightarrow 0^+} \Phi(\theta, t)x = x$, uniformly in θ . This means that for every $x \in X$ and every $\epsilon > 0$ there is a $\delta = \delta(x, \epsilon) > 0$ such that $\|\Phi(\theta, t)x - x\| \leq \epsilon$, for all $\theta \in \Theta$ and $0 \leq t \leq \delta$.
- (3) $\Phi(\theta, t)$ is a bounded linear operator from X into X that satisfies the cocycle identity:

$$\Phi(\theta, t+s) = \Phi(\theta_t, s)\Phi(\theta, t) \quad \theta \in \Theta, \quad 0 \leq s, t. \quad (2.11)$$

(4) for all $t \geq 0$ the mapping from \mathcal{E} into X given by

$$(x, \theta) \rightarrow \Phi(\theta, t)x$$

is continuous.

The properties (2) and (3) imply that for each $(x, \theta) \in \mathcal{E}$ the solution operator $t \rightarrow \Phi(\theta, t)x$ is right continuous for $t \geq 0$. In fact :

$$\|\Phi(\theta, t+h)x - \Phi(\theta, t)x\| = \|[\Phi(\theta, h) - I]\Phi(\theta, t)x\|$$

which goes to 0 as h goes to 0^+ .

For any subset $\mathcal{F} \subset \mathcal{E}$ we define the fiber $\mathcal{F}(\theta) := \{x \in X : (x, \theta) \in \mathcal{F}\}$, $\theta \in \Theta$.

So $\mathcal{E}(\theta) = X \times \{\theta\}$, $\theta \in \Theta$. Also, we define $\mathcal{E}_0 = \{(x, \theta) \in \mathcal{E} : x = 0\}$ as the zero fiber

2.2 The Stable and Unstable Sets

Definition 2.2 A point $(x, \theta) \in \mathcal{E}$ is said to have a negative continuation with respect to π if there exists a continuous functions $\phi : (-\infty, 0] \rightarrow \mathcal{E}$ satisfying the following properties:

- (1) $\phi(t) = (\phi^x(t), \theta_t)$ where $\phi^x : (-\infty, 0] \rightarrow X$
- (2) $\phi(0) = (x, \theta)$
- (3) $\pi(\phi(s), t) = \phi(s+t)$, for each $s \leq 0$, and $0 \leq t \leq -s$
- (4) $\pi(\phi(s), t) = \pi(x, \theta, t+s)$, for each $0 \leq -s \leq t$.

In this case the function ϕ is said to be a **negative continuation of the point** (x, θ) . For any negative continuation ϕ and any $\tau \leq 0$, we define $\phi_\tau(t) = \phi(\tau+t)$ for $-\infty < t < -\tau$.

Now we shall define the following sets:

$$\mathcal{M} := \{(x, \theta) \in \mathcal{E} : (x, \theta) \text{ has a negative continuation } \phi\}$$

$$\mathcal{U} := \{(x, \theta) \in \mathcal{M} : \text{there is a negative continuation } \phi \text{ of } (x, \theta) \text{ satisfying } \|\phi^x(t)\| \rightarrow 0 \text{ as } t \rightarrow -\infty\}$$

$$\mathcal{B}^+ := \{(x, \theta) \in \mathcal{E} : \sup_{t \geq 0} \|\Phi(\theta, t)x\| < \infty\}$$

$$\mathcal{B}_u^- := \{(x, \theta) \in \mathcal{M} : (x, \theta) \text{ has a unique bounded negative continuation } \phi\}$$

$$\mathcal{B}^- := \{(x, \theta) \in \mathcal{M} : \text{there is a bounded negative continuation } \phi \text{ of } (x, \theta)\}$$

$$\mathcal{S} := \{(x, \theta) \in \mathcal{E} : \|\Phi(\theta, t)x\| \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

$$\mathcal{B} := \mathcal{B}^+ \cap \mathcal{B}^-$$

The set \mathcal{U} is called **unstable set**, \mathcal{S} is the **stable set** and \mathcal{B} is the **bounded set**.

