UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS

NOTAS DE MATEMATICA

BOUNDED SOLUTIONS OF A SECOND ORDER ABSTRACT EQUATIONS AND APLICATIONS

POR

HUGO LEIVA

DEPARTAMENTO DE MATEMATICA

Universidad de los Andes
Facultad de Ciencias
Departamento de Matemática

Bounded Solutions of a Second Order Abstract Equations and Applications

HUGO LEIVA

Notas de Matemática
Serie: Pre-Print
No. 186

Mérida - Venezuela
1999

Notas de Matemática, No. 186
Mérida, January 1999.

Bounded Solutions of a Second Order Abstract Equations and Applications *

HUGO LEIVA

Abstract

In this paper we study the following abstract second order differential equations with dissipation in a Hilbert space H

$$u'' + cu' + dAu + kG(u) = P(t), \quad u \in H, \quad t \in I\!R,$$

where c, d and k are positive constants, $G : H \to H$ is a Lipschitzian function and $P : I\!R \to H$ is a continuous and bounded function. $A : D(A) \subset H \to H$ is an unbounded linear operator self-adjoint, positive definite and has compact resolvent. Under these conditions we prove that for some values of d, c and k this system has a bounded solution which is exponentilly asymptotically stable. Moreover; if $P(t)$ is almost periodic, then this bounded solution is also almost periodic. These results are applied to a very well known second order system partial differential equations; such as, The Sine Gordon equation, The suspension bridge equation proposed by Lazer and Mckenna, etc.

Key words. differential equations, bounded solutions, stability.

AMS(MOS) subject classifications. primary: 34; secondary: 45.

1 Introduction

In this paper, we study the existence and the asymptotic behavior of the bounded solutions of the following abstract second order differential equation with dissipation

$$u'' + cu' + dAu + kG(u) = P(t), \quad u \in H, \quad t \in I\!R, \tag{1.1}$$

where H is a Hilbert space, c, d and k are positive constants, $P : I\!R \to H$ is a continuous and bounded function, and $G : H \to H$ is a Lipschitzian function. i.e., there exists $L > 0$ such that

$$||G(U_1) - G(U_2)|| \leq L||U_1 - U_2||, \quad U_1, U_2 \in H. \tag{1.2}$$

$A : D(A) \subset H \to H$ is an unbounded linear operator, self-adjoint, positive definite and has compact resolvent. Under these conditions stated on A it follows that:

The spectrum $\sigma(A)$ of A consisting of isolated eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$$

* This research was partially supported by CDCHT-ULA

each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace and
a) there exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvector of A in H.

b) for all $x \in D(A)$ we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j x, \tag{1.3}$$

where $< \cdot, \cdot >$ is the inner product in H and

$$E_j x = \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k}.$$

So, $\{E_j\}$ is a family of complete orthogonal projections in H and

$$x = \sum_{j=1}^{\infty} E_j x, \quad x \in H.$$

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At} x = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j x, \quad x \in H.$$

Many very well known system of partial differential equations can be written in the form of system (1.1):

Example 1.1 The Sine-Gordon Equation with Dirichlet boundary conditions

$$\begin{cases} U_{tt} + cU_t - dU_{xx} + k \sin U = & p(t,x), \quad 0 < x < L, \quad t \in I\!R, \\ U(t,0) = U(t,L) = 0, & t \in I\!R, \end{cases} \tag{1.4}$$

where c and k are positive constants, $p : I\!R \times [0, L] \to I\!R$ is continuous and bounded. In this case we take: $H = L^2(0, L)$ and $A\phi = -\phi_{xx}$ with domain $D(A) = H^2 \cap H_0^1$. $G(u) = \sin u$ and $P(t) = p(t, \cdot)$

Example 1.2 The suspension bridge model proposed by Lazer and Mckenna(see, [4], [5]).

$$\begin{cases} U_{tt} + cU_t + dU_{xxxx} + kU^+ = & p(t,x), \quad 0 < x < L, \quad t \in I\!R, \\ U(t,0) = U(t,L) = U_{xx}(t,0) & = U_{xx}(t,L) = 0, \quad t \in I\!R, \end{cases} \tag{1.5}$$

where c, d and k are positive constants, $p : I\!R \times [0, L] \to I\!R$ is continuous and bounded. In this case we take: $H = L^2(0, L)$ and $A\phi = -\phi_{xxxx}$ with domain

$$D(A) = \{\phi \in H : \phi_{xxxx} \in H; \phi(0) = \phi(L) = \phi_{xx}(0) = \phi_{xx}(L) = 0\}.$$

$G(U) = U^+$ and $P(t) = p(t, \cdot)$.

