MARIO MILMAN

"SOME NEW FUNCTION SPACES

AND THEIR TENSOR PRODUCTS"

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"SOME NEW FUNCTION SPACES AND THEIR TENSOR PRODUCTS"

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ABSTRACT

In the theory of admissibility of integral operators a central role is played by Banach function spaces. Indeed, the behaviour of the solutions of integral equations can be simply described if we can ascertain that they belong to certain Banach spaces.

In this paper we study the admissibility problems which, by duality, are equivalent to the embedding of the projective tensor product of two function spaces into a third B.F.S..

A fundamental role in our development is played by the construction of function spaces following ideas of Calderón (9). Using these spaces we are able to extend unify and considerably simplify previous work of several mathematicians.

Notably we should mention the work of R. O'Neil (43) who studied similar problems in the context of Orlicz and L(p,q) spaces. The fundamental ideas here are, however, different. From our point of view complicated spaces are constructed from simpler spaces by means of a suitable functor. Thus, our embedding theorems follow once have obtained the results for a suitable set "simpler spaces".

This not only simplifies the proofs but clarifies the role of the conditions involved in O'Neil's paper. Moreover the method also gives embeddings for mixed norm spaces and our generalized setting allows us to prove results that hold simultaneously for a large class of spaces.

INTRODUCTION

In the theory of admissibility of integral operators a central rôle is played by Banach function spaces. Indeed, the behaviour of the solutions of integral equations can be simply described if we can ascertain that they belong to certain Banach spaces.

In this paper we study the admissibility problems which, by duality, are equivalent to the embedding of the projective tensor product of two function spaces into a third B.F.S. .

A fundamental role in our development is played by the construction of function spaces following ideas of Calderón (9). Using these spaces we are able to extend unify and considerably simplify previous work of several mathematicians.

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This not only simplifies the proofs but clarifies the rôle of the conditions involved in O'Neil's paper. Moreover the method also gives embeddings for mixed norm spaces and our generalized setting allows us to prove results that hold simultaneously for a lage class of spaces.

The paper is naturally divided in two parts. In the first part we introduce and study the relevant properties of our spaces, including duality, and interpolation theory. In the second we study the embedding theory for tensor products of these spaces.

The reader is referred to page 1 and page 50 for a detailed description of the results obtained in this paper.

Some of results were announced in (36) and (37).

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CHAPTER 0

PRELIMINARIES AND NOTATION

It is expected that the reader will have some familiarity with the theory of Banach Function Spaces as developed for example in Luxemburg [32] and Zaanen [61].

The purpose of this chapter is to provide a brief introduction to the theory of rearrangement invariant spaces and to indicate pertinent references to the literature. Moreover, we shall also develop the notation to be followed in this work.

0.1. YOUNG'S FUNCTIONS. Let a be a monotonic, non-decreasing function, $a: [0, \infty) \to [0, \infty]$, such that a is not constantly equal to zero or infinite, then let $A(x) = \int_0^x a(t)dt$, $x \in [0,\infty)$. A is called a Young's function.

In what follows we shall need to consider more general functions

(0.1.1) DEFINITION. Let A be a monotonic, non decreasing function, $A: [0, \infty) \to [0, \infty]$, such that (i) $A(0) = 0$; (ii) A is left continuous; (iii) there exists $t_0 \in (0, \infty)$ such that $0 < A(t_0) < \infty$. A will be called a generalised Young's function.

Let A be a generalised Young's function, the inverse of A is defined on $[0, \infty)$ by

$$A^{-1}(t) = \inf \{s: A(s) > t\} \quad , \inf \{\phi\} = \infty \quad .$$

It follows that A^{-1} is a monotone non-decreasing function which is right continuous, and that the following inequalities hold $\forall t > 0$

$$A(A^{-1}(t)) \leqslant t \leqslant A^{-1}(A(t)).$$

Moreover, A and A^{-1} are related by

$$A(t) = \sup \{s: A^{-1}(s) < t\} \quad , \sup \{\phi\} = 0 .$$

Let A be a generalised Young's function such that $A(t)t^{-1} \uparrow$ (i.e. $A(t)t^{-1}$ increases in a wide sense), then the Young's complement of A is defined by

$$\overline{A}(t) = \sup_{s \geqslant 0} \{st - A(s)\} \quad , \quad t \in [0, \infty) \quad .$$

From the above definition we readily get Young's inequality,

$$ts \leq A(t) + \bar{A}(s) \qquad \forall\, t,s \geq 0.$$

The inverses of A and $\bar{A}$ are related by the following inequality

$$t \leq A^{-1}(t)\, \bar{A}^{-1}(t) \leq 2t \quad , \; 0 \leq t < \infty .$$

The reader is referred to O'Neil [43] for a detailed account. However for the reader's convenience we shall review the definitions of certain conditions that we shall ocassionally impose on Young's functions.

(0.1.2) DEFINITION. THE Δ_2 CONDITION. A generalised Young's function A is said to satisfy the Δ_2 condition if there exists a constant $\theta \geq 1$ such that for all $x \geq 0$,

$$A(2x) \leq \theta\, A(x) .$$

A is said to satisfy the Δ_2 condition for large values if A is finite valued and if there exist constants $\theta \geq 1$, $x_0 \geq 0$, such that $\forall\; x \geq x_0$, $A(2x) \leq \theta\, A(x)$.

A is said to satisfy the Δ_2 condition for small values if there exist constants $\theta \geq 1$, $x_0 > 0$ such that $\forall\, x \in (0, x_0)$, $A(2x) \leq \theta\, A(x)$.

It can be proved that a Young's function A satisfies the Δ_2 condition if and only if $\exists\; p > 1$ such that $A(t).t^{-p} \downarrow$.

(0.1.3) DEFINITION. THE ∇_2 CONDITION. A generalised

Young's function A is said to satisfy the ∇_2 condition if there exists a constant $\theta \geq 1$ such that $\forall x \geq 0$

$$A(\theta x) \geq 2\,\theta\, A(x) .$$

It is known that if A is a Young's function then A satisfies the ∇_2 condition if and only if $\bar{A}$ satisfies the Δ_2 condition.

We define similarly the ∇_2 condition for small and large values.

(0.1.4) DEFINITION. THE Λ CONDITION. A generalised Young's function A is said to satisfy the Λ condition if there exist constants $\alpha \geq 1$, $\beta > 1$, such that $\forall x \geq 0$,

$$A(\alpha x) \geq \beta\, A(x).$$

0.2. ORLICZ SPACES.

In this work we shall only consider non-atomic, σ-finite and complete measure spaces. Thus, in all what follows, we shall make the following convention: when we say "a measure space" we shall mean "a non-atomic, σ-finite, complete measure space".

Let (Ω,μ) be a measure space and let us denote by $M(\Omega)$ the class of real valued μ-measurable functions. Let A be a generalised Young's function, and for $f \in M(\Omega)$ define

$$\|f\|_{L_A(\Omega)} = \inf \{\alpha > 0: \int_\Omega A(|f(x)|/\alpha)\, d\mu(x) \leq 1\} .$$

The Orlicz space $L_A(\Omega)$ is defined by

$$L_A(\Omega) = \{\bar{f} \in \bar{M}(\Omega): \|\bar{f}\|_{L_A(\Omega)} < \infty\}$$

where $\bar{M}(\Omega)$ is the quotient space of $M(\Omega)$ under the equivalence relation $\equiv$ defined as follows: $h \equiv g$, $h,g \in M(\Omega)$, if and only if $h = g$ μ.a.e. ; and $\|\bar{f}\|_{L_A(\Omega)} = \|f\|_{L_A(\Omega)}$, $f \in \bar{f}$.

In general we shall follow the standart practice of identifying a measurable function f with its equivalence class $\bar{f} = \{g \in M(\Omega): g \equiv f\}$.

Ocassionally, when we are willing to emphasise the rôle of μ , we shall write $L_A(d\mu)$ rather than $L_A(\Omega)$.

The reader is referred to the standart treatises on Orlicz spaces ([32],[27],[62]).

