

MULTIPLICITY THEORY OF PROJECTIONS  
IN ABELIAN VON NEUMANN ALGEBRAS

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NOTAS DE MATEMATICAS

Nº 79

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1985

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ABSTRACT

The spectral multiplicity theory is generalized for projections in an arbitrary abelian von Neumann algebra. The well known type  $I_n$  direct sum decomposition of a type I von Neumann algebra follows as a consequence.

RESUMEN

La teoría de multiplicidad espectral se generaliza para las proyecciones en un álgebra conmutativa de von Neumann. La bien conocida descomposición de un álgebra de tipo I de von Neumann en una suma directa de álgebras de tipo  $I_n$  se deduce como una consecuencia.

1.- **INTRODUCTION.** Suppose  $(X, S)$  is a measure space with  $S$  a  $\sigma$ -algebra. If  $E(\cdot)$  is a spectral measure on  $S$  with values in projections on a Hilbert space  $H$ , let  $W$  be the commutant of the range of  $E(\cdot)$  in  $B(H)$ . Then  $W$  is a von Neumann algebra on  $H$  with  $W'$ , the commutant of  $W$ , abelian. In

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(\*)Supported by C.D.C.H. Projects C-S-80-149, 150. Universidad de los Andes, Venezuela.

[2] Halmos develops a (spectral) multiplicity theory for projections in  $W'$  through §§54-64, making use of the countable additivity of  $E(\cdot)$ . In Section 6 we conclude the present article with a discussion on the results of Halmos [2] in the set up of the von Neumann algebra  $W$ .

By the results of Stone [5] the maximal ideal space  $M$  of an abelian von Neumann algebra  $A$  on a Hilbert space  $H$  is extremally disconnected. Thus it is possible to construct a spectral measure  $E(\cdot)$  on the Borel subsets of  $M$  such that the range of  $E(\cdot)$  coincides with the Boolean algebra of all projections in  $A$ . (Vide pp. 158-160, [3]). Consequently, by means of  $E(\cdot)$  thus constructed the spectral multiplicity theory developed in Halmos [2] can also be interpreted as a multiplicity theory for projections in  $A$ . The object of the present work is to give a direct alternate method of study for the multiplicity theory of projections in an arbitrary abelian von Neumann algebra and thus deduce the results of Halmos in §§54-64 of [2] as a particular case. Besides, the present study offers a new proof to the well known type  $I_n$  direct sum decomposition of a type I von Neumann algebra.

2.- PRELIMINARIES. This section is devoted to fix the terminology and notation to be followed in the sequel and to list a few definitions and results from the theory of von Neumann algebras which are repeatedly

made use of in the body of this article.

$H$  will denote a Hilbert space,  $W$  a von Neumann algebra on  $H$  with its commutant  $W'$  abelian,  $\mathcal{R}$  an arbitrary von Neumann algebra on  $H$  with  $Z$  its centre and  $I$  the identity operator on  $H$ . If  $M$  is a nonempty subset in  $H$ ,  $\overline{M}$  denotes the closed linear subspace spanned by  $M$ . If  $E$  is a projection,  $E$  is also denoted by its range and vice-versa. Thus for a vector  $x \in H$ ,  $[Rx]$  is the projection with its range given by  $\{Tx: T \in \mathcal{R}\}$  and is the cyclic projection generated by  $x$  under  $\mathcal{R}$ . Clearly,  $[Rx] \in \mathcal{R}'$  and  $[R'x] \in \mathcal{R}$ .

Let  $A \in \mathcal{R}$ . If  $F = \{Q \in Z: Q \text{ a projection, } QA = A\}$  then  $C_A = A\{Q: Q \in Z\}$  belongs to  $Z$  and is a projection.  $C_A$  is called the central support of  $A$ . For  $C_A$  it is true that  $C_A A = A$ .

**PROPOSITION 2.1.** If  $A$  is an operator in  $\mathcal{R}$  and  $Q$  a central projection of  $\mathcal{R}$ , then  $C_{QA} = QC_A$ .