Example 1.3 We consider a system of Sine-Gordon equation occurring in the Josephson junctions (see M. Levi [10])

$$\begin{cases} U_{tt} + cU_t - d\Delta U + ksinU + k(U - V) = & p_1(t,x), \\ V_{tt} + cV_t - d\Delta V + ksinV + k(V - U) = & p_2(t,x), \\ U(t,x) = V(t,x) = 0 \text{ on } \partial\Omega \times \mathbb{R}. \end{cases} \tag{1.6}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. In this case we take $H = L^2(\Omega) \times L^2(\Omega)$, $A(\phi_1, \phi_2) = (-\Delta\phi_1, -\Delta\phi_2)$ with domain $D(A) = (H_0^1(\Omega) \cap H^2(\Omega))^2$.

A finite dimensional version of the system (1.1)($H = \mathbb{R}^n$ and $A - n \times n$ matrix) has been studied in [2], [9] and [8], where they proved the existence of a bounded solution of this equation, which is exponentially stable, and applied those results to the spatial discretization of the systems (1.4) and (1.5).

Under the above conditions, we prove that for some $c > 0$, $d > 0$ and $k > 0$ the equation (1.1) has one and only one bounded solution $u(t)$ which is exponentially asymptotically stable. Moreover, if $P(t)$ is almost periodic, then such a solution is also almost periodic(see Theorem 3.1 and Lemma 3.2 in section 3).

Our method is very simple, we just rewrite the equation (1.1) as a first orden system of abstract ordinary differential equations. Next, we prove that the linear part of this equation generates a C_0-group which decays exponentially to zero. After that, we use the variation constans formula for the mild solutions of (1.1) and some ideas from [14] [15] to find a formula for the bounded solutions of (1.1). From this formula we can prove the existence and the stability of the bounded solution easily. Finally, we prove the smoothness of this bounded solution(see Theorem 3.2).

2 Preliminaries Results

Before we prove the main Theorems of this paper, we shall prove some preliminaries results to be use in the next section. The equation (1.1) can be rewritten as a first order system of ordinary differential equations in the space $W = H \times H$ as follow:

$$w' + \mathcal{A}w + k\mathcal{G}(w) = \mathcal{P}(t), \quad w \in W, \quad t \in \mathbb{R}, \tag{2.1}$$

where $v = u'$ and

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{G} = \begin{pmatrix} 0 \\ G(u) \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 0 \\ P(t) \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} 0 & -I_H \\ dA & cI_H \end{pmatrix}. \tag{2.2}$$

is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times H$.

In this section we shall study the linear part of the equation (2.1); that is to say, the equation

$$w' + \mathcal{A}w = 0, \quad w \in W, \quad t \in \mathbb{R}. \tag{2.3}$$

To this end, we shall define the following family of complete orthogonal family of projections in W

$$\hat{E}_j = \begin{pmatrix} E_j & 0 \\ 0 & E_j \end{pmatrix}, \quad j = 1, 2, \ldots. \tag{2.4}$$

and consider the family of 2×2 matrices

$$B_j = \begin{pmatrix} 0 & -1 \\ d\lambda_j & c \end{pmatrix}. \tag{2.5}$$

Then, from (1.3) we get that

$$\mathcal{A}w = \sum_{j=1}^{\infty} B_j \hat{E}_j w, \quad w \in D(\mathcal{A}). \tag{2.6}$$

On the other hand, the eigenvalues of the matriz B_j are given by

$$\rho(j) = \frac{c \pm \sqrt{c^2 - 4d\lambda_j}}{2}, \quad j = 1, 2, \ldots, \tag{2.7}$$

which are simples if $c \neq 2\sqrt{d\lambda_j}, \quad j = 1, 2, \ldots$.