2.3 Exponential Dichotomy and Dynamical Spectrum

A mapping $\mathbf{P} : \mathcal{E} \rightarrow \mathcal{E}$ is said to be a projector if \mathbf{P} is continuous and has the form $\mathbf{P}(x, \theta) = (P(\theta)x, \theta)$, where $P(\theta)$ is a bounded linear projection on the fiber $\mathcal{E}(\theta)$.

For any projector \mathbf{P} we define the range and null space by

$$\mathcal{R} = \mathcal{R}(\mathbf{P}) = \{(x, \theta) \in \mathcal{E} : P(\theta)x = x\}$$

$$\mathcal{N} = \mathcal{N}(\mathbf{P}) = \{(x, \theta) \in \mathcal{E} : P(\theta)x = 0\}$$

The continuity of \mathbf{P} implies that the fibers $\mathcal{R}(\theta)$ and $\mathcal{N}(\theta)$ vary continuously in θ . This also means that $P(\theta)$ varies continuously in the operator norm. The following result is elementary and can be found in Sacker-Sell [7].

Lemma 2.1 *Let \mathbf{P} be a projector on \mathcal{E} , then \mathcal{R} and \mathcal{N} are closed subsets in \mathcal{E} and we have*

$$\mathcal{R}(\theta) \cap \mathcal{N} = \{0\}, \quad \mathcal{R}(\theta) + \mathcal{N}(\theta) = \mathcal{E}(\theta) \quad \text{for all } \theta \in \Theta.$$

A projector \mathbf{P} on \mathcal{E} is said to be **invariant** if it satisfies the following property

$$P(\theta_t)\Phi(\theta, t) = \Phi(\theta, t)P(\theta) \quad t \geq 0, \quad \theta \in \Theta \quad (2.12)$$

We shall say that a linear skew-product semiflow π on \mathcal{E} has an **exponential dichotomy over an invariant set** $\hat{\Theta}$, where $\hat{\Theta} \subset \Theta$, if there are constants $k \geq 1$, $\beta > 0$ and an invariant projector \mathbf{P} such that the following inequalities hold :

$$\begin{aligned} \|\Phi(\theta, t)P(\theta)\| &\leq ke^{-\beta t} \quad t \geq 0, \quad \theta \in \hat{\Theta} \\ \|\Phi(\theta, t)(I - P(\theta))\| &\leq ke^{\beta t} \quad t \leq 0, \quad \theta \in \hat{\Theta}, \end{aligned}$$

and $\dim \text{Range}(I - P(\theta)) < \infty$, $\text{Range}(I - P(\theta)) \subset \mathcal{B}_u^-(\theta)$ for all $\theta \in \hat{\Theta}$.

Proposition 2.1 *(See [7]) If π is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ which admits an exponential dichotomy over Θ , then one has that $\mathcal{B} = \mathcal{E}_0$.*

Consider $\pi = (\Phi, \sigma)$ a linear skew-product semiflow on \mathcal{E} . Then for each $\lambda \in \mathbb{R}$ we define the shifted semiflow as follows:

$$\pi_\lambda = (\Phi_\lambda, \sigma), \quad \Phi_\lambda(\theta, t) = e^{-\lambda t}\Phi(\theta, t) \quad t \geq 0, \quad \theta \in \Theta$$

Let $\hat{\Theta}$ be an invariant subset of Θ under the flow σ . The **resolvent** $\rho(\hat{\Theta})$ of $\hat{\Theta}$ under the skew-product semiflow π is defined as follows

$$\rho(\hat{\Theta}) = \{\lambda \in \mathbb{R} : \pi_\lambda \text{ admits an exponential dichotomy over } \hat{\Theta}\}$$

and the **dynamical spectrum** $\Sigma(\hat{\Theta})$ of $\hat{\Theta}$ under π as follows

$$\Sigma(\hat{\Theta}) = \mathbb{R} \setminus \rho(\hat{\Theta}).$$

The following Lemma plays an important role along this work, it can be found in [1].