**DEFINITION 2.2.-** For two projections  $E$  and  $F$  in  $\mathcal{R}$  we say  $E \sim F$  if there exists a partial isometry  $V$  in  $\mathcal{R}$  such that  $V^*V = E$  and  $VV^* = F$ . We say  $E \leq F$  if there exists a subprojection  $E_1$  of  $F$  in  $\mathcal{R}$  such that  $E \sim E_1$  and  $E < F$  if  $E \leq F$  and  $E \not\sim F$ .

**DEFINITION 2.3.** A projection  $E$  in  $\mathcal{R}$  is said to be finite if there exists no subprojection  $E_1$  in  $\mathcal{R}$  of  $E$

such that  $F \sim E_1 < E$ . Otherwise  $E$  is called infinite.

**DEFINITION 2.4.** A projection  $E$  in  $\mathcal{R}$  is said to be abelian if  $E\mathcal{R}E$  is an abelian algebra.

**PROPOSITION 2.5.** If  $E$  and  $F$  are projections in  $\mathcal{R}$ , then the following assertions hold:

- (i) If  $E$  is finite and  $E \sim F$ , then  $F$  is also finite.
- (ii)  $E$  is abelian in  $\mathcal{R}$  if and only if every subprojection  $F$  of  $E$  in  $\mathcal{R}$  is of the form  $F = C_F E$ .
- (iii) If  $E$  and  $F$  are abelian in  $\mathcal{R}$  and  $C_E = C_F$ , then  $E \sim F$ .
- (iv) If  $E \leq F$ , then  $C_E \leq C_F$ ; if  $E \sim F$ , then  $C_E = C_F$ .
- (v) If  $E$  is abelian in  $\mathcal{R}$ , then  $E$  is finite.

**PROOF.**

- (i) Trivial.
- (ii) Use the observation that  $E$  is abelian in  $\mathcal{R}$  if and only if  $E\mathcal{R}E = ZE$ .
- (iii) An immediate consequence of Lemma 1, §3, Chapter III of Dixmier [1].
- (iv) Vide Corollary 1 to Proposition 7, §1, Chapter I and Proposition 1, §1, Chapter III of Dixmier [1].
- (v) If  $E \sim E_1 < E$ ,  $E_1$  a projection in  $\mathcal{R}$ , then by (iv)  $C_E = C_{E_1}$ . But, on the other hand, by (ii)  $E_1 = C_{E_1} E = E$ . This is a contradiction. Hence  $E$  is finite.

**PROPOSITION 2.6.**

(i) If  $\{E_\alpha\}$  is an orthogonal family of projections (respectively, (ii)

abelian projections) in  $\mathcal{R}$  with the property that  $\{C_{E_\alpha}\}$  is an orthogonal family and if  $E = \sum E_\alpha$ , then  $C_E = \sum C_{E_\alpha}$  (respectively, and  $E$  is abelian in  $\mathcal{R}$ ).

**PROOF.** If  $Q = \sum C_{E_\alpha}$ , then  $Q$  is a central projection of  $\mathcal{R}$  and  $Q \geq C_{E_\alpha} \geq E_\alpha$  for all  $\alpha$ . Thus  $Q \geq E$  and hence,  $Q \geq C_E$ .

Since  $C_E \geq E \geq E_\alpha$ ,  $C_E \geq C_{E_\alpha}$  for all  $\alpha$  and hence  $C_E \geq Q$ . Thus  $Q = C_E$ . In consequence,  $E R E = \sum C_{E_\alpha} E R E C_{E_\alpha} = \sum E_\alpha R E_\alpha$ . If each  $E_\alpha$  is abelian, then  $E R E$  is abelian and hence  $E$  is abelian. (Vide 2, §2, Chapter I of Dixmier [1], where  $\Pi$  is used instead of  $\sum$ ).