Therefore, there exists a complete system of orthogonal projections $\{P_i(j)\}_{i=1}^{2}$ in $\mathbb{R}^2$ such that

$$\begin{cases} B_j & = \rho_1(j)P_1(j) + \rho_2(j)P_2(j) \\ e^{B_j t} & = e^{\rho_1(j)t}P_1(j) + e^{\rho_2(j)t}P_2(j). \end{cases} \tag{2.8}$$

Moreover, we can compute these projections

$$\begin{aligned} P_1(j) &= \frac{1}{\sqrt{c^2 - 4d\lambda_j}}(B_j - \rho_2(j)I_{\mathbb{R}^2}), \\ P_2(j) &= \frac{-1}{\sqrt{c^2 - 4d\lambda_j}}(B_j - \rho_1(j)I_{\mathbb{R}^2}). \end{aligned}$$

Now, if we put

$$\begin{aligned} \alpha &= \alpha(c,d) = \max\left\{ \mathrm{Re}(\rho_j) = \mathrm{Re}\left(\frac{c \pm \sqrt{c^2 - 4d\lambda_j}}{2}\right) : j = 1, 2, \ldots. \ i = 1, 2. \right\}, \\ \beta &= \beta(c,d) = \min\left\{ \mathrm{Re}(\rho_j) = \mathrm{Re}\left(\frac{c \pm \sqrt{c^2 - 4d\lambda_j}}{2}\right) : j = 1, 2, \ldots. \ i = 1, 2. \right\}, \end{aligned}$$

we get the following estimates

$$\|e^{-B_j t}\| \leq e^{-\beta t}, \quad t \geq 0, \quad j = 1, 2, \ldots \tag{2.9}$$

$$\|e^{-B_j t}\| \leq e^{-\alpha t}, \quad t \leq 0, \quad j = 1, 2, \ldots. \tag{2.10}$$

Hence, from (2.9)-(2.10) we can easily prove that $-\mathcal{A}$ generates a C_0-group $\{e^{-\mathcal{A}t}\}_{t \in \mathbb{R}}$ given by

$$e^{-\mathcal{A}t}w = \sum_{j=1}^{\infty} e^{-B_j t}\hat{E}_j w \quad w \in W, \quad t \in \mathbb{R}. \tag{2.11}$$

Finally, putting $Q_i(j) = P_i(j)\hat{E}_j$ we obtain a complete system of orthogonal projections in W and

$$e^{-\mathcal{A}t}w = \sum_{j=1}^{\infty} \left\{ e^{\rho_1(j)t}Q_1(j)w + e^{\rho_2(j)t}Q_2(j)w \right\}, \quad w \in W, \quad t \in \mathbb{R}.$$

So, we have proved the following Theorem.

Theorem 2.1 *Suppose that $c \neq 2\sqrt{d\lambda_j}$, $j = 1, 2, \ldots$. Then, the operator $-\mathcal{A}$ generates a C_0-group $e^{-\mathcal{A}t}$ given by*

$$e^{-\mathcal{A}t}w = \sum_{j=1}^{\infty}\left\{e^{-\rho_1(j)t}Q_1(j)w + e^{-\rho_2(j)t}Q_2(j)w\right\}, \quad w \in W, \quad t \in I\!R, \tag{2.12}$$

where

$$\rho(j) = \frac{c \pm \sqrt{c^2 - 4d\lambda_j}}{2}, \quad j = 1, 2, \ldots. \tag{2.13}$$

and $\{Q_i(j) : i = 1, 2\}_{j=1}^{\infty}$ is a complete orthogonal system of projections in W.