0.3. BANACH FUNCTION SPACES. Let (Ω,μ) be a measure space. We consider normed (respect. quasi-normed) spaces $(X(\Omega), \| \|_{X(\Omega)})$ of measurable functions on (Ω,μ) such that:

(i) $|f| \leq |g|$ μ.a.e. and $g \in X(\Omega)$ implies $f \in X(\Omega)$ and $\|f\|_{X(\Omega)} \leq \|g\|_{X(\Omega)}$.

(ii) Let χ_E denote the characteristic function of a measurable set E, then $\chi_E \in X(\Omega)$ whenever $\mu(E) < \infty$.

(iii) $f \in X(\Omega)$ implies f is locally integrable (i.e. $f \in X(\Omega) \Rightarrow f \in L^1_{Loc}(\Omega)$) .

(iv) If $0 \leq f_n \uparrow f$ μ.a.e., $f_n \in X(\Omega)$ $\forall n \in N$, and $\sup_{n \in N} \|f_n\|_{X(\Omega)} < \infty$, then $f \in X(\Omega)$ and moreover $\lim_{n \to \infty} \|f_n\|_{X(\Omega)} = \|f\|_{X(\Omega)}$. (The Fatou Property (F.P.).)

A normed (respect. quasi-normed) function space $X(\Omega)$ verifying conditions (i) - (iv) will be called a Banach Function Space (B.F.S.) (respect. q.B.F.S.). Indeed, it can be shown that condition (iv) implies that $X(\Omega)$ is a Banach space (cf. [32]).

In general we shall write X and $\| \ \|_X$ rather than $X(\Omega)$ and $\| \ \|_{X(\Omega)}$, whenever no confusion arises.

Let X be a B.F.S. (respect. q.B.F.S.) the associate space of X is defined by

$$X' = \{g \in M(\Omega): \|g\|_{X'} = \sup_{\|f\|_X \leq 1} \int_\Omega |f(x)g(x)|d\mu(x) < \infty\} .$$

A deep result of Lorentz and Luxemburg (cf. [32]), states that X'' is isometrically isomorphic to X for any B.F.S. X.

(0.3.2) DEFINITION. Let $X(\Omega)$ be a B.F.S. (respect. q.B.F.S.), we shall say that X has absolutely continuous norm (a.c.n.) (respect. a.c.q.n.) whenever $\forall f \in X$ and every sequence of measurable sets $\{A_n\}$ such that $A_n \downarrow \emptyset$ we have $\lim_{n \to \infty} \|f\chi_{A_n}\|_X = 0$.

It can be shown (cf. [32], [48]) that $f_n \to f$ μ.a.e. and $|f_n| \leq |g|$, $g \in X$ implies $\lim_{n \to \infty} \|f_n - g\|_X = 0$, whenever X has a.c.n..

Let X be a B.F.S. and denote by X^* its dual space, define $\Gamma: X' \to X^*$ by $\Gamma(g)(f) = \langle f,g \rangle = \int_\Omega f(x)\, g(x)\, d\mu(x)$, $g \in X'$, $f \in X$. Then Γ defines an isometric isomorphism between X' and X^* if and only if X has a.c.n. (cf. [32]).

0.4. REARRANGEMENT INVARIANT SPACES.

Among the class of B.F.S. spaces (respect. q.B.F.S.) we are specially interested in the subclass of rearrangement invariant spaces.

Let (Ω,μ), (Ω', μ') be measure spaces such that $\mu(\Omega) = \mu'(\Omega')$, and let $f \in M(\Omega)$, $g \in M(\Omega')$, then we shall say that f and g are equimeasurable (we write $f \sim g$) if for all $t > 0$,

$$\mu\{ x \in \Omega: |f(x)| > t\} = \mu'\{x' \in \Omega': |g(x')| > t\} .$$

(0.4.1) DEFINITION. Let $X(\Omega)$ be a B.F.S. (respect.q.B.F.S.), we shall say that X is a rearrangement invariant space (r.i. space)(respect. q.r.i. space) if $\|f\|_X = \|g\|_X$ $\forall$ $f,g \in M(\Omega)$ such that $f \sim g$.

The Orlicz spaces L_A, where A is a Young's function, are examples of r.i. spaces.

Let $f \in M(\Omega)$, the non-increasing rearrangement of f is defined by

$$f^*(t) = \inf \{s: \lambda_f(s) \leqslant t\} \quad , \quad \inf \{\phi\} = 0$$

where $\lambda_f(s) = \mu\{x: |f(x)| > s\}$, $0 \leqslant t \leqslant \mu(\Omega)$.

In [33] it is shown that for every r.i.space $X(\Omega)$ there exists a r.i. space $\hat{X}(0,\mu(\Omega))$ such that $\|f\|_X = \|f^*\|_{\hat{X}}$. We shall call $\hat{X}$ the Luxemburg representation of X . A similar result holds for q.r.i. spaces.

It can be shown (cf. [33]) that if X is a r.i.space then X' is a r.i. space, and moreover

$$\|g\|_{X'} = \sup_{\|f\|_X \leq 1} \int_0^{\mu(\Omega)} f^*(t) g^*(t) dt.$$

Let $X(\Omega)$ be a r.i. space (respect. q.r.i. space), the fundamental function of X is defined by

$$\phi_X(t) = \begin{cases} \|\chi_E\|_X & \mu(E) = t, \quad \text{if} \quad t < \mu(\Omega). \\ 0 & \text{if} \quad t > \mu(\Omega). \end{cases}$$

It can be proved (cf. [63]) that a r.i. space X can be renormed in such a way that the resulting fundamental function is concave on $(0,\mu(\Omega))$.

We assume, without loss of generality, throughout this work that all spaces have been renormed in this manner.

The fundamental function ϕ_X is absolutely continuous on $(0,\mu(\Omega))$ and $\phi_X(t).t^{-1} \downarrow$. The fundamental functions of X and X' are related by

$$(0.4.2) \quad \phi_X(t)\, \phi_{X'}(t) = t \qquad \forall t \geq 0 .$$

From (0.4.2) it follows that

$$(0.4.3) \qquad \frac{d\phi_X(t)}{dt} \leq \phi_X(t).t^{-1} \qquad \text{a.e..}$$

For example the fundamental function associated with an Orlicz space $L_A(0,\infty)$ is given by $\phi_{L_A}(t) = 1/A^{-1}(1/t)$.

We shall consider some conditions on r.i. spaces that shall be impossed ocassionally in the sequel.

(0.4.4) DEFINITION. THE δ_2 CONDITION. Let $X(\Omega)$ be a r.i. space, we shall say that X satisfies the δ_2 condition if there exists a constant $\theta \geq 1$ such that

$$\int_0^t \phi_X(u)\frac{du}{u} \leq \theta\, \phi_X(t), \qquad 0 < t < \mu(\Omega).$$

(0.4.5.) EXAMPLE. Let A be a Young's function such that A satisfies the Δ_2 condition, then $L_A(0,\infty)$ satisfies the δ_2 condition.

(0.4.6) DEFINITION. THE N_2 CONDITION. Let $X(\Omega)$ be a r.i. space, $\mu(\Omega) = \infty$. We shall say that X satisfies the n_2 condition if $\xi(t) = \min\{1, \frac{1}{t}\}$ belongs to $\hat{X}(0,\infty)$. Similarly if $X(0,\infty)$ is a B.F.S. we shall say that it satisfies the n_2 condition if $\xi(t) = \min\{\frac{1}{t}, 1\} \in X$.

(0.4.7) EXAMPLE. Let A be a Young's function satisfying the ∇_2 condition then $L_A(0,\infty)$ satisfies the n_2 condition.

0.5. SPACES $\Lambda(X)$, $M(X)$, $\widetilde{M}(X)$. Let $X(\Omega)$ be a r.i. space, the Lorentz space associated with X is defined by

$$\Lambda(X) = \{f \in M(\Omega): \|f\|_{\Lambda(X)} = \int_0^{\mu(\Omega)} f^*(t)\, \phi_X(t)\frac{dt}{t} < \infty\} .$$

If X satisfies the δ_2 condition then $\Lambda(X)$ is a r.i. space . (In general $\Lambda(X)$ is not a r.i. space according to our definitions since it may happen that $\chi_E \notin \Lambda(X)$ for some $E \subseteq \Omega$, $0 < \mu(E) < \infty$.)

