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LINEAR REACTION-DIFFUSION SYSTEMS

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Abstract

In this paper we study a linear reaction-diffusion system of the form:

$$u_t(t,x) = D\Delta u(t,x) + Bu(t,x), \quad t > 0,$$

where x belong to a domain $\Omega \subset I\!R^N$ and subject to the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. The main point, is that the $n \times n$ matrices D and B are not necessarily diagonalizable, but the eigenvalues $\lambda \in \boldsymbol{C}$ of D are assuming to be such that $\mathrm{Re}\lambda \geq 0$ (some of them could have zero real part). We generalize a result from [3] where they assume that the eigenvalues λ of D have strictly positive real parts ($\mathrm{Re}\lambda > 0$). Under our condition($\mathrm{Re}\lambda \geq 0$) we prove that this system generates an C_0-semigroup in an appropriate Hilbert space. Finally, we apply our result to the following linear thermoelastic bar problem

$$\begin{cases} u_{tt} - au_{xx} + b\theta_x &= 0, \quad 0 < x < 1, \quad t > 0, \\ \theta_t - \theta_{xx} + bu_{tx} &= 0, \quad 0 < x < 1, \quad t > 0, \end{cases}$$

with some Dirichlet boundary conditions.

Key words: reaction diffusion system, C_0-semigroup, energy decay.

AMS(MOS) subject classifications. primary: 34G10; secondary: 35B40.

1 Introduction

In this paper we shall study the following reaction-diffusion system with Dirichlet boundary conditions

$$\begin{cases} u_t &= D\Delta u + Bu, \quad t > 0, \quad u \in I\!R^n, \\ u &= 0, \quad t > 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $I\!R^N$, D and B are $n \times n$ matrices. One of the points that make this work different from the work done by many authors, is that most of them assume that the diffusion matrix D is diagonal with positive entries(see [4,8,13]). However, cross-diffusion phenomena are not uncommon, for the case that $B = 0$ one can find in [3] several mathematical models in which D is not even diagonalizable, but the eigenvalues $\lambda \in \boldsymbol{C}$ of the matrix D are assuming to be such that $\mathrm{Re}(\lambda) > 0$. Here, we consider a matrix D so general that it could have aeigenvalues with zero real part. We impose some condition on D and B, which is more general than the others authors conditions. Namely, our main hypothesis is:

H) If $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ are the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary conditions, then we will assume the existence of a continuous function $g : [0, \infty] \to I\!R$ such that

$$\|e^{(-\lambda_n D+B)t}\| \leq g(t), \quad t \geq 0, \quad n = 1, 2, 3, \ldots. \tag{1.2}$$

A particular case is $g(t) = Me^{Rt}$, $M \geq 1$, $R \in I\!R$. Under the hypothesis (1.2) we prove that this system (1.1) generates a C_0-semigroup in the Hilbert space $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$; that is to say, the equation (1.1) is well posed.

Finally; as an example, we study the following linear thermoelastic bar problem

$$\begin{cases} u_{tt} - au_{xx} + b\theta_x = 0, & 0 < x < 1, \quad t > 0, \\ \theta_t - \theta_{xx} + bu_{tx} = 0, & 0 < x < 1, \quad t > 0, \end{cases} \tag{1.3}$$

with the following boundary and initial conditions

$$u(t,0) = u(t,1) = u_t(t,0) = u_t(t,1) = \theta_x(t,0) = \theta_x(t,1) = 0, \tag{1.4}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \text{ and } \theta_x(0,x) = \theta_0(x) \quad 0 < x < 1. \tag{1.5}$$

This problem has been studied by many authors, in [7] Kim show under some kind of Dirichlet boundary conditions that the energy of the solutions of this problem decays exponentially fast, he used the energy method, combined with a multiplier technique and compactness property. In [5] Hansen studied a similar problem with several natural boundary conditions. He shows that the system (1.3)-(1.4) generates a strongly continuous semigroup that can be expanded through a Riesz basis on the Hilbert spaces of finite energy states and that this energy decays exponentially for some kind of Dirichlet boundary conditions.