Corollary 2.1 *The spectrum $\sigma(-\mathcal{A})$ of the matrix $-\mathcal{A}$ is given by*

$$\sigma(-\mathcal{A}) = \{\frac{-c \pm \sqrt{c^2 - 4d\lambda_j}}{2}, \quad j = 1, 2, \ldots\}.$$

Corollary 2.2 *Under the hypothesis of Theorem 2.1 we have that*

$$\|e^{-\mathcal{A}t}\| \leq e^{-\beta t}, \quad t \geq 0, \tag{2.14}$$

$$\|e^{-\mathcal{A}t}\| \leq e^{-\alpha t}, \quad t \leq 0, \tag{2.15}$$

where α, β are positive numbers and

$$-\beta = -\beta(c,d) = max\left\{Re(\rho_j) = Re\left(\frac{-c \pm \sqrt{c^2 - 4d\lambda_j}}{2}\right) : j = 1, 2, \ldots. \quad i = 1, 2.\right\},$$

$$-\alpha = -\alpha(c,d) = min\left\{Re(\rho_j) = Re\left(\frac{-c \pm \sqrt{c^2 - 4d\lambda_j}}{2}\right) : j = 1, 2, \ldots. \quad i = 1, 2.\right\}.$$

Proof From formula (2.12) we get that

$$\begin{aligned}
\|e^{-\mathcal{A}t}w\|^2 &= \sum_{j=1}^{\infty}\left\{e^{2\mathrm{Re}(-\rho_1(j))t}\|Q_1(j)w\|^2 + e^{2\mathrm{Re}(-\rho_2(j))t}\|Q_2(j)w\|^2\right\} \\
&\leq \sum_{j=1}^{\infty} e^{-2\beta t}\left\{\|Q_1(j)w\|^2 + \|Q_2(j)w\|^2\right\} \\
&= e^{-2\beta t}\|w\|^2, \quad w \in W, \quad t \geq 0.
\end{aligned}$$

Therefore, $\|e^{-\mathcal{A}t}\| \leq e^{-\beta t}$, $t \geq 0$. Analogously, we prove the other inequality. □

Corollary 2.3 *The initial value problem*

$$\begin{cases} w' + \mathcal{A}w = 0 \\ w(t_0) = w_0, \quad w_0 \in D(\mathcal{A}), \end{cases}$$

has the unique solution

$$w(t) = e^{-\mathcal{A}(t-t_0)}w_0 = \sum_{j=1}^{\infty}\left\{e^{-\rho_1(j)(t-t_0)}Q_1(j)w_0 + e^{-\rho_2(j)(t-t_0)}Q_2(j)w_0\right\}. \tag{2.16}$$

3 Main Results

In this section we shall prove the main Theorems of this paper, under the hypothesis of Theorem 2.1($c \neq 2\sqrt{d\lambda_j}$, $j = 1, 2, \ldots$).

Definition 3.1 (Mild Solution) For mild solution $w(t)$ of (2.1) with initial condition $w(t_0) = w_0 \in W$, we understand a function given by

$$w(t) = e^{-\mathcal{A}(t-t_0)} w_0 + \int_{t_0}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\} \, ds, \quad t \in I\!R. \tag{3.1}$$

We shall consider $W_b = C_b(I\!R, W)$ the space of bounded and continuous functions defined in $I\!R$ taking values in $W = H \times H$. W_b is a Banach space with suprem norm

$$\|w\|_b = \sup\{\|w(t)\|_W : t \in I\!R\}, \quad w \in W_b.$$

A ball of radio $\rho > 0$ and center zero in this space is given by

$$B_\rho^b = \{w \in W_b : \|w(t)\|_b \leq \rho, \quad t \in I\!R\}.$$

Lemma 3.1 *Let w be in W_b. Then, w is a mild solution of (2.1) if and only if w is given by*

$$w(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\} \, ds, \quad t \in I\!R. \tag{3.2}$$

Proof Suppose that w is a mild solution of (2.1). Then,

$$w(t) = e^{-\mathcal{A}(t-t_0)} w(t_0) + \int_{t_0}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\} \, ds, \quad t \geq t_0. \tag{3.3}$$