**THEOREM 2.7.** Suppose  $E, F$  are non-zero projections in  $\mathcal{R}$ ,  $F$  being finite. If  $(F_\alpha)_{\alpha \in A}$  and  $(G_\beta)_{\beta \in B}$  are two maximal orthogonal families of projections in  $\mathcal{R}$  such that  $F \sim F_\alpha \leq E$  and  $F \sim G_\beta \leq E$  for all  $\alpha \in A$  and  $\beta \in B$ , then  $\text{Card.}A = \text{Card.}B$ .

**PROOF.** Vide Ringrose [4] where this theorem is known as theorem of generalizad invariance of dimension.

3.- **SOME LEMMAS.** As mentioned in §2  $W$  will denote a von Neumann algebra on  $H$  with its commutant  $W'$  abelian. In this section we shall give two lemmas which generalize respectively Theorem 2, §60 and theorem 3, §61 of Halmos [2] for projections in  $W$ .

**LEMMA 3.1.** For each vector  $x$  in  $H$  the cyclic projection  $E = [W'x]$  is abelian in  $W$ .

**PROOF.** Suppose  $x$  is a non-zero vector in  $H$ . If  $E = [W'x]$ , then  $W'E$  is an abelian von Neumann algebra on  $E(H)$  and has a generating vector  $x$ . Then from Corollary 2 to Proposition 4, §6, Chapter I of Dixmier [1] it follows that  $(W'E)' = W'E$  and hence  $(W'E)'$  is abelian. But by Proposition 1(i), §2 Chapter I of Dixmier [1],  $(W'E)' = EWE$  so that  $EWE$  is abelian and hence  $E$  is abelian in  $W$ .

**LEMMA 3.2.** Given a projection  $E$  in  $W$ , there exists an abelian projection  $F \leq E$  in  $W$  such that  $C_F = C_E$ .

**PROOF.** Let  $E$  be a non-zero projection in  $W$ . By Lemma 3.1 and Zorn's lemma there exists a maximal orthogonal family  $\{E_\alpha\}$  of non-zero abelian subprojections of  $E$  in  $W$  such that  $\{C_{E_\alpha}\}$  is an orthogonal family. Then  $E \leq \sum C_{E_\alpha}$ . In fact, on the contrary, there exists a non-zero vector  $x$  in the range of  $E$  such that  $C_{E_\alpha}x = 0$  for all  $\alpha$ . Then for  $T' \in W'$  and for all  $\alpha$ ,  $C_{E_\alpha}T'x = T'C_{E_\alpha}x = 0$  and hence  $C_{E_\alpha}[W'x] = 0$  for all  $\alpha$ . Then by Proposition 2.1  $C_{[W'x]}C_{E_\alpha} = C_{[W'x]}C_{E_\alpha} = 0$  for all  $\alpha$ . As  $[W'x]$  is abelian in  $W$  by Lemma 3.1, this contradicts the maximality of  $\{E_\alpha\}$ . As  $E_\alpha \leq E$ ,  $C_{E_\alpha} \leq C_E$  for all  $\alpha$  and thus  $C_E = \sum C_{E_\alpha}$ . If  $F = \sum E_\alpha$ , then by Proposition 2.6(ii)  $F$  is abelian in  $W$  and  $C_F = \sum C_{E_\alpha} = C_E$ .

4.- **MULTIPLICITY OF PROJECTIONS.** Making use of the lemmas of §3 and Theorem 2.7 we shall associate with each projection  $E'$  in  $W'$  a well defined cardinal number  $u(E')$  called the multiplicity of  $E'$  and prove that  $u(\sum E'_j) = \min\{u(E'_j)\}$ , whenever  $\{E'_j\}$  is an orthogonal family of non-zero projections in  $W'$ .

**LEMMA 4.1.** Let  $E'$  be a projection in  $W'$ . Then there exists a maximal orthogonal family  $\{E_\alpha\}$  of abelian projections in  $W$  such that  $C_{E_\alpha} = E'$  for all  $\alpha$ .