Here we treat the problem (1.3)-(1.4) as the equation (1.1) in which the matrix D has zero as one of its eigenvalues; we prove that the system (1.3)-(1.4) generates a strongly continuous semigroup. That is to say, this problem is well posed. Also, we prove that the eigenvalues of the infinitesimal generator of this semigroup have negative real part bounded away from zero, which implies the energy decay. Finally, our method can be apply to others mathematical models, see for example [9,10,12].

2 Preliminaries

In this section we shall choose the space where this problem will be set and consider some notations.

Let $Z = (L^2(\Omega))^n = L^2(\Omega, I\!R^n)$ be the Hilbert space of the square integrable functions $u : \Omega \to I\!R^n$ with the usual inner product.

Now, we shall use some notation from [4 and 8] to write the system (1.1) as an abstract ordinary differential equation in the space Z.

Let $H = L^2(\Omega) = L^2(\Omega, I\!R)$ and $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ be the eigenvalues of $-\Delta$ with Dirichlet boundary conditions, each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Therefore

a) there exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvector of $-\Delta$.

b) for all ξ in $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ we have

$$-\Delta\xi = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j \xi, \tag{2.1}$$

where $< \cdot, \cdot >$ is the inner product in H and

$$E_j \xi = \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k}. \tag{2.2}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in H and

$$\xi = \sum_{j=1}^{\infty} E_j \xi, \quad \xi \in H.$$

c) Δ generates an analytic semigroup $\{e^{\Delta t}\}$ given by

$$e^{\Delta t}\xi = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j \xi. \tag{2.3}$$

Now, we define the following operators

$$A : D(A) \subset Z \to Z, \quad A\psi = -D\Delta\psi, \tag{2.4}$$

with

$$D(A) = H^2(\Omega, \mathbb{R}^n) \cap H_0^1(\Omega, \mathbb{R}^n)$$

and

$$\mathcal{B} : Z \to Z, \quad \mathcal{B}z(x) = Bz(x), \quad x \in \Omega.$$

Also, we shall consider the following family of complete orthogonal projections in Z.

$$P_j = \text{diag}(E_j, E_j, \cdots, E_j) = \begin{pmatrix} E_j & 0 & \dots & 0 \\ 0 & E_j & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E_j \end{pmatrix}. \tag{2.5}$$

Therefore, for $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j D P_j z, \tag{2.6}$$

and

$$z = \sum_{j=1}^{\infty} P_j z, \quad \|z\|^2 = \sum_{j=1}^{\infty} \|P_j z\|^2, \quad z \in Z. \tag{2.7}$$

3 Abstract Formulation of the Problem

With the above notation the system (1.1) can be written as an abstract linear ordinary differential equation in the Hilbert space Z as follows:

$$\begin{cases} z' &= -Az + \mathcal{B}z, \quad t > 0, \\ z(0) &= z_0, \quad z_0 \in D(A). \end{cases} \tag{3.1}$$

Theorem 3.1 *The operator $-A + \mathcal{B}$ generates an C_0-semigroup $T = \{T(t)\}_{t\geq 0}$ given by*

$$T(t)z = \sum_{j=1}^{\infty} e^{(-\lambda_j D + B)t} P_j z, \quad t \geq 0, \quad z \in Z \tag{3.2}$$

with domain $D(-A + \mathcal{B}) = D(A)$.

Proof Suppose that $z(t)$ is a solution of the initial value problem (2.7). Then, using the orthogonal projections $\{P_j\}_{j\geq 1}$ given by the formula (2.5) we obtain

$$z(t) = \sum_{j=1}^{\infty} P_j z(t) = \sum_{j=1}^{\infty} z_j(t), \quad t \geq 0 \tag{3.3}$$

where z_j is the solution of the finite dimensional ordinary differential equation

$$\begin{cases} y' = (-\lambda_j D + B)y, \quad t \in \mathbb{R}, \\ y(0) = P_j z_0. \end{cases} \tag{3.4}$$

Therefore, $z_j(t) = e^{(-\lambda_j D + B)t} P_j z_0$. Then, from (3.3) and our main hypothesis (1.2) we obtain that

$$z(t) = \sum_{j=1}^{\infty} e^{(-\lambda_j D + B)t} P_j z_0, \quad t \geq 0, \quad z_0 \in Z \tag{3.5}$$