On the other hand, from (2.14) we obtain that

$$\|e^{-\mathcal{A}(t-t_0)} w(t_0)\| \leq e^{-\beta(t-t_0)} \|w(t_0)\|, \quad t \geq t_0,$$

and since $\|w(t)\| \leq M$, $t \in I\!R$, we get the following estimate

$$\|e^{-\mathcal{A}(t-t_0)} w(t_0)\| \leq M e^{-\beta(t-t_0)}, \quad t \geq t_0,$$

which implies that

$$\lim_{t_0 \to -\infty} \|e^{-\mathcal{A}(t-t_0)} w(t_0)\| = 0.$$

Therefore, passing to the limit in (3.3) when t_0 goes to $-\infty$ we conclude that

$$w(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\} \, ds, \quad t \in I\!R.$$

Suppose that w is a solution of the integral equation (3.2). Then

$$\begin{aligned} w(t) &= \int_{-\infty}^{0} e^{-\mathcal{A}(t-s)} \{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\} \, ds \\ &+ \int_{0}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\} \, ds. \end{aligned}$$

On the other hand, we have that

$$\| \int_{-\infty}^{0} e^{-\mathcal{A}(t-s)} \{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\} ds\| \leq \int_{-\infty}^{0} e^{-\beta(t-s)} \{kR_w + L_p\} ds = \frac{kR_w + L_p}{\beta},$$

where R_w and L_p are constants such that

$$\|\mathcal{G}(w(s))\| \leq R_w, \quad \|\mathcal{P}(s)\| \leq L_p, \quad s \in I\!R.$$

Hence, the following improper integral is well defined

$$w_0 = \int_{-\infty}^{0} e^{-\mathcal{A}(t-s)} \{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\} ds,$$

and

$$w(t) = e^{-\mathcal{A}(t-s)} w_0 + \int_{0}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\} ds.$$

This concludes the proof of the lemma. □

Theorem 3.1 *Consider a function G satisfying (1.2). Suppose $\rho > 0$ big enough such that*

$$0 < L_p + k\|G(0)\| = \sup_{s \in I\!R} \|P(s)\| + k\|G(0)\| < (\beta(c,d) - kL)\rho. \tag{3.4}$$

Then the equation (2.1) has one and only one mild solution $w_b(t)$ which belong to the ball B_ρ^b in W_b.

Moreover, this bounded solution is the only bounded solution of the equation (3.1) and is exponentially stable in the large.

Proof For the existence of such solution, we shall prove that the following operator has a unique fixed point in the ball B_ρ^b, $T : B_\rho^b \to B_\rho^b$

$$(Tw)(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} \{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\} ds, \quad t \in I\!R.$$

In fact, for $w \in B_\rho^b$ we have

$$\|Tw(t)\| \leq \int_{-\infty}^{t} e^{-\beta(t-s)} \{kL\|w(s)\| + k\|G(0)\| + L_p\} \leq \frac{(kL)\rho + k\|G(0)\| + L_p}{\beta}.$$

The condition (3.4) implies that

$$kL\rho + k\|G(0)\| + L_p < \beta\rho \iff \frac{kL\rho + k\|G(0)\| + L_p}{\beta} < \rho.$$

Therefore, $Tw \in B_\rho^b$ for all $w \in B_\rho^b$.

Now, we shall see that T is a contraction mapping. In fact, for all $w_1, w_2 \in B_\rho^b$ we have that

$$\|Tw_1(t) - Tw_2(t)\| \leq \int_{-\infty}^{t} e^{-\beta(t-s)} kL\|w_1(s) - w_2(s)\| ds \leq \frac{kL}{\beta}\|w_1 - w_2\|_b, \quad t \in I\!R.$$

Hence,

$$\|w_1 - Tw_2\|_b \leq \frac{kL}{\beta}\|Tw_1 - w_2\|_b, \quad w_1, w_2 \in B^b_\rho.$$

The condition (3.4) implies that

$$0 < \beta - kL \iff kL < \beta \iff \frac{kL}{\beta} < 1.$$

Therefore, T has a unique fixed point w_b in B^b_ρ

$$w_b(t) = (Tw_b)(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)}\left\{-k\mathcal{G}(w_b(s)) + \mathcal{P}(s)\right\} ds, \quad t \in I\!R,$$

From Lemma 3.1, w_b is a bounded solution of the equation (3.1). Since condition (3.4) holds for any $\rho > 0$ big enough independent of $kL < \beta(c,d)$, then w_b is the unique bounded solution of the equation (3.1).