**PROOF.** Suppose  $E'$  is a non-zero projection in  $W'$ . By Lemma 3.2 there exists an abelian projection  $F$  in  $W$  such that  $C_F = E'$ . Then the collection of all orthogonal families of abelian projections in  $W$  such that each of their members has the central support  $E'$  is a non-empty partially ordered set under set inclusion and every linearly ordered set in this collection has an upper bound which corresponds to the union of all the members of the family. Appealing to Zorn's lemma we obtain the result.

**PROPOSITION 4.2.** Let  $E'$  be a projection in  $W'$ . Then for two maximal orthogonal families  $(E_\alpha)_{\alpha \in J}$  and  $(F_\beta)_{\beta \in K}$  of abelian projections in  $W$  such that  $C_{E_\alpha} = C_{F_\beta} = E'$  for all  $\alpha \in J$  and  $\beta \in K$ ,  $\text{Card.} J = \text{Card.} K$ .

**PROOF.** By Proposition 2.5(iii)  $E_\alpha \sim F_\beta$  for all  $\alpha \in J$  and  $\beta \in K$ . Besides, by Proposition 2.5(v) they are all finite projections. Now the proposition follows

from Theorem 2.7.

By virtue of the above proposition we are justified in giving the following

**DEFINITION 4.3.** A non-zero projection  $E'$  in  $W'$  is said to have multiplicity  $n$  if there exists a maximal orthogonal family of  $n$  abelian projections  $\{E_i\}$  in  $W$  with  $C_{E_i} = E'$  for each  $i$ . The multiplicity of the zero projection is defined to be zero.

**NOTATION 4.4.** Let  $\mathcal{B}$  denote the collection of all projections in  $W'$ .

Since  $W'$  is abelian and strongly closed,  $\mathcal{B}$  is a complete Boolean algebra of projections on  $H$ .

**THEOREM 4.4.** Let  $u$  be the function associating each projection in  $W'$  with its multiplicity and  $N = \{n : n \text{ a cardinal number } \leq \dim H\}$ . Then:

- (i)  $u$  is well defined, the domain of  $u$  is  $\mathcal{B}$  and the range of  $u$  is a subset of  $N$ .
- (ii) If  $E', F'$  are members of  $\mathcal{B}$  and  $0 < E' \leq F'$ , then  $u(E') \leq u(F')$ .
- (iii) If  $\{F'_j\}$  is an orthogonal family of non-zero members in  $\mathcal{B}$  and if  $F' = \bigvee F'_j$ , then  $u(F') = \min \{u(F'_j)\}$ .

**PROOF.**

- (i) Immediate from Lemma 4.1, Proposition 4.2 and Definition 4.3.
- (ii) If  $\{F_j\}$  is a maximal orthogonal family of abelian projections in  $W$  such that  $C_{F_j} = F'$ , then by

Proposition 2.1  $C_{F_j E'} = E'$  and  $F_j E'$  is abelian for all  $j$ . Hence the conclusion(ii).

(iii) As  $\mathcal{B}$  is a complete Boolean algebra of projections,  $F'$  belongs to  $\mathcal{B}$ . Let  $\min\{u(F'_j)\} = n$ . By (ii) it is obvious that  $u(F') \leq n$ . To prove the reverse inequality let  $I_0$  be a set with  $\text{Card. } I_0 = n$ . As  $u(F'_j) \geq n$ , for each  $j$  there exists an orthogonal family  $\{F_{jk}\}_{k \in I_0}$  of abelian projections in  $W$  with  $C_{F_{jk}} = F'_j$  for all  $k \in I_0$ . Since  $F'_j F'_{j'} = 0$  for  $j \neq j'$ , it follows from Proposition 2.6 that  $G_k = \sum_j F_{jk}$  is abelian in  $W$  with  $C_{G_k} = F'$ , for all  $k \in I_0$ . In other words,  $u(E') \geq n$  and hence the conclusion(iii).