On the other hand, it is not hard to show that the family of operator

$$T(t)z = \sum_{j=1}^{\infty} e^{(-\lambda_j D + B)t} P_j z, \quad t \geq 0, \quad z \in Z,$$

is a C_0-semigroup whose generator is $-A + \mathcal{B}$ with $D(-A + \mathcal{B}) = D(A)$. □

Corolary 3.1 *The spectrum $\sigma(-A + \mathcal{B})$ of the operator $-A + \mathcal{B}$ is given by*

$$\sigma(-A + \mathcal{B}) = \bigcup_{j=1}^{\infty} \sigma(-\lambda_j D + B) \tag{3.6}$$

Corolary 3.2 *For all $z_0 \in D(A)$ the initial value problem (2.7) has a unique solution given by*

$$z(t) = \sum_{j=1}^{\infty} e^{(-\lambda_j D + B)t} P_j z_0, \quad t \geq 0 \tag{3.7}$$

4 Application

In this section we shall apply Theorem 3.1 to study the existence of the solution of the linear thermoelastic bar problem (1.3)-(1.5), where $a > 0$ and $b \neq 0$ are constants u, θ denote the displacement and the temperature respectively.

Now, making the following change of variables $v = u_t$ and $w = \theta_x$ we can write (1.3)-(1.5) as follows

$$\begin{cases} u_t = v \\ v_t = au_{xx} - bw \\ \theta_t = w_x - bv_x \end{cases} \Leftrightarrow \begin{cases} u_t = v \\ v_t = au_{xx} - bw \\ \theta_{tx} = w_{xx} - bv_{xx} \end{cases} \Leftrightarrow \begin{cases} u_t = v \\ v_t = au_{xx} - bw \\ w_t = w_{xx} - bv_{xx} \end{cases}$$

Therefore, system (1.3)-(1.4) can be written as (1.1) in $\mathbb{R}^3$

$$\begin{cases} z_t = D\Delta z + Bz, \quad t > 0 \qquad 0 < x < 1 \\ z(t,0) = z(t,1) = 0 \\ z(0,x) = z_0(x), \quad 0 < x < 1 \end{cases} \tag{4.1}$$

where

$$z = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad z_0 = \begin{pmatrix} u_0 \\ u_1 \\ \theta_0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & -b & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -b \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4.1 *For u_0, u_1, $\theta_0 \in H^2 \cap H_0^1$ there exists a unique solution (u,θ) of (1.3)-(1.5) such that*

$$u,\ u_t,\ \theta_x \in C^1\left(0,\infty; H^2 \cap H_0^1\right).$$

Moreover, there exists $R > 0$ such that

$$E(u,\theta)(t) \leq e^{-Rt} E(u,\theta)(0), \quad t \geq 0$$

where $E(u,\theta)$ is the energy function given by

$$E(u,\theta)(t) = \|u(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|_2^2 + \|\theta_x(t,\cdot)\|_2^2.$$

Proof In order to apply Theorem 3.1, we shall verify the main hypothesis (H). In fact, the eigenvalues of D are $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$. Now, we shall verify (1.2), for that purpose we consider the matrix $A_n = -\lambda_n D + B$ and compute the spectrum $\sigma(A_n)$ of this matrix, which is given by

$$A_n = \begin{pmatrix} 0 & 1 & 0 \\ -\lambda_n a & 0 & -b \\ 0 & \lambda_n b & -\lambda_n \end{pmatrix}.$$

Therefore, the characteristic equation of A_n is

$$\lambda^3 + \lambda_n \lambda^2 + \lambda_n\left(a + b^2\right)\lambda + \lambda_n^2 a = 0 \tag{4.2}$$

Hence, from Routh-Hurwitz Theorem we get that the roots of this equation have negative real parts. It is easy to see that these roots are simple. Then, there exists a family of orthogonal projections $\{q_i(n)\}_{i=1}^{3}$ on $\mathbb{R}^3$ which are complete and

$$e^{A_n t}y = \sum_{i=1}^{3} e^{\alpha_i(n)t} q_i(n)\, y, \quad t \in \mathbb{R}, \quad y \in \mathbb{R}$$

where $\alpha_i(n)$'s are the eigenvalues of A_n.