Let $f \in M(\Omega)$ and define the maximal rearrangement of f by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds = \sup_{\mu(E)=t} \{ \frac{1}{t} \int_E |f(x)|d\mu(x)\} \quad , t > 0.$$

It follows that f^{**} is non-increasing and $f^*(t) \leq f^{**}(t)$, $\forall$ $t > 0$.

We notice for future reference the following useful equation,

(0.5.1) $$\int_{f^*(r)}^{\infty} \lambda_f(u)du = r\, f^{**}(r) - r\, f^*(r)$$

valid $\forall$ $r > 0$.

The Marcinkiewicz space M(X) associated with X is defined by

$$M(X) = \{f \in M(\Omega): \|f\|_{M(X)} = \sup_{t>0} \{f^{**}(t)\phi_X(t)\} < \infty\} .$$

Suppose that $\mu(\Omega) = \infty$, then we have

(0.5.2) $\Lambda(X) \subseteq X \subseteq M(X)$

with continuous embeddings (cf. [63]). A similar result holds if $\mu(\Omega) < \infty$.

We shall also consider the q.r.i. spaces $\tilde{M}(X)$ defined by

$$\tilde{M}(X) = \{f \in M(\Omega): \|f\|_{\tilde{M}(X)} = \sup_{t > 0} \{f^*(t)\phi_X(t)\} < \infty\} .$$

It is easy to see that $X \subseteq \tilde{M}(X)$, indeed let $f \in X$, then $\forall\ t > 0$

$$f^*(t)\phi_X(t) \leqslant \|f^*\chi_{(0,t)}\|_{\hat{X}} \leqslant \|f^*\|_{\hat{X}} = \|f\|_X$$

and the assertion is proved.

0.6. The symbol + stands for the end of a proof, or if used after the statement of a theorem means that the proof is elementary or contained in the given reference.

To simplify the formulation of our results we shall use the notation $F(h) \approx G(h)$, where h belongs to some specified class C, to mean that there exist positive numbers a,b such that

$$F(h) \leqslant a\ G(h) \quad \text{and} \quad G(h) \leqslant b\ F(h)$$

for all $h \in C$ (infinite values of $F(h)$, $G(h)$ being permited). Finally we write $X \cong Y$, where $X(\Omega)$ $Y(\Omega)$ are B.F.S. spaces to mean that $\|f\|_X \approx \|f\|_Y$ $\forall\ f \in M(\Omega)$.

The reader is referred to page 122 for other notational conventions and symbols used in the text.

PART I

FUNCTION SPACES

INTRODUCTION

In this part of the thesis we develop the theory of the so called Calderón spaces. Let $X(\Omega)$ be a function space and $\xi(x, t)$ be a positive function defined on $\Omega \times R^+$ such that

(i) $\xi(x, t)$ increases in t for each $x \in \Omega$ fixed.

(ii) $\xi(x, 0) = 0 \quad \forall x \in \Omega$.

(iii) For each $x \in \Omega$, $\xi_x: R^+ \to R^+$, defined by $\xi_x(t) = \xi(x, t)$ is concave.

For a pair $(X(\Omega), \xi)$ define

$$\xi(X)(\Omega) = \{f \in M(\Omega): \exists\, \lambda > 0,\ g \in \Sigma(X) \text{ such that } |f(x)| \leq \lambda \xi(x, |g(x)|)\}$$

and put

$$\|f\|_{\xi(X)} = \inf\{\lambda > 0: \exists\, g \in \Sigma(X),\ |f(x)| \leq \lambda\, \xi(x, |g(x)|)\}\ .$$

Then $\xi(X)(\Omega)$ is a Banach space (cf. Calderón [9]). For example if A is a Young's function and $\xi(x, t) = A^{-1}(t)$ then $A^{-1}(L^1) = L_A$, the classical Orlicz spaces, if $\xi(x,t) = t$, then $\xi(X) = X$.

We shall also consider spaces constructed, by means of the maximal rearrangement, as follows: let $X(0, \infty)$ be a B.F.S. of Lebesgue measurable functions on $(0, \infty)$, then $X^0 = \{f \in M(0, \infty): \|f\|_{X^0} = \|f^{**}\|_X < \infty\}$.

We shall single out a class of spaces, constructed by means of a combination of the methods described above, which contains among others the Orlicz spaces, the L(p,q) spaces and several of its generalisations.

The general theory of B.F.Spaces which started to flourish in the early fifties has concerned itself with unifying certain general common features shared by the spaces which analysts use in every day life. Moreover, new classes of function spaces were invented and studied in a systematic way. This is the case for example of the so called Lorentz Λ_α spaces [29], the $\Lambda(\alpha,p)$ spaces studied by Lorentz [30], [31] and Halperin [21], the Marcinkiewicz spaces studied by Lorentz [31] and Halperin [21], the Orlicz spaces studied by Orlicz [46] and others. These spaces are generalisations of the classical L^p spaces (F. Riesz [50]).

It should be noted that these developments were closely connected with the problems analysts were studying at that time. Moreover, most of the "new" spaces, were at the time of their "invention", already implicit in the formulation of several problems in analysis.

Not surprisingly then, these theories led to new and profound developments in functional analysis. Let us mention, as an example, the theory of interpolation of operators developed among others by M. Riesz [51], Marcinkiewicz [35], Zygmund [64], Calderón [9], [10], Peetre [47], Lorentz [31], Lions [28], Oklander [42] (cf. [8] for a detailed account of the theory and a detailed source of contributions). The theory of interpolation of operators has had an enormous influence on classical and modern Harmonic Analysis (cf. [64]).

The general systematic development of abstract function spaces - initiated by Ellis and Halperin [17] and complemented by decisive contributions by Luxemburg [32], Lorentz [31] and Luxemburg and Zaanen [34] - in interaction with the "concrete" field of possible applications led to the study of special subcategories of function spaces. Thus, rearrangement invariant spaces, translation invariant spaces, harmonically invariant spaces, etc (cf. [33], [52], [26], [54], [15]), have been studied, and in general the methods used in each case depend on the particular subclass under consideration.

In this thesis we introduce a new class of function spaces in order to study some problems which arise in the qualitative theory of integral operators. These problems are formulated here in terms of the possibility of embedding the projective tensor product of two B.F. spaces into a third B.F.S..

Our spaces provide a generalised setting on which to formulate our results. Moreover, the O.C.L.H.Z. spaces introduced in Chapter 2 are interesting in their own since the characteristic features of the theory of Lorentz spaces and Orlicz spaces are blended.

This part of the thesis has been divided in three chapters: in §1 we discuss the general theory of Càlderőn spaces, in §2 we introduce the class of O.C.L.H.Z. spaces and compute their duals and associate spaces and in §3 we extend the interpolation theorems of Marcinkiewicz-Calderőn-Hunt (cf. [10]) to the context of O.C.L.H.Z. spaces.

These results will be used in the second part of our work.

CHAPTER 1

GENERAL THEORY OF CALDERON SPACES

Let A be a Young's function (respect. generalised Young's function) and $X(\Omega)$ be a B.F.S.. The spaces $A^{-1}(X)(\Omega)$ (respect. $GA^{-1}(X)$) are introduced and their elementary properties established in §1.1. and §1.2. The functor ^{0} is described in §1.3 and it is used to give a unified discussion on Hardy's inequalities for $A^{-1}(X)$ spaces. In §1.4 several examples, illustrating the scope of our theory, are presented. In §1.5 we prove that if X is a B.F.S. with a.c.n. then $A^{-1}(X)(\Omega)$ is separable if and only if $\mu(\Omega) = \infty$ and A satisfies the Δ_2 condition or $\mu(\Omega) < \infty$ and A satisfies the Δ_2 condition for large values ((Ω,μ) is assumed to be separable). In §1.6 the reader will find a brief discussion on the associate spaces $A^{-1}(X)'$ and bibliographical information.

1.1. THE SPACES $A^{-1}(X)$. Let (Ω,μ) be a measure space, $X(\Omega)$ be a B.F.S. and let A be a generalised Young's function, we define

$$A^{-1}(X)(\Omega) = \{f \in M(\Omega): \|f\|_{A^{-1}(X)} < \infty\}$$

$$\|f\|_{A^{-1}(X)} = \inf \{\lambda > 0: \|A(\tfrac{|f|}{\lambda})\|_X \leq 1\}$$

where $A(\frac{|f|}{\lambda})(x) = A(\frac{|f(x)|}{\lambda})$.