Lemma 2.2 *Let $\hat{\Theta}$ be a compact invariant subset of Θ and $\lambda \in \mathbb{R}$.*

If $\|\Phi_\lambda(\theta, t)\| \rightarrow 0$ as $t \rightarrow \infty$ for all $\theta \in \hat{\Theta}$, then $\lambda \in \rho(\hat{\Theta})$, $\Sigma(\hat{\Theta}) \subseteq (-\infty, \lambda)$, and $\mathcal{S}_\mu = \mathcal{E}(\hat{\Theta}) = X \times \hat{\Theta}$ for all $\mu \geq \lambda$.

3 Dynamical Spectrum for Scalar ODE

In this section we shall characterize the dynamical spectrum for the following very simple time dependent ODE

$$\dot{x} = b(t)x, \quad x \in X, \quad (3.13)$$

where X is a Banach space and $b(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying the following property

$$\lim_{t \rightarrow \infty} b(t) = \beta \geq \lim_{t \rightarrow -\infty} b(t) = \alpha. \quad (3.14)$$

Proposition 3.1 *Under the above conditions the function $a(\cdot)$ is uniformly continuous and bounded in \mathbb{R} .*

The function $b(\cdot)$ belong to the space $W = C(\mathbb{R})$ endowed with the topology of uniform convergence on compact subsets of \mathbb{R} . The following set plays an important role along this work, it is called the Hull of $b(\cdot)$:

$$H(b) := \text{Hull}(b) = \text{cl}\{b_\tau : \tau \in \mathbb{R}\}, \quad (3.15)$$

where $b_\tau \in W$ is given by $b_\tau(t) = b(t + \tau)$ and cl denotes the closure in the topology of W . It is known from classical topological dynamical systems Theory [8] that $H(b)$ is a compact metrizable subset of W , and the metric is given by:

$$\rho(h, g) := \sum_{k=1}^{\infty} 2^{-k} \frac{\rho_k(h, g)}{1 + \rho_k(h, g)}, \quad h, g \in H(b), \quad (3.16)$$

where

$$\rho_k(h, g) := \sup\{|h(t) - g(t)| : -k \leq t \leq k\}$$

Moreover, the mapping $\sigma(h, t) = h_t$ is a flow on $H(b)$.

Proposition 3.2 *Under the condition (3.14) we have that*

$$H(b) = \{b_\tau, h(t) = \alpha, g(t) = \beta : \tau \in \mathbb{R}\}.$$

Proof It follows from the formula (3.16). □

Instead of concentrating on the single equation (3.13) we shall consider the family of equations

$$\dot{x} = h(t)x, \quad h \in H(b). \quad (3.17)$$

Then the mapping $\pi : X \times H(b) \times \mathbb{R} \rightarrow X \times H(b)$ given by

$$\pi(x, h, t) = (\Phi(h, t)x, h_t), \quad (3.18)$$

defines a linear skew-product flow on $X \times H(b)$, where

$$\Phi(h, t)x = \exp\left(\int_0^t h(s)ds\right)x. \quad (3.19)$$

The following Lemma is the key of this work.

Lemma 3.1 *The dynamical spectrum of the skew-product flow given by (3.18) generated by the equation (3.13) is:*

$$\Sigma(H(b)) = [\alpha, \beta].$$

Proof

Case 1. If $\lambda > \beta$, then $\lambda \in \Sigma(H(b))$. In fact, consider $h \in H(b)$, $h \neq \alpha$. Then $\lim_{t \rightarrow \infty} h(t) = \beta$. Let $\epsilon > 0$ be small enough such that $\beta + \epsilon < \lambda$. Then there exists $N > 0$ with $h(t) < \beta + \epsilon$ for all $t \geq N$. So, for $x \in X$ and $t \geq N$ we have

$$\begin{aligned} \Phi_\lambda(h, t)x &= e^{-\lambda t} \exp\left(\int_0^t h(s) ds\right) x \\ &= \exp\left(\int_0^N h(s) ds\right) \exp\left(\int_N^t h(s) ds\right) e^{-\lambda t} x. \end{aligned}$$