Therefore we can find $L \geq 0$(L could be zero) such that

$$\mathrm{Re}\,(\alpha_i(n)) \leq -L, \quad i = 1, 2, 3; \quad n = 1, 2, \dots \tag{4.3}$$

Hence,

$$\left\| e^{A_n t} y \right\|^2 = \sum_{i=1}^{3} e^{2\mathrm{Re}(\alpha_i(n))t} \left\| q_i(n)\, y \right\|^2 \leq \sum_{i=1}^{3} e^{-2Lt} \left\| q_i(n) \right\|^2 \leq e^{-2Lt} \left\| y \right\|^2$$

and (1.2) is verified.

Therefore, the system (4.1) generates a C_0-semigroup $\{T(t)\}_{t \geq 0}$ given by

$$T(t)\,\xi = \sum_{n=1}^{\infty} e^{A_n t} P_n \xi, \quad \xi \in Z, \quad t \geq 0. \tag{4.4}$$

Now, we shall prove that the eigenvalues of the infinitesimal generator of this semigroup are bounded away from zero, which implies the energy decay of the solutions of (1.3)-(1.4). The characteristic equation (4.2) of A_n can be written as follows

$$\left(\frac{\lambda}{\lambda_n}\right)^3 + \left(\frac{\lambda}{\lambda_n}\right)^2 + \frac{1}{\lambda_n}\left(a + b^2\right)\left(\frac{\lambda}{\lambda_n}\right) + \frac{a}{\lambda_n} = 0. \tag{4.5}$$

Next, putting $z = \frac{\lambda}{\lambda_n}$, $a_n = \frac{a+b^2}{\lambda_n}$ and $b_n = \frac{a}{\lambda_n}$ we obtain the equation

$$z^3 + z^2 + a_n z + b_n = 0 \tag{4.6}$$

with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.

Making the change of variable $y = z + \frac{1}{3}$ we get the cubic equation

$$y^3 + 3p_n y + 2q_n = 0 \tag{4.7}$$

where

$$2q_n = \frac{2}{27} - \frac{a_n}{3} + b_n, \quad 3p_n = \frac{3a_n - 1}{3}$$

Hence,

$$\lim_{n\to\infty} q_n = \frac{1}{27}, \quad \lim_{n\to\infty} p_n = -\frac{1}{9} \quad \text{and,} \quad \lim_{n\to\infty} q_n^2 + p_n^3 = 0.$$

Also, $p_n < 0$ and

$$\begin{aligned} D_n = q_n^2 + p_n^3 &= -\left(\tfrac{27}{2624}a_n^2 + \tfrac{a_n b_n}{6}\right) + \tfrac{b_n}{27} + \tfrac{b_n^2}{4} + \tfrac{a_n^3}{27} \\ &= \tfrac{1}{\lambda_n}\left\{-\tfrac{27}{2624}\tfrac{(a+b^2)^2}{\lambda_n} - \tfrac{(a+b^2)a}{6\lambda_n} + \tfrac{a}{27} + \tfrac{a^2}{4\lambda_n} + \tfrac{(a+b^2)^3}{27\lambda_n^2}\right\} > 0. \end{aligned} \tag{4.8}$$

Then following formulas for the roots $y_1(n)$, $y_2(n)$, $y_3(n)$ of the equation (4.7) can be found in [2] pg. 156.

$$\begin{cases} y_1(n) = -2r_n \cosh\frac{\varphi_n}{3}, \\ y_2(n) = r_n \cosh\frac{\varphi_n}{3} + i\sqrt{3}r_n \sinh\frac{\varphi_n}{3}, \\ y_3(n) = r_n \cosh\frac{\varphi_n}{3} - i\sqrt{3}r_n \sinh\frac{\varphi_n}{3}, \end{cases} \tag{4.9}$$

where $r_n = \sqrt{|p_n|}$ and $cosh\varphi_n = \frac{q_n}{r_n^3}$. Then

$$\lim_{n\to\infty} r_n = \frac{1}{3} \quad \text{and} \quad \lim_{n\to\infty} \varphi_n = 0.$$

Cardano's formula:

$$\begin{cases} y_1(n) &= \sqrt[3]{-q_n + \sqrt{D_n}} + \sqrt[3]{-q_n - \sqrt{D_n}} \\ y_j(n) &= \frac{-1}{2}\left\{\sqrt[3]{-q_n + \sqrt{D_n}} + \sqrt[3]{-q_n - \sqrt{D_n}}\right\} \\ & \quad \pm i\frac{\sqrt{3}}{2}\left\{\sqrt[3]{-q_n + \sqrt{D_n}} - \sqrt[3]{-q_n - \sqrt{D_n}}\right\}, \quad j = 1, 2. \end{cases} \tag{4.10}$$