5. - **UNIFORM MULTIPLICITY OF PROJECTIONS.** The concept of uniform multiplicity for projections in  $W'$  is introduced. We give a theorem characterizing those projections in  $W'$  which have uniform multiplicity and deduce the well known type  $I_n$  decomposition of a type I von Neumann algebra as a corollary to another theorem on uniform multiplicity. (Vide Theorems 5.4, 5.6 and Corollary 5.7)

**DEFINITION 5.1.** A non-zero projection  $E'$  in  $W'$  is said to have uniform multiplicity  $n$  if every non-zero projection  $F' \leq E'$  in  $W'$  has multiplicity  $n$ .

**DEFINITION 5.2.** Let  $\mathcal{B}$  be a von Neumann algebra on  $H$ . An orthogonal family  $\{E_\alpha\}$  of projections in  $\mathcal{B}$  is said to be a complete orthogonal system for a central

projection  $Q$  of  $\mathcal{R}$  if  $\sum E_\alpha = Q$ .

**LEMMA 5.3.** Let  $E'$  be a non-zero projection in  $W'$  and  $\{E_\alpha\}$  a maximal orthogonal family of abelian projections in  $W$  such that  $C_{E'} = E'$  for all  $\alpha$ . Then there exists a non-zero vector  $x$  in  $H$  such that  $\{C_{[W',x]} E_\alpha\}$  is a complete orthogonal system of abelian projections in  $W$  for  $C_{[W',x]}$  and the central support of  $C_{[W',x]} E_\alpha$  is  $C_{[W',x]}$  for each  $\alpha$ .

**PROOF.** If  $E_0 = \sum E_\alpha$ , and  $F = E' - E_0$ , then by the maximality of  $\{E_\alpha\}$  and by Lemma 3.2  $G' = E' - C_F \neq 0$ . Then  $G' C_F = 0$  and hence by Proposition 2.1  $G' F = 0$ . Consequently,  $G' = G' E' = G' E_0 = \sum G' E_\alpha$ ,  $\{G' E_\alpha\}$  is an orthogonal family of abelian projections in  $W$  and  $C_{G' E_\alpha} = G' C_{E_\alpha} = G' E_\alpha = G'$  by Proposition 2.1. Now let  $x$  be a non-zero vector in the range of  $G'$ . Then as  $W'$  is abelian, and  $G' \in W'$  it follows that  $[W', x] \leq G'$ . Evidently  $\{C_{[W',x]} E_\alpha\}$  is a complete system of abelian projections in  $W$  for  $C_{[W',x]}$ . Besides,  $C_{[W',x]} C_{[W',x]} = C_{[W',x]} C_{E'} = C_{[W',x]} E' = C_{[W',x]}$  since  $C_{[W',x]} \leq G' \leq E'$ .

The following theorem gives a necessary and sufficient condition for a non-zero projection  $E'$  in  $W'$  to have uniform multiplicity.

**THEOREM 5.4.** A non-zero projection  $E'$  in  $W'$  has uniform multiplicity  $n$  if and only if there exists a complete orthogonal system  $\{E_i\}$  of  $n$  abelian projections in  $W$  for  $E'$  with  $C_{E_i} = E'$  for all  $i$ . Thus  $E'$  has uniform

multiplicity  $n$  if and only if  $WE'$  is a von Neumann algebra of type  $I_n$ .

**PROOF.** Suppose there exists an orthogonal family  $\{E_\alpha\}$  of abelian projections in  $W$  such that  $C_E = E'$  for all  $\alpha$  and such that  $\sum E_\alpha = E'$ . Let  $F'$  be a non-zero subprojection of  $E'$  in  $W'$ . As  $C_{F'}E_\alpha = F'$ ,  $F'E_\alpha \neq 0$  for each  $\alpha$ . Evidently  $\{F'E_\alpha\}$  is a complete orthogonal system of abelian projections in  $W$  for  $F'$ . Since a complete orthogonal system of projections for  $F'$  is also a maximal orthogonal system of projections for  $F'$  and since  $C_{F'}E_\alpha = F'C_{E_\alpha} = F'E_\alpha = F'$  for all  $\alpha$  by Proposition 2.1, it follows that  $u(F') = n$ . Thus the condition is sufficient.