Let A be a Young's function, then it is not difficult to see that $A^{-1}(X)(\Omega)$ is exactly the Calderón space generated by the pair (X, ξ_A), where $\xi_A(x, t) = A^{-1}(t)$. Thus, if A is a Young's function, $A^{-1}(X)$ is a Banach space and moreover, $|f| \leq |g|$ a.e. implies

$$\|f\|_{A^{-1}(X)} \leq \|g\|_{A^{-1}(X)} \text{ (cf. [9]).}$$

The purpose of this section is to prove some elementary results concerning the $A^{-1}(X)$ spaces which shall be useful in the sequel.

(1.1.1) LEMMA. Let A be a generalised Young's function and let X be a B.F.S., then

$$\|A(|f| / \|f\|_{A^{-1}(X)})\|_X \leq 1, \quad \forall f \in A^{-1}(X).$$

Proof. Follows readily from the fact that A is left continuous and X has the F.P.. +

In the remainder of this section we shall assume that X is a B.F.S. and A is a Young's function.

(1.1.2) THEOREM. $A^{-1}(X)$ is a B.F.S..

Proof. We shall prove that: (i) $A^{-1}(X)$ has the F.P.; (ii) $\chi_E \in A^{-1}(X)$, whenever $\mu(E) < \infty$, (iii) $f \in A^{-1}(X)$ implies $f \in L^1_{LOC}$.

(i) Suppose that $0 < f_n \uparrow f$ μ.a.e., $f_n \in A^{-1}(X)$, $n = 1$ ...., and $\lim_{n \to \infty} \|f_n\|_X = \alpha < \infty$. It must be shown that $\|f\|_{A^{-1}(X)} = \alpha$.

It is clear that $\|f\|_{A^{-1}(X)} \geq \alpha$. Let $\alpha_n = \|f_n\|_{A^{-1}(X)}$, we may assume without loss that $0 < \alpha_n < \infty$, $n = 1, \ldots$. Then since $\alpha \geq \alpha_n$ and $A \uparrow$, we get using (1.1.1), $\|A(|f_n| / \alpha)\|_X \leq 1$, $n = 1, \ldots$. Therefore using the F.P. of X, we get $\|A(|f| / \alpha)\|_X \leq 1$, which implies $\|f\|_{A^{-1}(X)} \leq \alpha$.

(ii) Let $E \subseteq \Omega$, be a measurable set such that $\mu(E) < \infty$. Since $\lim_{x \to 0} A(x) = 0$, there exists $0 < \alpha < \infty$ such that $A(\frac{1}{\alpha}) \|\chi_E\|_X \leq 1$. Then $\|\chi_E\|_{A^{-1}(X)} \leq \alpha$.

(iii) Let $f \neq 0 \in A^{-1}(X)$, then $A(|f| / \|f\|_{A^{-1}(X)}) \in X$ and since X is a B.F.S. we conclude that $A(|f| / \|f\|_{A^{-1}(X)})$ is locally integrable.

Let $E \subseteq \Omega$, be a measurable set, $0 < \mu(E) < \infty$, then by Jensen's inequality

$$A\left(\frac{1}{\mu(E)\|f\|_{A^{-1}(X)}} \int_E |f(x)|\, d\mu(x)\right) \leq \frac{1}{\mu(E)} \int_E A\left(\frac{|f(x)|}{\|f\|_{A^{-1}(X)}}\right) d\mu(x) < \infty.$$

Therefore $\int_E |f(x)|\, d\mu(x) < \infty$. +

We collect, without proof, some elementary properties of the $\| \ \|_{A^{-1}(X)}$ norm.

(1.1.3) PROPOSITION. (i) $\|f\|_{A^{-1}(X)} \leq 1 \iff \|A(|f|)\|_X \leq 1$.

(ii) $\|A(|f|)\|_X = 1 \Rightarrow \|f\|_{A^{-1}(X)} = 1$ and if $\|A(k|f|)\|_X < \infty$, for some constant $k > 1$, then $\|f\|_{A^{-1}(X)} = 1 \Rightarrow \|A(|f|)\|_X = 1$.

(iii) $f_n \to f$ in $A^{-1}(X)$ if and only if $\lim_{n\to\infty} \|A(k|f_n - f|)\|_X = 0$ $\forall\, k \geq 0$, whenever X is a B.F.S. with a.c.n.. +

In our work we shall be interested in $A^{-1}(X)$ spaces generated by r.i. spaces X. The following result will be important in what follows

(1.1.4) THEOREM. Let $X(\Omega)$ be a r.i. space, and let $\hat{X}(0,\mu(\Omega))$ be its Luxemburg representation, then

(i) $A^{-1}(X)$ is a r.i. space.

(ii) $A^{-1}(X)\hat{\ }(0,\mu(\Omega)) = A^{-1}(\hat{X})(0,\mu(\Omega))$.

(iii) The fundamental function of $A^{-1}(X)$ is given by

$$\phi_{A^{-1}(X)}(t) = 1/A^{-1}(1/\phi_X(t)),\ 0 \leq t \leq \mu(\Omega).$$

Proof. (i) Let f, g be measurable functions such that $f \sim g$, then $\forall \alpha > 0$ $A(\frac{|f|}{\alpha}) \sim A(\frac{|g|}{\alpha})$ (cf. [12]). Thus,

$$\|A(|f|/\|g\|_{A^{-1}(X)})\|_X = \|A(|g|/\|g\|_{A^{-1}(X)})\|_X \leqslant 1.$$

which implies $\|f\|_{A^{-1}(X)} \leq \|g\|_{A^{-1}(X)}$ and similarly we get

$$\|g\|_{A^{-1}(X)} \leqslant \|f\|_{A^{-1}(X)}.$$

(ii) Let $f \in A^{-1}(X)$, then $A(\frac{f^*}{\alpha}) \sim A(\frac{|f|}{\alpha})$ $\forall \alpha > 0$. Therefore, $\|f\|_{A^{-1}(X)} = \|f^*\|_{A^{-1}(\hat{X})} = \|f^*\|_{A^{-1}(X)\hat{\ }}$.

(iii) Let $E \subseteq \Omega$ be a measurable set, $0 < \mu(E) < \mu(\Omega)$, then

$$\phi_{A^{-1}(X)}(\mu(E)) = \phi_{A^{-1}(\hat{X})}(\mu(E))$$

$$= \inf\{\alpha > 0 : \| A(\frac{\chi(0,\mu(E))}{\alpha}) \|_{\hat{X}} \leq 1\}$$

$$= \inf\{\alpha > 0 : A(\frac{1}{\alpha})\ \phi_X(\mu(E)) \leqslant 1\}$$

$$= 1/A^{-1}(1/\phi_X(\mu(E))). \quad +$$

1.2. THE SPACES $GA^{-1}(X)$. Let (Ω,μ) be a measure space, let A be a generalised Young's function and let $X(\Omega)$ be a B.S.F.. In this section we consider a generalisation of the spaces $A^{-1}(X)$.

(1.2.1) DEFINITION. Let $f \in M(\Omega)$, define

$$\|f\|_{GA^{-1}(X)} = \inf\{k > 0 : \| A(|f|/k)\|_X \leq k\}$$

$$GA^{-1}(X) = \{f \in M(\Omega) : \|f\|_{GA^{-1}(X)} < \infty\}.$$

It is not difficult to see that $\| \ \|_{GA^{-1}(X)}$ defines a metric under which $GA^{-1}(X)$ is a complete metric space. However, the spaces $GA^{-1}(X)$ are not, in general, topological vector spaces.

In this section we compare the spaces $A^{-1}(X)$ and $GA^{-1}(X)$ and obtain necessary and sufficient conditions for scalar multiplication to be continuous on $GA^{-1}(X)$.

For future reference we point out that (1.1.1) can be generalised as follows

$$(1.2.2) \quad \| A(|f|/\|f\|_{GA^{-1}(X)})\|_X \leq \|f\|_{GA^{-1}(X)}.$$

(1.2.3) PROPOSITION. Let A be a Young's function, then

(i) $A^{-1}(X) = GA^{-1}(X)$ as sets.