To prove that $w_b(t)$ is exponentially stable in the large, we shall consider any other solution $w(t)$ of (3.1) and consider the following estimate

$$\begin{aligned} \|w(t) - w_b(t)\| &\leq \|e^{-\mathcal{A}t}(w(0) - w_b(0)) + \int_0^t e^{-\mathcal{A}(t-s)}\left\{k\mathcal{G}(w(s)) - k\mathcal{G}(w_b(s))\right\} ds\| \\ &\leq e^{-\beta t}\|(w(0) - w_b(0))\| + \int_0^t e^{-\beta(t-s)}kL\|w(s) - w_b(s)\|ds. \end{aligned}$$

Then,

$$e^{\beta t}\|w(t) - w_b(t)\| \leq \|(w(0) - w_b(0))\| + \int_0^t e^{\beta s}kL\|w(s) - w_b(s)\|ds.$$

Hence, applying the Gronwall's inequality we obtain

$$\|w(t) - w_b(t)\| \leq e^{(kL_\rho - \beta)t}\|(w(0) - w_b(0))\|, \quad t \geq 0.$$

From (3.4) we get that $kL - \beta < 0$ and therefore $w_b(t)$ is exponentially stable in the large. □

Corollary 3.1 *The bounded solution $w_b(\cdot, P)$ of (3.1) given by Theorem 3.1 depends continuously on $P \in C_b(I\!R, H)$.*

Proof Let $P_1, P_2 \in C_b(I\!R, H)$ and $w_b(\cdot, P_1), w_b(\cdot, P_2)$ be the bounded functions given by Theorem 3.1. Then

$$\begin{aligned} w_b(\cdot, P_1) - w_b(\cdot, P_2) &= \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)}k\left\{\mathcal{G}(w_b(s, P_2)) - \mathcal{G}(w_b(s, P_1))\right\} ds \\ &+ \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)}k\left\{\mathcal{P}_1(s) - \mathcal{P}_1(s)\right\} ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \|w_b(\cdot, P_1) - w_b(\cdot, P_2)\|_b &\leq \frac{kL}{\beta}\|w_b(\cdot, P_1) - w_b(\cdot, P_2)\|_b \\ &+ \frac{1}{\beta}\|P_1 - P_2\|_b. \end{aligned}$$

Hence,

$$\|w_b(\cdot,P_1)-w_b(\cdot,P_2)\|_b \leq \frac{1}{\beta-kL}\|P_1-P_2\|_b.$$

□

We conclude this part with the following lemma about almost periodicity of the bounded solutions of the equation (3.1).

Lemma 3.2 *If $P(t)$ is almost periodic, then the unique bounded solution of the equation (3.1) given by Theorem 3.1 is also almost periodic.*

Proof To prove this lemma, we shall use the following well known fact, due to S. Bohr. A function $f \in C(\mathbb{R}; W)$ is almost periodic (a.p according to S. bohr) if and only if the Hull $H(f)$ of f is compact in the topology of uniform convergence.

Where $H(f)$ is the closure of the set of translates of f under the topology of uniform convergence

$$H(f)=\overline{\{f_\tau : \tau \in \mathbb{R}\}}, \quad f_\tau(t)=f(t+\tau), \ \ t\in \mathbb{R}.$$

Since the limit of a uniformly convergent sequence of a.p. functions is a.p., then the set A_ρ of a.p. functions in the ball B_ρ^b is closed, where ρ is given by Theorem 3.1.