Conversely, suppose  $E'$  has uniform multiplicity  $n$ . By Lemma 5.3 and Zorn's lemma there exists a maximal orthogonal family  $\{F'_j\}_{j \in J}$  of non-zero subprojections of  $E'$  in  $W$  such that each  $F'_j$  has a complete orthogonal system  $\{F'_{jk}\}_{k \in I_0}$  of abelian projections in  $W$  with  $C_{F'_{jk}} = F'_j$  for all  $k \in I_0$ , where  $\text{Card. } I_0 = n$ . In consequence,

$C_{F'_{jk}} = F'_j, j \in J$ . Then by the maximality of  $\{F'_j\}_{j \in J}$  and by Lemma 5.3 we conclude  $\sum_{j \in J} F'_j = E'$ . Let  $G_k = \sum_{j \in J} F'_{jk}$ . Then by Proposition 2.0  $G_k$  is abelian in  $W$  and  $C_{G_k} = \sum_{j \in J} F'_j = E'$  for all  $k \in I_0$ . Besides,  $C_{G_k} = \sum_{j \in J} C_{F'_{jk}} = \sum_{j \in J} F'_j = E'$ .

Hence the **necessity** of the condition in the first part

of the theorem.

The second part of the theorem is an immediate consequence of the first if one appeals to Proposition 2.5(iv) and the definition of type  $I_n$  von Neumann algebras in [1]. This completes the proof of the theorem.

**PROPOSITION 5.5.** Suppose  $\{E_j\}$  is an orthogonal family in  $\mathcal{B}$  such that each  $E_j$  has uniform multiplicity  $n$ . If  $E' = \sum E_j$ , then  $E' \in \mathcal{B}$  and  $E'$  has uniform multiplicity  $n$ .

**PROOF.** As  $\mathcal{B}$  is a complete Boolean algebra of projections and  $E' = \sum E_j$ , it follows that  $E' \in \mathcal{B}$ . Let  $E_0$  be a non-zero subprojection of  $E'$  in  $\mathcal{B}$ . Then  $E_0 = \sum E_0 E_j$ , where we consider only those indices  $j$  for which  $E_0 E_j \neq 0$ . By hypothesis, for each of such  $j$   $u(E_0 E_j) = n$  and the proposition is an immediate consequence of Theorem 4.4.(iii).

**THEOREM 5.6.** Suppose, for each cardinal number  $n$  not exceeding the dimension of  $H$ ,  $P'_n$  is the supremum of all those projections in  $\mathcal{W}'$  which have uniform multiplicity  $n$ . Then  $\{P'_n\}$  is an orthogonal family of projections in  $\mathcal{W}'$ ,  $\sum P'_n = 1$  and for each  $n$  either  $P'_n = 0$  or  $P'_n$  has uniform multiplicity  $n$ .

**PROOF.** For a fixed cardinal number  $n \leq \dim H$ , let  $\{E'_\alpha\}$  be a maximal orthogonal family of projections in  $\mathcal{W}'$  such that each  $E'_\alpha$  has uniform multiplicity  $n$ . If no such family exists we take the supremum  $P'_n$  to be the zero projection. If  $P'_n = \sum E'_\alpha > 0$ , then there

exists some non-zero projection  $F'$  in  $W'$  such that  $0 < F' \leq P'_n - VE'_\alpha$  and such that  $F'$  has uniform multiplicity  $n$ . Since this contradicts the maximality of  $\{E'_\alpha\}$ , we conclude  $P'_n = VE'_\alpha$ . Thus, if  $P'_n \neq 0$ , then by Proposition 5.5  $P'_n$  has uniform multiplicity  $n$ . In consequence for  $n \neq m$  with  $P'_n P'_m \neq 0$ , we have  $n =$  the multiplicity of  $P'_n =$  the multiplicity of  $P'_n P'_m =$  the multiplicity of  $P'_m = m$ . This contradiction establishes that  $\{P'_n\}$  is an orthogonal family. Finally,  $\sum P'_n = I$  by Lemmas 4.1 and 5.5. This completes the proof of the theorem.