(ii) $\| \ \|_{GA^{-1}(X)}$ and $\| \ \|_{A^{-1}(X)}$ determine the same uniform topologies on $A^{-1}(X)$.

Proof. Suppose that $0 < \|f\|_{GA^{-1}(X)} \leq 1$, then by (1.2.1),

$\|A(|f| / \|f\|_{GA^{-1}(X)})\|_X \leq \|f\|_{GA^{-1}(X)} \leq 1$, which implies $\|f\|_{A^{-1}(X)} \leqslant \|f\|_{GA^{-1}(X)}$. Now since $A(x).x^{-1} \uparrow$ we have for $0 < r \leq 1$,

$$\frac{A(|f(x)|/ r^{1/2})}{|f(x)|/ r^{1/2}} \leq \frac{A(|f(x)|/r)}{|f(x)|/ r}, \quad \forall x \text{ such that } |f(x)| > 0.$$

Therefore, $\|A(|f|/ r^{1/2})\|_X \leq r^{1/2}$ and if we choose $r = \|f\|_{A^{-1}(X)}$ we get $\|f\|^2_{GA^{-1}(X)} \leq \|f\|_{A^{-1}(X)}$.

Suppose that $\|f\|_{A^{-1}(X)} \leq 1$, then it follows readily that $\|f\|_{GA^{-1}(X)} \leq 1$. Thus, we have proved,

$$(1.2.4) \quad \|f\|_{GA^{-1}(X)} \leq 1 \vee \|f\|_{A^{-1}(X)} \leq 1 \Rightarrow \|f\|^2_{GA^{-1}(X)} \leq$$
$$\leqslant \|f\|_{A^{-1}(X)} \leqslant \|f\|_{GA^{-1}(X)}.$$

Consider the case where $\|f\|_{GA^{-1}(X)} > 1$, then $\|f\|_{A^{-1}(X)} > 1$ and moreover $\|f\|_{GA^{-1}(X)} \leq \|f\|_{A^{-1}(X)}$. Let $r = \|f\|_{GA^{-1}(X)}$ then $r^2 \geq r$, and since $A(x).x^{-1}\uparrow$ we obtain

$$\|r\, A(|f|/ r^2)\|_X \leq \|A(|f|/ r)\|_X \leq r.$$

Thus, $r^2 \geq \|f\|_{A^{-1}(X)}$ and we have proved

(1.2.5) $\|f\|_{GA^{-1}(X)} \geq 1 \vee \|f\|_{A^{-1}(X)} \geq 1 \Rightarrow \|f\|_{GA^{-1}(X)} \leq$

$$\leq \|f\|_{A^{-1}(X)} \leq \|f\|^2_{GA^{-1}(X)} \quad .$$

The required result follows readily from (1.2.4) and (1.2.5).+

The above result can be extended to generalised Young's functions. We shall omit the details of the proof of the following

(1.2.6) PROPOSITION. Let X be a B.F.S. with a.c.n., and let A be a generalised Young's function. Then,

(i) $A^{-1}(X) = GA^{-1}(X)$ if and only if $\lim_{t \to 0} A(t) = 0$.

(ii) If A satisfies the Λ condition and $\lim_{t \to 0} A(t) = 0$, then the topologies of $A^{-1}(X)$ and $GA^{-1}(X)$ coincide. +

We shall now consider necessary and sufficient conditions on A for scalar multiplication to be continuous on $GA^{-1}(X)$.

(1.2.7) PROPOSITION. Let X be a q. B.F.S. with a.c.q.n. and let A be a generalised Young's function, then scalar multiplication is continuous on $GA^{-1}(X)$ if and only if $\lim_{t \to 0} A(t) = 0$, $\lim_{t \to \infty} A(t) = \infty$.

Proof. Suppose that $\lim_{t \to 0} A(t) = 0$, $\lim_{t \to \infty} A(t) = \infty$, then $f \in GA^{-1}(X)$ implies $|f| < \infty$ μ.a.e.. Let $\varepsilon > 0$, and $\theta > 0$, then since $\lim_{\theta \to 0} A(\theta|f|) = 0$ μ.a.e., and X has a.c.q.n. we get $\lim_{\theta \to 0} \|A(\frac{\theta|f|}{\varepsilon})\|_X = 0$, which implies $\|\theta f\|_{GA^{-1}(X)} < \varepsilon$ if θ is sufficiently small.

Suppose that $\lim_{t \to 0} A(t) = \alpha > 0$. Let $f \neq 0 \in GA^{-1}(X)$, then there exist $E \subseteq \Omega$, $0 < \mu(E) < \infty$, $r > 0$, such that $|f(x)| \geq r.\ \chi_E(x)$. Then $||\theta f||_{GA^{-1}(X)} \geq \alpha ||\chi_E||_X$, $\forall\, \theta > 0$. Thus, scalar multiplication is not continuous.

Suppose that $\lim_{t \to \infty} A(t) = \alpha < \infty$. Let E be a measurable set such that $0 < \mu(E) < \infty$ and define $f = \infty.\chi_E$. Then, $f \in GA^{-1}(X)$ and $||\theta f||_{GA^{-1}(X)} \geqslant \alpha ||\chi_E||_X, \forall\, \theta > 0$. +

The condition that X be a q.B.F.S. with a.c.q.n. cannot be weakened in general. However , for the $GA^{-1}(\tilde{M}(X))$ spaces the following result holds,

(1.2.8) PROPOSITION. Let $X(\Omega)$ be a r.i. space and let A be a generalised Young's function. Then scalar multiplication is continuous on $GA^{-1}(\tilde{M}(X))$ if A satisfies the Λ condition. Conversely if $\mu(\Omega) = \infty$ and Range $(\phi_X) = [0,\infty)$, then scalar multiplication is continuous on $GA^{-1}(\tilde{M}(X))$ if and only if A satisfies the Λ condition.

Proof. Suppose that A satisfies the Λ condition. There exist $\alpha \geq 1$, $\beta > 1$ such that $A(\alpha\, x) \geq \beta\, A(x)$, therefore $\forall\, n \in N$, $A(\alpha^n x) \geq \beta^n A(x)$. Let $f \neq 0$ be a function in $GA^{-1}(X)$, $||f||_{GA^{-1}(X)} = r$, let $\varepsilon > 0$ and choose $n \in N$, and $\theta \in (0,\infty)$ such that $r \leq \beta^n.\varepsilon^{-1}$, $0 < \theta < \varepsilon\, r^{-1}.\ \alpha^{-n}$, then $\forall\ t > 0$

$$\begin{aligned} A(\theta f^*(t)/\varepsilon)\ \phi_X(t) &= \beta^{-n}\ [\beta^n A\ (\theta f^*(t)/\varepsilon)]\ \phi_X(t) \\ &\leq \beta^{-n}\ A(\alpha^n\ \theta\varepsilon^{-1} f^*(t))\ \phi_X(t) \\ &\leq \beta^{-n}\ A(f^*(t)/\ r)\ \phi_X(t). \end{aligned}$$

Therefore,

$$\sup_{t > 0} \{A(\frac{\theta f^*(t)}{\varepsilon}) \ \phi_X(t)\} \leq r.\ \beta^{-n} \leq \varepsilon.$$

$$\lim_{\theta \to 0} ||\theta f||_{GA^{-1}(X)} = 0.$$

The converse can be proved by observing that one can assume that $\Omega = (0,\infty)$ and considering $f(t) = 1/A^{-1}(1/\phi_X(t))$. +

1.3. SPACES X^0. Let $\ell \in (0,\infty]$, let $Y(0,\ell)$ be a B.F.S. of Lebesgue measurable functions on $(0,\ell)$ and let (Ω,μ) be a measure space such that $\mu(\Omega) = \ell$, then we define

$$Y^0(\Omega) = \{f \in M(\Omega): ||f||_{Y^0} = ||f^{**}||_Y < \infty\}.$$

It is easy to see that Y^0 is a Banach space (cf. [9]) and moreover if Y satisfies the n_2 condition (c.f. Chapter 0) then Y^0 is a r.i. space. (The n_2 condition implies that $\chi_E \in Y^0$ if $\mu(E) < \infty$.)