Then

$$\|\Phi_\lambda(h, t)x\| \leq \exp\left(\int_0^N h(s) ds\right) e^{(\beta + \epsilon - \lambda)t} \|x\|.$$

Hence, $\|\Phi_\lambda(h, t)\| \rightarrow 0$, as $t \rightarrow \infty$, for all $h \in H(b)$, $h \neq \alpha$. Now, if $h = \alpha$, then

$$\|\Phi_\lambda(h, t)x\| = e^{-\lambda t} e^{\alpha t} \|x\| = e^{(\alpha - \lambda)t} \|x\| \rightarrow 0, \text{ as } t \rightarrow \infty.$$

Therefore, $\|\Phi_\lambda(h, t)\| \rightarrow 0$, as $t \rightarrow \infty$, for all $h \in H(b)$. So, from Lemma 2.2 we get that $\lambda \in \rho(H(b))$.

Case 2. If $\lambda < \alpha$, then $\lambda \notin \Sigma(H(b))$. This case is similar to case 1.

Case 3. $\alpha, \beta \in \Sigma(H(b))$. In fact, for all $x \in X$

$$\Phi_\alpha(\alpha, t)x = \Phi_\beta(\beta, t)x = x,$$

which means that, the bounded sets $\mathcal{B}_\alpha, \mathcal{B}_\beta$ are not trivial. So, from Proposition 2.1 we have that $\alpha, \beta \in \Sigma(H(b))$.

Case 4. If $\lambda \in (\alpha, \beta)$, then $\lambda \in \Sigma(H(b))$. In fact, for all $x \in X$ we have

$$\lim_{t \rightarrow \infty} \Phi_\lambda(\alpha, t)x = \lim_{t \rightarrow \infty} e^{(\alpha - \lambda)t} x = 0 \text{ and } \lim_{t \rightarrow \infty} \Phi_\lambda(\beta, t)x = \lim_{t \rightarrow \infty} e^{(\beta - \lambda)t} x = \infty.$$

Hence, from Lemma 2.2 we get that $\lambda \in \Sigma(H(b))$.

□

Lemma 3.2 *If the function b in the equation (3.13) is periodic of period $\tau > 0$, then the dynamical spectrum of the skew-product flow given by (3.18) and generated by the equation (3.13) is:*

$$\Sigma(H(b)) = \left\{ \frac{1}{\tau} \int_0^\tau b(s) ds \right\}.$$

Proof It follows from the following fact: for $t > \tau$, there exists $n = n(t) \in \mathbb{N}$ such that $t = n\tau + r$, $0 \leq r < \tau$. Then, for all $h \in H(b)$ we get:

$$\begin{aligned} \int_0^t h(s) ds &= n \int_0^\tau h(s) ds + \int_0^r h(s) ds \\ &= \frac{t-r}{\tau} \int_0^\tau b(s) ds + \int_0^r h(s) ds \\ &= \left(\frac{1}{\tau} \int_0^\tau b(s) ds \right) t - \frac{r}{\tau} \int_0^\tau b(s) ds + \int_0^r h(s) ds \\ &= \left(\frac{1}{\tau} \int_0^\tau b(s) ds \right) t + B(t, h). \end{aligned}$$

Where $B(t, h)$ is uniformly bounded in t , h . Therefore,

$$\Phi(h, t) = e^{B(t, h)} e^{\left(\frac{1}{\tau} \int_0^\tau b(s) ds\right)t}, \quad \forall h \in H(b).$$

(If $t < -\tau$ the proof follows in the same way) □

Given a point $(x, h) \in X \times H(b)$, $x \neq 0$, we shall define the **four Lyapunov characteristic exponents** of (x, h) as follows:

$$\lambda_s^+(x, h) = \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \|\Phi(h, t)x\| \quad (3.20)$$

$$\lambda_i^+(x, h) = \liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \|\Phi(h, t)x\| \quad (3.21)$$

$$\lambda_s^-(x, h) = \limsup_{t \rightarrow -\infty} \frac{1}{t} \ln \|\Phi(h, t)x\| \quad (3.22)$$

$$\lambda_i^-(x, h) = \liminf_{t \rightarrow -\infty} \frac{1}{t} \ln \|\Phi(h, t)x\| \quad (3.23)$$