Hence, the roots of the equation (4.6) are given by

$$z_i(n) = y_i(n) - \frac{1}{3}, \quad i = 1, 2, 3; \quad n = 1, 2, 3, \ldots$$

with $\mathrm{Re}(z_i(n)) < 0$, and the roots $\rho_1(n)$, $\rho_2(n)$, $\rho_3(n)$ of the equation (4.2) are given by

$$\rho_i(n) = \lambda_n z_i(n), \quad i = 1, 2, 3; \quad n = 1, 2, 3, \ldots \tag{4.11}$$

From (4.9) and (4.10) we get the followig result.

Claim 4.1.

$$\begin{cases} A) \lim_{n\to\infty} \rho_1(n) = -\infty, \\ B) \lim_{n\to\infty} \mathrm{Im}(\rho_2(n)) = -\infty, \\ C) \lim_{n\to\infty} \mathrm{Re}(\rho_2(n)) = -\frac{b^2}{2}. \end{cases}$$

In fact, A) is not hard to prove. Let us prove part B).

$$\begin{aligned} \lim_{n\to\infty} \mathrm{Im}(\rho_2(n)) &= \frac{-\lambda_n\sqrt{3}}{2}\left\{\sqrt[3]{-q_n + \sqrt{D_n}} - \sqrt[3]{-q_n - \sqrt{D_n}}\right\} \\ &= \frac{-\lambda_n\sqrt{3}}{2} \cdot \frac{2\sqrt{D_n}}{\sqrt[3]{(-q_n + \sqrt{D_n})^2} - \sqrt[3]{-q_n^2 + D_n} + \sqrt[3]{(q_n + \sqrt{D_n})^2}} = -\infty. \end{aligned}$$

Part C). We shall use the following facts:

$$\lim_{n\to\infty} \cosh\frac{\varphi_n}{3} = 1 \quad \text{and} \quad \cosh\frac{\varphi_n}{3} \equiv 1+\frac{\varphi_n^2}{9\times 2},$$

$$\varphi_n = \ln\left(\frac{q_n}{r_n^3}+\sqrt{\left(\frac{q_n}{r_n^3}\right)^2-1}\right) \equiv \frac{q_n}{r_n^3}+\sqrt{\left(\frac{q_n}{r_n^3}\right)^2-1}-1,$$

Now, we are ready to compute this limit:

$$\begin{aligned}\lim_{n\to\infty}\operatorname{Re}(\rho_2(n)) &= \lim_{n\to\infty}\left\{\lambda_n r_n \cosh\frac{\varphi_n}{3}-\frac{\lambda_n}{3}\right\}\\ &= \lim_{n\to\infty}\lambda_n\left\{r_n-\frac{1}{3}\right\}+\lim_{n\to\infty}\lambda_n r_n\left\{\cosh\frac{\varphi_n}{3}-1\right\} = L_1+L_2.\end{aligned}$$

Then

$$L_1 = \lim_{n\to\infty}\lambda_n\left\{\sqrt{\frac{1}{9}-\frac{a+b^2}{3\lambda_n}}-\frac{1}{3}\right\} = -\frac{(a+b^2)}{2}.$$