Let $X(\Omega)$ be a q.r.i. space and A be a Young's function, the purpose of this section is to study the relationship between $A^{-1}(X)(\Omega)$ and $A^{-1}(\hat{X})^0(\Omega)$.

(1.3.1) THEOREM. Suppose that the operator P given by $P(f)(t) = \frac{1}{t} \int_0^t f(s)\, ds$ defines a bounded linear operator $P: \hat{X} \to \hat{X}$, then $A^{-1}(X) \cong A^{-1}(\hat{X})^0$.

Proof. Let $f \in A^{-1}(X)^0$ then, since $f^*(t) \leq f^{**}(t)$ $\forall\, t > 0$, we have

$$\|f\|_{A^{-1}(X)} = \|f^*\|_{A^{-1}(\hat{X})} \leq \|f^{**}\|_{A^{-1}(\hat{X})}$$

$$= \|f\|_{A^{-1}(\hat{X})^0} .$$

Suppose that $f \in A^{-1}(X)$, $0 < \|f\|_{A^{-1}(X)} = r$, and let $\theta = \max\{1, \|P\|_{\hat{X} \to \hat{X}}\}$, then $\|f\|_{A^{-1}(\hat{X})^0} = \|P(f^*)\|_{A^{-1}(\hat{X})}$.

Now, using Jensen's inequality we get

$$\|A(P(f^*/\theta r))\|_{\hat{X}} \leq \|P(A(f^*/\theta r))\|_{\hat{X}}$$

$$\leq \|P\|_{\hat{X} \to \hat{X}} \|A(f^*/\theta r)\|_{\hat{X}}.$$

But since $A(x).x^{-1} \uparrow$ we get $\|A(f^*/\theta r)\|_{\hat{X}} \leq \theta^{-1} \|A(f^*/r)\|_{\hat{X}} \leq \theta^{-1}$, hence

$$\|Pf^*\|_{A^{-1}(\hat{X})} \leq \theta \|f\|_{A^{-1}(X)} . +$$

(1.3.2) COROLLARY. Let $X(0,\infty)$ be a r.i. space, then

(i) $A^{-1}(\Lambda(X)) \cong A^{-1}(\Lambda(X))^0$, whenever $\exists \theta > 0$ such that

$$\int_t^\infty \phi_X(u) \frac{du}{u^2} \leq \theta \phi_X(t).t^{-1} , \quad \forall t > 0.$$

(ii) $A^{-1}(\tilde{M}(X)) \cong A^{-1}(\tilde{M}(X))^0$, whenever X' satisfies the δ_2 condition.

Proof. (i) It is easy to see that the condition on ϕ_X implies that $P: \Lambda(X) \to \Lambda(X)$, continuously, in fact this can be proved using Fubini's theorem. Then apply (1.3.1).

(ii) We shall prove that the condition on X' implies that

$P: \widetilde{M(X)} \to \widetilde{M(X)}$ continuously. In fact $\forall\, t > 0$,

$$P(f)^*(t)\ \phi_X(t) \le P(f^*)(t)\ \phi_X(t)$$

$$\le \frac{1}{\phi_{X'}(t)} \int_0^t f^*(s)\ \phi_X(s)\phi_{X'}(s)\frac{ds}{s}$$

$$\le \|f\|_{\widetilde{M(X)}} \left(\frac{1}{\phi_{X'}(t)} \int_0^t \phi_{X'}(s)\frac{ds}{s}\right)$$

$$\le \|f\|_{\widetilde{M(X)}} \cdot C$$

where C is an absolute constant.

Thus,

$$\|P(f)\|_{\widetilde{M(X)}} \le C\ \|f\|_{\widetilde{M(X)}} .$$

The result follows by (1.3.1). +

Let us note that $A^{-1}(\widetilde{M(X)}) \cong M(A^{-1}(X))^{\sim}$ for every r.i. space X, therefore if the conditions of (1.3.2) hold we have $A^{-1}(\widetilde{M(X)}) \cong$ $\cong A^{-1}(\widetilde{M(X)})^0 \cong M(A^{-1}(X))^{\sim 0} \cong M(A^{-1}(X))$.

(1.3.3) REMARK. It is well known and easy to see that $P: \hat{X} \to \hat{X}$ whenever $\int_0^1 h(s,\hat{X})\ ds < \infty$, where $h(s,\hat{X}) = \|E_s\|_{\hat{X} \to \hat{X}}$, $(E_s\, f)(t) = f(t.s)$. The following result holds:

$h(s, A^{-1}(\Lambda(X))) \le \sup_{t>0} \{A^{-1}(t)/A^{-1}(\phi_X(s)\ t)\}$, whenever ϕ_X is strictly increasing, $\lim_{t\to\infty} \phi_X(t) = \infty$, and there exists a constant $\theta > 0$ such that $\phi_X(u)\ \phi_X(v) \le \theta\ \phi_X(u.r)\ \forall\, u, v > 0$. In the case that $X = L^p$, then $h(s,A^{-1}(L(p,1))) = \sup_{t>0} \{A^{-1}(t)/A^{-1}(t.s^{1/p})\}$, $1 \le p < \infty$. These results can be proved using similar methods to those of Boyd [7], where the case p = 1 is considered.

1.4. EXAMPLES. We consider several examples which illustrate the scope of our results. For simplicity we assume $\Omega = (0,\infty)$, μ = Lebesgue measure.

(1.4.1) Let $X = L^1$, then $A^{-1}(X) = L_A$, and we obtain the classical Orlicz spaces.

(1.4.2) Let $X = \Lambda(X)$, then if $A(t) = t^p$, $1 \leq p < \infty$, $A^{-1}(\Lambda(X)) = \Lambda(\phi_X, p)$ (cf. Lorentz [31]). Thus the $A^{-1}(\Lambda(X))$ spaces generalise the Lorentz spaces $\Lambda(\phi, p)$ as well as the Orlicz spaces L_A.

(1.4.3) Let $X = L^1$, then $\tilde{M}(X) = L(1, \infty)$ ("weak L^1") and $M(L_A) = M(A^{-1}(L^1)) = A^{-1}(L(1, \infty))^0$. Moreover $GA^{-1}(L(1,\infty))$ coincides with the O'Neil space W_A (cf [43]). Notice that if A satisfies the ∇_2 condition then $M(L_A) \cong M(L_A)^{\sim} \cong A^{-1}(L(1, \infty))$.

(1.4.4) Let $A(t) = t^r(\log^+ t)^s$ and $X = L(p,q)$, where $0 < r < \infty$, $0 \leq s < \infty$, $1 \leq p < \infty$, $0 < q \leq \infty$. Then $f \in A^{-1}(L(p,q))$ if $\exists\, \theta > 0$ such that

$$\int_0^\infty |f^*(t)|^{rq} \left(\log^+ \left|\frac{f^*(t)}{\theta}\right|\right)^{sq} t^{q/p} \frac{dt}{t} < \infty, \text{ if } q < \infty$$

$$\sup_{t>0} \{f^*(t)^r \log^+ \left|\frac{f^*(s)}{\theta}\right|^s t^{1/p}\} < \infty \text{ if } q = \infty .$$

These spaces provide a generalisation of the $L^r(\mathrm{Log}^+ L)^s$ spaces as well as the $L(p,q)$ spaces.

(1.4.5) Let $A_1(t) = \begin{cases} 0 \text{ if } & 0 \leq t \leq 1 \\ \infty \text{ if } & t > 1 \end{cases}$, then A_1 is a Young's function and $A^{-1}(X) = L^{\infty}$ for every B.F.S. X.

(1.4.6) Let $A_2(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$, then A_2 is a generalised Young's function, and for every r.i. space $X(\Omega)$ we have

$$\|f\|_{GA^{-1}(X)} = \phi_X\left(\lim_{t\to 0} \lambda_f(t)\right).$$

Then $GA_2^{-1}(X) = \{f \in M(0,\infty): f \text{ is of bounded support}\}$ whenever $\lim_{t\to\infty} \phi_X(t) = \infty$; if $\lim_{t\to\infty} \phi_X(t) < \infty$ then $GA_2^{-1}(X) = M(0,\infty)$. In either case scalar multiplication is not continuous on $GA_2^{-1}(X)$.