Definition 3.1 For all $h \in H(b)$ we define the upper and lower Lyapunov exponents $\lambda_s^+(h)$ and $\lambda_i^+(h)$ as follows:

$$\lambda_s^+(h) := \sup\{\lambda_s^+(x, h) : x \in X, x \neq 0\}$$

$$\lambda_i^+(h) := \inf\{\lambda_i^+(x, h) : x \in X, x \neq 0\}$$

The following Theorem is proved in [1] for a general skew-product semiflow.

Theorem 3.1 *The upper and the lower Lyapunov exponents $\lambda_s^+(h)$ and $\lambda_i^+(h)$ associated with $h \in H(b)$ are given respectively by :*

$$\lambda_s^+(h) = \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \|\Phi(h, t)\| \in [\alpha, \beta] \quad (3.24)$$

$$\lambda_i^+(h) = - \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \|\Phi(h_{-t}, t)\| \in [\alpha, \beta] \quad (3.25)$$

The following Corollary can be used to study the stability of the solutions of the equation (3.13).

Corolary 3.1 *Under the condition of Lemma 3.1 we have the following:
For each $\epsilon > 0$ there exists a constant $M = M(\epsilon) \geq 1$ such that*

$$\|\Phi(h, t)\| \leq Me^{(\beta+\epsilon)t}, \quad t \geq 0, \quad h \in H(b) \quad (3.26)$$

$$\|\Phi(h, t)\| \leq Me^{(\alpha-\epsilon)t}, \quad t \leq 0, \quad h \in H(b) \quad (3.27)$$

Corolary 3.2 (a) *If $\beta < 0$, then the solutions of (3.13) are exponentially stable*

(b) *If $\alpha > 0$, then the solutions of (3.13) are unstable.*

(c) *If $0 \in (\alpha, \beta)$, then the solution of (3.13) blow up in both directions.*

4 Dynamical Spectrum Parabolic PDE

We begin this section with a simple time-dependent scale parabolic equation with Neumann boundary conditions

$$u_t = a(t)u_{xx} \quad x \in (0, 1) \quad (4.28)$$

$$u_x(t, 0) = u_x(t, 1) = 0, \quad (NB), \quad (4.29)$$

where $a(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a uniformly continuous and bounded function with the following properties:

$$\lim_{t \rightarrow \infty} a(t) = \gamma \geq \lim_{t \rightarrow -\infty} a(t) = \sigma > 0 \quad \text{and} \quad a(t) > 0. \quad (4.30)$$

The equation (4.28) can be written as an abstract equation in the Hilbert space $Z = L_2[0, 1]$ as follow:

$$\dot{z} = a(t)Az, \quad t > 0 \quad (4.31)$$

where $A\phi(x) = \ddot{\phi}(x)$ is a linear unbounded operator with domain

$$D(A) = \{z \in L_2[0, 1] : z_{xx} \in L_2[0, 1], \quad z_x(0) = z_x(1) = 0\} \quad (4.32)$$

It is well known that $A = \frac{\partial^2}{\partial x^2}$ is selfadjoint with compact resolvent; so the spectrum of A consist only of a contable number of eigenvalues:

$$\lambda_n = -(n-1)^2\pi^2, \quad n = 1, 2, 3, \dots, \dots, \quad (4.33)$$

and the corresponding eigenfunctions:

$$\phi_n(x) = \sqrt{2} \cos(n-1)\pi x, \quad n = 1, 2, 3, \dots, \dots \quad \text{and} \quad \phi_1(x) = 1. \quad (4.34)$$