Claim. The contraction mapping T given in Theorem 3.1 leaves A_ρ invariant. In fact; if $w\in A_\rho$, then $f(t)=-k\mathcal{G}(w(t))+\mathcal{P}(t)$ is also an a.p. function. Now, consider the function

$$\begin{aligned} F(t)=(Tw)(t) &= \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)}\{-k\mathcal{G}(w(s))+\mathcal{P}(s)\}\,ds \\ &= \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} f(s)ds, \quad t\in\mathbb{R}. \end{aligned}$$

Then, it is enough to establish that $H(F)$ is compact in the topology of uniform convergence. Let $\{F_{\tau_k}\}$ be any sequence in $H(F)$. Since f is a.p. we can select from $\{f_{\tau_k}\}$ a Cauchy subsequence $\{f_{\tau_{k_j}}\}$, and we have that

$$\begin{aligned} F_{\tau_{k_j}}(t)=F(t+\tau_{k_j}) &= \int_{-\infty}^{t+\tau_{k_j}} e^{-\mathcal{A}(t+\tau_{k_j}-s)} f(s)ds \\ &= \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)} f(s+\tau_{k_j})ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|F_{\tau_{k_j}}(t)-F_{\tau_{k_i}}(t)\| &\leq \int_{-\infty}^{t} \|e^{-\mathcal{A}(t+-s)}\|\|f(s+\tau_{k_j})-f(s+\tau_{k_i})\|ds \\ &\leq \|f_{\tau_{k_j}}-f_{\tau_{k_i}}\|_b\int_{-\infty}^{t} e^{-\beta(t-s)}ds=\frac{1}{\beta}\|f_{\tau_{k_j}}-f_{\tau_{k_i}}\|_b. \end{aligned}$$

Therefore, $\{F_{\tau_{k_j}}\}$ is a Cauchy sequence. So, $H(F)$ is compact in the topology of uniform convergence, F is a.p. and $TA_\rho \subset A_\rho$.

Now, the unique fixed point of T in the ball B_ρ^b lies in A_ρ. Hence, the unique bounded solution $w_b(t)$ of the equation (3.1) given in Theorem 3.1 is also almost periodic. □

3.1 Smoothness of the Bounded Solution

In this part, we shall prove that the bounded solution $w_b(t)$ of the equation (3.1)(mild solution of (2.1)) is also a solution of the equation (2.1).

Theorem 3.2 *Under the hypotheses of Theorem 3.1. If G and P are of C^1 class, then $w_b(t)$ satisfies (2.1) and*

$$w_b(\cdot) \in C_b(\mathbb{R}; D(\mathcal{A})).$$

Moreover, if $w_b(t) = (u_b(t), v_b(t))^T$, then $v_b(t) = u_b'(t)$,

$$u_b \in C_b(\mathbb{R}; D(A)), \quad u_b' \in C_b(\mathbb{R}; D(A^{\frac{1}{2}})), \quad u_b'' \in C_b(\mathbb{R}; H),$$

and

$$u_b'' + cu_b' + dAu_b + kG(u_b) = P(t), \quad t \in \mathbb{R}.$$

Proof Define the function $F(t) = -k\mathcal{G}(w_b(t)) + \mathcal{P}(t)$. Then

$$F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix},$$

where $f(t) = -kG(u_b(t)) + P(t), \quad w_b(t) = (u_b(t), v_b(t))^T$ and $f \in C_b(\mathbb{R}; H)$.

Now, consider the second order equation

$$u'' + cu' + dAu = f(t), \quad u \in H, \quad t \in \mathbb{R}. \tag{3.5}$$

Then, from Proposition 1.3 in [13], pg 182, the equation (3.5) admits a unique solution u which satisfies

$$u \in C_b(\mathbb{R}; H), \quad u' \in C_b(\mathbb{R}; H).$$

Therefore, $w(t) = (u(t), u'(t))^T$ is a bounded solution of the integral equation

$$w(t) = e^{-\mathcal{A}(t-t_0)}w(t_0) + \int_{t_0}^{t} e^{-\mathcal{A}(t-s)}F(s)ds, \quad t \in \mathbb{R}.$$

Then, taking limit as t_0 goes to $-\infty$ we get that

$$w(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)}F(s)ds = \int_{-\infty}^{t} e^{-\mathcal{A}(t-s)}\left\{-k\mathcal{G}(w(s)) + \mathcal{P}(s)\right\}ds.$$