(1.4.7) Let $A_3(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t > 1 \end{cases}$. Then A_3 is a generalised Young's function and for every r.i. space X we have

$$\|f\|_{GA_3^{-1}(X)} = \inf\{\alpha : \phi_X(\lambda_f(\alpha)) \le \alpha\}.$$

Thus, if $\lim_{t\to\infty} \phi_X(t) = \infty$, $GA_3^{-1}(X) = L^{\infty} + BS$ where $BS = \{f \in M(0,\infty): f \text{ has bounded support}\}$.

1.5. CONDITIONS FOR SEPARABILITY.

Let (Ω,μ) be a separable measure space, let $X(\Omega)$ be a B.F.S. with a.c.n. and let A be a generalised Young's function; in this section we give necessary and sufficient conditions for $GA^{-1}(X)(\Omega)$ and $A^{-1}(X)$ (Ω) to be separable.

Our main results are given by

(1.5.1) THEOREM. (i) Suppose that $\mu(\Omega) = \infty$, and A is a

Young's function, then $A^{-1}(X)(\Omega)$ is separable if and only if A satisfies the Δ_2 condition.

(ii) Suppose that $\mu(\Omega) < \infty$, and A is a Young's function then $A^{-1}(X)(\Omega)$ is separable if and only if A satisfies the Δ_2 condition for large values.

The above result can be generalised, to the spaces $GA^{-1}(X)$, as follows

(1.5.2) THEOREM. Let A be a generalised Young's function, then

(i) If $\mu(\Omega) = \infty$, then $GA^{-1}(X)(\Omega)$ is separable if and only if A satisfies the Δ_2 condition and $\lim_{t \to 0} A(t) = 0$.

(ii) If $\mu(\Omega) < \infty$, then $GA^{-1}(X)(\Omega)$ is separable if and only if A satisfies the Δ_2 condition for large values and $\lim_{t \to 0} A(t) = 0$.

We shall only give a proof of (1.5.1) (i) since (1.5.1) (ii) can be proved similary and moreover (1.5.2) can be proved using similar methods to those of reference [43] .

Proof of (1.5.1) (i). Let A be a Young's function verifying the Δ_2 condition, we shall prove that $\forall f \in A^{-1}(X)$ and for every sequence of measurable sets $\{A_n\}$, such that $A_n \downarrow \phi$ we have $\lim_{n \to \infty} \|f \chi_{A_n}\|_X = 0$.

Let $f \in A^{-1}(X)$, $\|f\|_{A^{-1}(X)} = r \neq 0$ and let $\{A_n\}$ be a sequence of measurable sets such that $A_n \downarrow \phi$. There exists $\theta \geq 1$ such that $A(2x) \leq \theta\, A(x)$ $\forall\, x \geq 0$. Let $\varepsilon > 0$, and choose $\alpha \in N$ such that $2^{\alpha} > r.\ \varepsilon^{-1}$, then $\forall\, n \in N$ we have

$$A(|f|\chi_{A_n}/\varepsilon) = A(2^{\alpha}|f|\chi_{A_n}/2^{\alpha}\,\varepsilon)$$

$$\leq \theta^{\alpha}A(|f|\chi_{A_n}/r)$$

$$\leq \theta^{\alpha}A(|f|/r).$$

Hence,

$$||A(|f|\chi_{A_n}/\varepsilon)||_X \leq \theta^{\alpha} \qquad \forall\, n \in N.$$

Therefore, since $\lim_{t\to 0} A(t) = 0$ and X has a.c.n., if follows that, if n is sufficiently large, $||A(|f|\chi_{A_n}/\varepsilon)||_X \leq 1$. From the above it follows readily that $\lim_{n\to\infty} ||f\chi_{A_n}||_X = 0$.

Thus, $A^{-1}(X)$ has a.c.n. and therefore is separable (cf Luxemburg [33]).

Let us now prove that the conditions on A are necessary. We consider two cases:

(I) Suppose that there exists $x_0 \varepsilon (0,\infty)$ such that $A(x_0) = \infty$, then we shall prove that $A^{-1}(X)$ is not separable. Indeed for $E \subsetneq \Omega$, $0 < \mu(E) < \infty$, define $f_E = \chi_E$, then $||f_E||_{A^{-1}(X)} = 1/A^{-1}(1/\,||\chi_E||_X)$, and therefore $\lim_{\mu(E)\to 0} ||f_E||_{A^{-1}(X)} = 1/A^{-1}(\infty) \geq x_0^{-1}$. Thus, $A^{-1}(X)$ does not have a.c.n. and therefore is not separable.

(II) Suppose that A is finite valued but A does not satisfy the Δ_2 condition. Suppose to the contrary that $A^{-1}(X)$ is separable and let $\{f_n\}_{n=0}^{\infty}$ be dense in $A^{-1}(X)$.

Since A does not satisfy the Δ_2 condition there exists a sequence $\{s_n\}_{n=0}^{\infty}$ such that $A(s_n) > 2^n A(2s_n)$, $n = 0, \ldots\ldots\ldots$ Let $\{E_n\}_{n=0}^{\infty}$ be a sequence of disjoint measurable sets such that $\|\chi_{E_n}\|_X = 2^{-n}[A(s_n)]^{-1}$ and f_n has constant sign on E_n. For $n = 0, \ldots\ldots,$ define

$$r(n) = \begin{cases} 1 & \text{if} \quad f_n \chi_{E_n} \geq 0 \\ -1 & \text{if} \quad f_n \chi_{E_n} < 0 \end{cases}$$

Let $f = -\Sigma r(n) s_n \chi_{E_n}$, then $\|A(|f|)\|_X = 1$, but

$$\|A(|f_n - f|2)\|_X \geq \|A(2s_n)\chi_{E_n}\|_X > 1, \quad n = 0 \ldots\ldots\ldots$$

Therefore $\|f_n - f\|_{A^{-1}(X)} \geq 1/2$, $n = 0 \ldots\ldots,$ a contradiction. +

(1.5.3) COROLLARY. Let $X(\Omega)$ be a r.i. space verifying the δ_2 condition, then

(i) if $\mu(\Omega) = \infty$, then $A^{-1}(\Lambda(X))(\Omega)$ is separable if and only if A satisfies the Δ_2 condition.

(ii) If $\mu(\Omega) < \infty$, then $A^{-1}(\Lambda(X))(\Omega)$ is separable if and only if A satisfies the Δ_2 condition for large values. +

1.6. NOTES TO CHAPTER 1. The theory of Orlicz spaces seems to have originated in Orlicz [46] and Birnbaum and Orlicz [3]. An excellent modern account of the theory is given in Luxemburg [32] (see also [27]).

The theory of $\xi(X)$ spaces was initiated by Beurling [2] and Calderón [9].

The author has been informed that the spaces $A^{-1}(X)$ have been also studied by Eijnsbergen in his unpublished thesis under Professor A. Zaanen.

However, the results presented here seem to be new. Most of the material presented in this chapter extends and unifies work by O'Neil [43].

We have omitted a discussion of the duality theory of Calderón spaces. Some results in this direction will be presented in the next chapter. However, we wish to point out here a general result communicated to the author by Professor W.A.J. Luxemburg.

(1.6.1) THEOREM. Let $X(\Omega)$ be a B.F.S., and let A be a Young's function, then

$$(A^{-1}(X))' = \{f \in M(\Omega) : \|f\|_{A^{-1}(X)'} = \inf_{k>0} \sup_{\|w\|_{X'} \leq 1} (\frac{1}{k}\int_\Omega \bar{A}(\frac{|f|k}{|w|})|w|d\mu + \frac{1}{k}) < \infty\}$$

The reader is referred to §2.4 where we compute the associate spaces of $A^{-1}(\Lambda(X))$ spaces and some more general spaces using a different method.

CHAPTER 2

O.C.L.H.Z. SPACES

Using the constructions of Chapter 1 we introduce the class of "O.C.L.H.Z. spaces" which generalise the spaces $A^{-1}(\Lambda(Y))$ as well as other well known classes of spaces.

In §2.2. and §2.3. Hardy type inequalities are proved in this generalised setting. In §2.4 we compute the associate spaces and the dual spaces of O.C.L.H.Z. spaces, and moreover we give sufficient conditions for reflexivity.

The last section, §2.5 , contains a brief comparison of our results with those already avalaible in the literature.