Instead of concentrating on the single equation (4.31) we shall consider the family of equations

$$\dot{z} = h(t)Az, \quad t > 0, \quad h \in H(a), \quad (4.35)$$

where $H(a)$ denote the Hull of a given by (3.3). Then the mapping $\pi : Z \times H(a) \times \mathbb{R}_+ \rightarrow Z \times H(a)$ given by

$$\pi(z, h, t) = (\Phi(h, t)z, h_t), \quad (4.36)$$

defines a linear skew-product semiflow on $Z \times H(a)$, where

$$\Phi(h, t)z = T \left(\exp \int_0^t h(s) ds \right) z, \quad (4.37)$$

and $\{T(t)\}_{t \geq 0}$ is the c_0 -semigroup generated by the operator A which is given by

$$T(t)z(x) = \sum_{n=1}^{\infty} \tilde{z}_n e^{\lambda_n t} \phi_n(x), \quad z \in Z,$$

and $\tilde{z}_n = (z, \phi_n) = \int_0^1 \phi_n(y) z(y) dy$. Since, $\{\phi_n\}_{n \geq 1}$ yield an orthonormal base of the Hilbert space Z , then we can write $\Phi(h, t)z$ as follow

$$\Phi(h, t)z = \sum_{n=1}^{\infty} e^{\int_0^t \lambda_n h(s) ds} P_n z, \quad (4.38)$$

where P_n s are orthonormal projections in the Hilbert space Z , which are given by: $P_n z = (z, \phi_n) \phi_n$.

Lemma 4.1 *The dynamical spectrum of the skew-product semiflow given by (4.36) generated by the equation (4.28) is:*

$$\Sigma = \bigcup_{n=1}^{\infty} [-(n-1)^2 \pi^2 \sigma, -(n-1)^2 \pi^2 \gamma],$$

Proof From (4.38) we get that

$$\Phi(h, t)z = \sum_{n=1}^{\infty} \Phi_n(h, t)z, \quad h \in H(a), \quad z \in Z, \quad (4.39)$$

where

$$\Phi_n(h, t)z = \exp \left(\int_0^t \lambda_n h(s) ds \right) P_n z, \quad h \in H(a), \quad z \in Z. \quad (4.40)$$

Then, for each n we consider the skew-product semiflow $\pi_n : Z \times H(a) \times \mathbb{R}_+ \rightarrow Z \times H(a)$ given by

$$\pi_n(z, h, t) = (\Phi_n(h, t)z, h_t),$$

which is generated by the following ODE

$$\dot{z} = \lambda_n a(t)z, \quad z \in Z.$$

From Lemma 3.1 the dynamical spectrum of the flow π_n is $\Sigma_n(H(a)) = [\lambda_n \sigma, \lambda_n \gamma]$. Clearly the dynamical spectrum of the semiflow π is given by

$$\Sigma(H(a)) = \bigcup_{n=1}^{\infty} \Sigma_n = \bigcup_{n=1}^{\infty} [-(n-1)^2 \pi^2 \sigma, -(n-1)^2 \pi^2 \gamma].$$

□

Now, we are ready to prove Lemmas 1.1 and 1.2.

Proof of Lemma 1.1. The equation (1.1) generates a skew-product semiflow $\pi : Z \times H(a, b) \times \mathbb{R}_+ \rightarrow Z \times H(a, b)$ given by

$$\pi_n(z, h, g, t) = (\Phi(h, g, t)z, h_t, g_t),$$

where $H(a, b) = H(a) \times H(b)$ and

$$\Phi(h, g, t)z = \sum_{n=1}^{\infty} \Phi_n(h, g; t)z, \quad h, g \in H(a, b), \quad z \in Z, \quad (4.41)$$

with

$$\Phi_n(h, t)z = \exp\left(\int_0^t [\lambda_n h(s) + b(s)] ds\right) P_n z, \quad h \in H(a), \quad z \in Z. \quad (4.42)$$