**COROLLARY 5.7.** Let  $R$  be a type I von Neumann algebra on  $H$ . Then  $R = \sum_{i \in I_0} \otimes R_i$ , where  $R_i$  are von Neumann algebras of type  $I_{n_i}$  where  $n_i \neq n_{i'}$  for  $i \neq i'$  in  $I_0$ .

**PROOF.** By the definition of a type I von Neumann algebra, there exists a von Neumann algebra  $W$  with  $W'$  abelian on a Hilbert space  $H'$  such that  $R$  and  $W$  are  $*$ -isomorphic. Let  $\phi$  be a  $*$ -isomorphism from  $R$  onto  $W$ . Let  $I_0 = \{n_i : n_i \leq \dim H' \text{ with } P'_{n_i} \neq 0\}$  in Theorem 5.6 with respect to  $W$ . Then  $W = \sum_{i \in I_0} \otimes WP'_{n_i}$  and by Theorem 5.4  $WP'_{n_i}$  is of type  $I_{n_i}$  for each  $i \in I_0$ . As a consequence,  $R = \phi^{-1} W = \sum_{i \in I_0} \otimes \phi^{-1}(WP'_{n_i}) = \sum_{i \in I_0} \otimes RQ_{n_i}$ , where  $Q_{n_i} = \phi^{-1}(P'_{n_i})$  is a central projection of  $R$  and each  $RQ_{n_i}$  is a von Neumann algebra of type  $I_{n_i}$ .

6.- **CONCLUDING REMARKS.**

Suppose  $(X, S)$  is a measure space with  $S$  a  $\sigma$ -algebra

and  $E(\cdot)$  a spectral measure on  $S$  with values in projections on  $H$ . In [2] Halmos uses the following terminology and notation:

$$E = \{ E(M) : M \in S \}$$

$$P = \{ F : FE(M) = E(M)F \text{ for all } M \in S, F \text{ a projection} \}$$

$$F = \{ G : GP = PG \text{ for all } P \in P, G \text{ a projection} \}.$$

For  $P \in P$ , the column generated by  $P$ , in symbols  $C(P)$ , is defined as  $C(P) = \Lambda \{ F : P \leq F \in F \}$ . A row projection  $R$  in  $P$  is one such that if  $R \geq P \in P$ , then  $P = C(P)R$ . For a vector  $x$  in  $H$  the cycle generated by  $x$ , in symbols  $Z(x)$ , is the projection  $[E(M)x : M \in S]$  and it is shown that  $Z(x) \in P$ .

The above definitions of rows, cycles and generated columns can be related to abelian projections, cyclic projections and central supports respectively if we define the von Neumann algebra  $W$  suitably. In fact, let  $W$  be the von Neumann algebra of all operators  $T$  in  $B(H)$  which commute with the members of  $P$ . Then the commutant  $W'$  of  $W$  is the von Neumann algebra generated by  $F$  and  $F$  is the Boolean algebra of all projections in  $W'$ . Thus for a projections  $P \in P$  the column  $C(P)$  generated by  $P$  coincides with the central support  $C_p$  since the centre of  $W$  is  $W'$ . By Proposition 2.5 (11), a row  $P \in P$  is nothing but an abelian projection in  $W$ . Finally, the cycle  $Z(x)$  is the cyclic projection  $[W'x]$  since  $W'$  is closed in the strong topology of operators. Consequently,  $Z(x) \in W'' = W$  and hence  $Z(x)$

belongs to P.

As a consequence of the above remarks, it is evident, as commented in Section 1, that our results subsume those of Halmos in §§54-64 of [2].

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