Hence, $w_b(t) = w(t) = (u(t), u'(t))$. so,

$$f(t) = -kG(u(t)) + P(t) \quad \text{and} \quad f \in C_b^1(\mathbb{R}; H).$$

Then, using the second part of Proposition 1.3 in [13], pg 182, we get that

$$u \in C_b(\mathbb{R}; D(A)), \quad u' \in C_b(\mathbb{R}; D(A^{\frac{1}{2}})) \quad \text{and} \quad u'' \in C_b(\mathbb{R}; H).$$

□

As an application of these results we can consider the Sin-Gordon equation with Dirichlet boundary condition (1.4). In the same way, one can consider many others examples like (1.5) and (1.6).

Corollary 3.2 *If in the system (1.4) the function $t \to p(t,\cdot) \in L^2(0,L)$ is of C^1 class, then for some values of c, d and k the system (1.4) admits a unique solution u such that*

$$u \in C_b(\mathbb{R}; H^2 \cap H_0^1), \quad u' \in C_b(\mathbb{R}; H_0^1) \quad \textit{and} \quad u'' \in C_b(\mathbb{R}; L^2(0,L)).$$

References

[1] J.M. ALONSO AND R. ORTEGA, "Boundedness and global asysmptotic stability of a forced oscillator", Nonlinear Anal. **25** (1995), 297-309.

[2] J.M. ALONSO AND R. ORTEGA, "Global asysmptotic stability of a forced newtonian system with dissipation", J. Math. Anal. and Applications **196** (1995), 965-986.

[3] J.M. ALONSO, J. MAWHIN AND R. ORTEGA, "Bounded solutions of second order semilinear evolution equations and applications to the telegraph equation", Rapport n0 284-Juillet (1998). Séminaire Mathématique(nouvelle série)

[4] Y.S. CHOI, K.C. JEN, AND P.J. McKENNA, " The structure of the solution set for periodic oscillations in a suspension bridge model", IMA J. Appl. Math. **47** (1991), 283-306.

[5] J. GLOVER, A.C. LAZER AND P.J. McKENNA, " Existence and stability of large scale nonlinear oscillations in suspension bridges" J. Appl. Math. Phys. **40** (1989), 172-200.

[6] J.K.HALE "Ordinary Differential Equations", Pure and Applied Math. Vol. XXI, Wiley-Interscience(1969).

[7] R. ORTEGA, "A boudedness result of Landesman-Lazer type, Differential Integral Equations", J. Math. Anal. and Applications **8** (1995), 729-734.

[8] L. GARCIA and H. LEIVA, "Center Manifold and Exponentially Bounded Solutions of a Forced Newtonian System with Dissipation "(to apear).

[9] H. LEIVA, "Existence of Bounded Solutions of a Second Order System with Dissipation"(to apear).

[10] M. LEVI, "Beating Modes in the Josephson Junction", in Chaos in Nonlinear Dyanamical Systmes, J. Chandra (Ed), SIAM, Philadelphia(1984).

[11] J. MAWHIN AND J.R.Jr. WARD, "Bounded solutions of some nonlinear differential equations", Rapport n0 262-June 1996. Séminaire Mathématique(nouvelle série). Inst. de Math. Pure et Appl. Univ. Cath. de Louvain.

[12] J. MAWHIN, "Bounded solutions of nonlinear ordinary differential equations", Recherches de mathématique n0 54, June 1996. Inst. de Math. Pure et Appl. Univ. Cath. de Louvain.

[13] R. TEMAM, "Infinite-dimensional dynamical systems in mechanics and physic" Applied Math. Sciences, vol. 68, Springer-Verlag(1988).

[14] A. VANDERBAUWHEDE, "Center Manifolds, Normal Forms and Elementary Bifurcations" Dyns. Reported (2), 89-170(1987).

[15] A. VANDERBAUWHEDE "Center Manifolds, Normal and Contractions on a Scale of B-Spaces", J. Funct. Analy. 72, 209-224(1987).

Universidad de los Andes Facultad de Ciencias Departamento de Matemática Mérida 5101-Venezuela.
E-mail: hleiva@ciens.ula.ve