Then, for each n we consider the skew-product flow $\pi_n : Z \times H(a, b) \times \mathbb{R}_+ \rightarrow Z \times H(a, b)$ given by

$$\pi_n(z, h, g; t) = (\Phi_n(h, t)z, h_t, g_t),$$

which is generated by the following ODE

$$\dot{z} = (\lambda_n a(t)z + b(s)), \quad z \in Z.$$

From Lemma 3.1 the Sacker-Sell spectrum of this flow π_n is $\Sigma_n(H(a, b)) = [\lambda_n \sigma + \alpha, \lambda_n \gamma + \beta]$. Therefore the dynamical spectrum of the semiflow π is given by

$$\Sigma(H(a, b)) = \bigcup_{n=1}^{\infty} \Sigma_n(H(a, b)) = \bigcup_{n=1}^{\infty} [-(n-1)^2 \pi^2 \sigma + \alpha, -(n-1)^2 \pi^2 \gamma + \beta].$$

For the equation (1.7) with Dirichlet boundary conditions we have to consider the operator $A = \frac{\partial^2}{\partial x^2}$ with the domain

$$D(A) = \{z \in L_2[0, 1] : z_{xx} \in L_2[0, 1], \quad z(0) = z(1) = 0\}$$

It is well known that A is selfadjoint with compact resolvent; so the spectrum of A consist only of a contable number of eigenvalues:

$$\lambda_n = -n^2 \pi^2, \quad n = 1, 2, 3, \dots, \dots,$$

and the corresponding eigenfunctions:

$$\phi_n(x) = \sqrt{2} \sin(n\pi x), \quad n = 1, 2, 3, \dots, \dots$$

Which yield to an orthonormal base of the Hilbert space Z . The c_0 -semigroup $\{S(t)\}_{t \geq 0}$ generated by A in this case is given by

$$S(t)z(x) = \sum_{n=1}^{\infty} \tilde{z}_n e^{\lambda_n t} \phi_n(x), \quad z \in Z,$$

and $\tilde{z}_n = (z, \phi_n) = \int_0^1 \phi_n(y) z(y) dy$. From here everything follows in the same way as above. □

Proof of Lemma 1.2. Again the equation (1.1) with τ -periodic coefficients $a(t), b(t)$ and Neumann boundary conditions generates a skew-product semiflow π on $Z \times H(a, b)$ given by

$$\pi_n(z, h, g, t) = (\Phi(h, g, t)z, h_t, g_t),$$

where $H(a, b) = \{a_t : t \in \mathbb{R}\} \times \{b_t : t \in \mathbb{R}\}$ and

$$\Phi(h, g, t)z = \sum_{n=1}^{\infty} \Phi_n(h, g; t)z, \quad h, g \in H(a, b), \quad z \in Z, \quad (4.43)$$

with

$$\Phi_n(h, t)z = \exp\left(\int_0^t [\lambda_n h(s) + b(s)] ds\right) P_n z, \quad h \in H(a), \quad z \in Z. \quad (4.44)$$

Then, for each n we consider the skew-product flow $\pi_n : Z \times H(a, b) \times \mathbb{R}_+ \rightarrow Z \times H(a, b)$ given by

$$\pi_n(z, h, g; t) = (\Phi_n(h, t)z, h_t, g_t),$$

which is generated by the following ODE

$$\dot{z} = (\lambda_n a(t)z + b(s)), \quad z \in Z.$$

From Lemma 3.2 the dynamical spectrum of this flow π_n is

$$\Sigma_n(H(a, b)) = \left\{ \frac{\lambda_n}{\tau} \int_0^\tau a(s) ds + \frac{1}{\tau} \int_0^\tau b(s) ds \right\} = \left\{ \frac{-(n-1)^2 \pi^2}{\tau} \int_0^\tau a(s) ds + \frac{1}{\tau} \int_0^\tau b(s) ds \right\}.$$

Therefore the dynamical spectrum of the semiflow π is given by

$$\begin{aligned} \Sigma(H(a, b)) &= \{ \Sigma_n(H(a, b)) : n = 1, 2, \dots \} \\ &= \left\{ \frac{-(n-1)^2 \pi^2}{\tau} \int_0^\tau a(s) ds + \frac{1}{\tau} \int_0^\tau b(s) ds : n = 1, 2, \dots, \dots \right\}. \end{aligned}$$