and

$$\begin{aligned}L_2 &= \lim_{n\to\infty}\frac{\lambda_n r_n\varphi_n^2}{2\times 9} = \frac{1}{2\times 9}\left(\lim_{n\to\infty}\sqrt{\lambda_n r_n}\varphi_n\right)^2\\ &= \frac{1}{2\times 9}\left(\lim_{n\to\infty}\sqrt{\lambda_n r_n}\left\{\frac{q_n}{r_n^3}+\sqrt{\left(\frac{q_n}{r_n^3}\right)^2-1}-1\right\}\right)^2\\ &= \frac{1}{2\times 9}\left(\lim_{n\to\infty}\frac{-2\sqrt{\lambda_n r_n}\sqrt{\left(\frac{q_n}{r_n^3}\right)^2-1}}{-2}\right)^2\\ &= \frac{1}{2\times 9}\lim_{n\to\infty}\lambda_n r_n\left\{\left(\frac{q_n}{r_n^3}\right)^2-1\right\}\\ &= \frac{1}{2\times 9}\lim_{n\to\infty}\lambda_n r_n\left\{\left(\frac{q_n}{r_n^3}-1\right)\left(\frac{q_n}{r_n^3}+1\right)\right\}\\ &= \frac{1}{9}\lim_{n\to\infty}\frac{\lambda_n(q_n-r_n^3)}{r_n^2} = \lim_{n\to\infty}\lambda_n(q_n-r_n^3)\\ &= \lim_{n\to\infty}\lambda_n\left\{\frac{1}{27}-\frac{a+b^2}{6\lambda_n}+\frac{a}{2\lambda_n}-\sqrt{\left(\frac{1}{9}-\frac{a+b^2}{3\lambda_n}\right)^3}\right\}\end{aligned}$$

$$= -\frac{a+b^2}{6}+\frac{a}{2}+\lim_{n\to\infty}\lambda_n\left\{\frac{1}{27}-\sqrt{\left(\frac{1}{9}-\frac{a+b^2}{3\lambda_n}\right)^3}\right\}$$

$$= -\frac{a+b^2}{6}+\frac{a}{2}+\lim_{n\to\infty}\lambda_n\left\{\frac{\left(\frac{1}{27}\right)^2-\left(\frac{1}{9}-\frac{a+b^2}{3\lambda_n}\right)^3}{\frac{2}{27}}\right\}$$

$$= -\frac{a+b^2}{6}+\frac{a}{2}+\lim_{n\to\infty}\frac{\lambda_n 3(\frac{1}{9})^2\frac{a+b^2}{3\lambda_n}}{\frac{2}{27}}$$

$$= -\frac{a+b^2}{6}+\frac{a}{2}+\frac{a+b^2}{6}=\frac{a}{2}.$$

Therefore,

$$\lim_{n\to\infty}\mathrm{Re}(\rho_2(n)=L_1+L_2=-\frac{b^2}{2}.$$

This completes the proof of the Claim.

Then, there exists $\beta>0$ such that

$$\mathrm{Re}\left(\rho_i\left(n\right)\right)\le-\beta,\quad i=1,2,3;\quad n=1,2,3,\ldots,$$

and

$$\|T\left(t\right)\|\le e^{-\beta t},\quad t\ge 0 \tag{4.12}$$

Therefore, the solution $z(t)=(u(t),v(t),w(t))^T$ of (4.1) satisfies

$$u\left(\cdot\right),v\left(\cdot\right),w\left(\cdot\right)\in C^1\left(0,\infty;H^2\cap H_0^1\right),$$

$$u\left(t,0\right)=u\left(t,1\right)=v\left(t,0\right)=v\left(t,1\right)=w\left(t,0\right)=w\left(t,1\right)=0$$

and

$$\|z\left(t\right)\|=\left\|\left(u\left(t\right),v\left(t\right),w\left(t\right)\right)^T\right\|\le e^{-\beta t}\|z_0\|,\quad t\ge 0.$$

Then, taking $z_0=(u_0,u_1,\theta_0)^T$ and changing the variable back, $\theta_x=w$, $u_t=v$ we get that (u,θ) is the solution of the problem (1.3)-(1.5) with

$$u,\ u_t,\ \theta_x\in C^1\left(0,\infty;H^2\cap H_0^1\right)$$

and

$$\|u\left(t,\cdot\right)\|_2^2+\|u_t\left(t,\cdot\right)\|_2^2+\|\theta_x\left(t,\cdot\right)\|_2^2\le e^{-2\beta t}\left\{\|u_0\|_2^2+\|u_1\|_2^2+\|\theta_0\|_2^2\right\},\quad t\ge 0.$$

From here the proof can be completed. □

Remark 4.1 *Condition (4.12) follows also from Lemma 2.2 of [5], wich said that the spectrum radius of $\{T(t)\}_{t\ge 0}$ coincides with the spectrum readius of the hyperbolic equation*

$$u_{tt}+b^2u_t=au_{xx},\quad 0<x<1,\quad t\in I\!R.$$

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