The last part of the Lemma can be proved in the same way. \square

Proof of Lemmas 1.3 and 1.4. We want to compute the dynamical spectrum of the non-autonomous linear system of differential equations

$$\dot{u}_i = a(t)n^2(u_{i-1} - 2u_i + u_{i+1}) + b(t)u_i \quad (4.45)$$

$$u_{-1} = u_0, \quad u_n = u_{n+1}. \quad (4.46)$$

Which is a discretization on space of the following non-autonomous parabolic equation with Neumann boundary conditions

$$u_t = a(t)u_{xx} + b(t)u \quad x \in (0, 1) \quad (4.47)$$

$$u_x(t, 0) = u_x(t, 1) = 0. \quad (4.48)$$

where $a, b \in C(\mathbb{R})$ with $a(t) \geq \sigma > 0$ with the following property:

$$\lim_{t \rightarrow \infty} a(t) = \gamma \geq \lim_{t \rightarrow -\infty} a(t) = \sigma \quad (4.49)$$

$$\lim_{t \rightarrow \infty} b(t) = \beta \geq \lim_{t \rightarrow -\infty} b(t) = \alpha. \quad (4.50)$$

Let us introduce some notation that will allow (4.45) - (4.46) to be rewritten in a more compact form. Consider the vector space of dimension $n + 1$ given by

$$U_n = \{(u_0, u_1, \dots, u_n)^T : u_i \in \mathbb{R}, i = 1, 2, \dots, n\},$$

where "T" denote transposition. Let A be the following $(n + 1) \times (n + 1)$ tridiagonal matrix

$$\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix}$$

Then the equations (4.45)-(4.46) can be written in a short way as follow

$$\dot{u} = n^2 a(t) A u + b(t) u. \quad (4.51)$$

It is well known that the eigenvalues of A are given by the following formula

$$\{\lambda_j = -2 \left(1 - \cos \frac{j\pi}{n+1}\right) : j = 0, 1, \dots, n\}.$$

Since the function $\cos x$ is 1 to 1 in $[0, \pi)$, then all the eigenvalues of the matrix A are simple. Hence, A is diagonalizable. i.e., There exists a non-singular $(n + 1) \times (n + 1)$ matrix P such that $A = PJP^{-1}$ where $J = \text{diag}(\lambda_0, \lambda_1, \dots, \lambda_n)$ is diagonal.

Therefore, the equation (4.51) can be written as follow

$$\begin{aligned} \dot{u} &= n^2 a(t) PJP^{-1} u + b(t) u \\ &= P \left(n^2 a(t) J + b(t) I \right) P^{-1} u. \end{aligned}$$

Now, making the change of variable $v = P^{-1} u$ we get the equivalent system

$$\dot{v} = \left(n^2 a(t) J + b(t) I \right) v. \quad (4.52)$$

The dynamical of this system is given by the following scale equations

$$\dot{v}_j = \left(\lambda_j n^2 a(t) + b(t) \right) v_j, \quad j = 0, 1, 2, \dots, n. \quad (4.53)$$

So, from Lemma 3.1 we get that the dynamical spectrum for the equation (4.53) is

$$\Sigma_j = [\lambda_j n^2 \sigma + \alpha, \lambda_j n^2 \gamma + \beta].$$

Therefore, the dynamical spectrum for the system (4.53) is

$$\begin{aligned} \Sigma &= \bigcup_{j=0}^n \Sigma_j = \bigcup_{j=0}^n [\lambda_j n^2 \sigma + \alpha, \lambda_j n^2 \gamma + \beta] \\ &= \bigcup_{j=0}^n \left[-2n^2 \left(1 - \cos \frac{j\pi}{n+1}\right) \sigma + \alpha, -2n^2 \left(1 - \cos \frac{j\pi}{n+1}\right) \gamma + \beta \right]. \end{aligned}$$

The proof of Lemma 1.4 follows in the same way by using Lemma 3.2 instead of Lemma 3.1. \square

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