2.1. O.C.L.H.Z. SPACES. In this section we introduce a class of spaces which can be obtained using the constructions outlined in Chapter 1.

In order to avoid notational complications we shall assume throughout this chapter that all spaces are B.F.Spaces of Lebesque measurable functions on $(0,\infty)$.

Let A be a generalised Young's function and let ξ_1, ξ_2 be concave non-decreasing functions $\xi_i: R^+ \to R^+$, $i = 1,2$, moreover define $\xi: R^+ \times R^+ \to R^+$ by $\xi(x, t) = A^{-1}(t) / \xi_1(x)$. Let $d\mu(t) = \xi_2(t)\frac{dt}{t}$ and consider the spaces $L^1(d\mu) = \{f \in M(0, \infty): \|f\|_{L^1(d\mu)} = \int_0^\infty |f(x)|d\mu(x) < \infty\}$. Then we define

$$\Lambda(A, \xi_1, \xi_2) = [\xi(L^1(d\mu))]^0 .$$

It follows that

$$\Lambda(A, \xi_1,\xi_2) = \{f \in M(0,\infty): \|f\|_{\Lambda(A,\xi_1,\xi_2)} < \infty\}$$

$$\|f\|_{\Lambda(A,\xi_1,\xi_2)} = \inf \{\alpha > 0: \int_0^\infty A(f^{**}(t)\ \xi_1(t)\alpha^{-1})\ d\mu(t) \leq 1\} .$$

We shall also consider the spaces $\Lambda(A,\xi_1, \xi_2)^\sim$ defined by

$$\Lambda(A,\xi_1,\xi_2) = \{f \in M(0,\infty): \|f\|^\sim_{\Lambda(A,\xi_1,\xi_2)} < \infty\}$$

$$\|f\|^\sim_{\Lambda(A,\xi_1,\xi_2)} = \inf \{\alpha > 0: \int_0^\infty A(f^*(t)\ \xi_1(t)\alpha^{-1})\ d\mu(t) \leq 1\} .$$

In particular if ϕ_X, ϕ_Y are the fundamental functions of r.i. spaces X and Y, the spaces $\Lambda(A, \phi_X, \phi_Y)$, $\Lambda(A, \phi_X, \phi_Y)^\sim$ shall be referred as "O.C.L.H.Z. spaces" (thinking in the names of Orlicz, O'Neil, Calderón, Lorentz, Luxemburg, Halperin, Zaanen).

(2.1.1) EXAMPLE. Let $X = L^{\infty}$ so that $\phi_X \equiv 1$, then $\Lambda(A, 1, \phi_Y)^{\sim} = A^{-1}(\Lambda(Y))$ and if the condition of (1.3.2) (i) is satisfied then $\Lambda(A, 1, \phi_Y) \cong A^{-1}(\Lambda(Y))$. In particular if $Y = L^1$, $\Lambda(Y) = L^1$ and $\Lambda(A, 1, \phi_{L_1})^{\sim} = \Lambda(A, 1, t)^{\sim} = L_A$; $\Lambda(A, 1, t) \cong L_A$ if A satisfies the ∇_2 condition.

(2.1.2) EXAMPLE. Let $Y = L^{\infty}$ and $X = L^p$, $1 < p < \infty$, moreover let $A(t) = t^q \cdot q^{-1}$, $1 \leqslant q < \infty$, then

$$\Lambda(A, \phi_X, \phi_Y) \doteq \Lambda(t^q q^{-1}, t^{1/p}, 1) \cong L(p, q).$$

(2.1.3) EXAMPLE. Let A be the Young's function defined in (1.4.5), then $\Lambda(A, \phi_X, \phi_Y) \cong M(X)$, and $\Lambda(A, \phi_X, \phi_Y)^{\sim} \cong \tilde{M}(X)$.

It is easy to see that the $\Lambda(A, \phi_X, \phi_Y)$ spaces are Banach spaces whenever A is a Young's function, and moreover they verify all the conditions set out in 0.3 except perhaps that it may happen that $|E| < \infty$ ($|E|$ denotes the Lebesgue measure of E) but $\chi_E \notin \Lambda(A, \phi_X, \phi_Y)$.

2.2. RELATIONSHIP BETWEEN $\Lambda(A, \phi_X, \phi_Y)$ AND $\Lambda(A, \phi_X, \phi_Y)^{\sim}$.

Let X and Y be r.i. spaces, in this section we compare the spaces $\Lambda(A, \phi_X, \phi_Y)$ and $\Lambda(A, \phi_X, \phi_Y)^{\sim}$.

(2.2.1) THEOREM. Let A be a Young's function verifying the Δ_2 condition and suppose that there exists a constant $\theta \gtrsim 1$ such that

$$\text{(i)} \int_0^t \phi_{X'}(u)\frac{du}{u} \leq \theta\, \phi_{X'}(t), \qquad \forall\ t > 0.$$

$$\text{(ii)} \int_t^\infty \phi_X(u)\phi_Y(u)\frac{du}{u^2} \leq \theta\, \phi_X(t)\phi_Y(t)t^{-1}, \qquad \forall\ t > 0.$$

Then, $\forall\ f \in M(0,\infty)$

$$\|f\|^{\sim}_{\Lambda(A,\phi_X,\phi_Y)} \sim \|f\|_{\Lambda(A,\phi_X,\phi_Y)} .$$

Proof. $\|f\|^{\sim}_{\Lambda(A,\phi_X,\phi_Y)} \leq \|f\|_{\Lambda(A,\phi_X,\phi_Y)}$ since $f^*(t) \leq f^{**}(t)$, $\forall\ t > 0$. Suppose that $0 < \|f\|^{\sim}_{\Lambda(A,\phi_X,\phi_Y)} = r < \infty$.

Since A satisfies the Δ_2 condition $\exists\ k \geq 1$ such that $A(2x) \leq k\, A(x)\ \forall\ x \geq 0$; choose $n \in N$ such that $\theta \,.\, 2^{-n} \leq 1$. Then,

$$f^{**}(t)\phi_X(t) = \frac{1}{\theta\phi_{X'}(t)} \int_0^t \theta\ f^*(s)\phi_X(s)\phi_{X'}(s)\frac{ds}{s}$$

therefore by Jensen's inequality

$$A(f^{**}(t)\phi_X(t)r^{-1}k^{-n}) \leq \frac{1}{\theta\phi_{X'}(t)} \int_0^t A(f^*(s)\phi_X(s)\theta r^{-1}k^{-n})\phi_{X'}(s)\frac{ds}{s} .$$

Thus, if we let $d\mu(t) = \phi_Y(t)\frac{dt}{t}$,

$$\begin{aligned}
\int_0^\infty A(f^{**}(t)\phi_X(t)r^{-1}k^{-n})d\mu(t) &\leq \int_0^\infty A(f^*(s)\phi_X(s)\theta r^{-1}k^{-n})\phi_{X'}(s)\theta^{-1} \int_s^\infty \phi_X(t)\frac{d\mu(t)}{t}\ \frac{ds}{s} \\
&\leq \int_0^\infty A(f^*(s)\phi_X(s)\theta r^{-1}k^{-n})d\mu(s) \\
&= \int_0^\infty A(2^n(f^*(s)\phi_X(s)2^{-n}\theta r^{-1}k^{-n}))d\mu(s) \\
&\leq k^n \int_0^\infty A(f^*(s)\phi_X(s)r^{-1}k^{-n})d\mu(s) \\
&\leq \int_0^\infty A(f^*(s)\phi_X(s)r^{-1})d\mu(s) \\
&\leq 1.
\end{aligned}$$

Therefore, $\|f\|_{\Lambda(A,\phi_X,\phi_Y)} \leq k^n \|f\|^{\sim}_{\Lambda(A,\phi_X,\phi_Y)}$. +

The above result can be extended to generalised Young's functions A such that $A(t).t^{-1}\downarrow$. We shall need the following (cf. [58])

(2.2.2) LEMMA. Let A be a generalised Young's function such that $A(t).t^{-1}\downarrow$ and let ϕ be a concave increasing function, then $\forall\, t > 0$

$$\int_0^t f^*(s)\ \phi(s) \frac{ds}{s} \leq \frac{\log 2}{2}\ A^{-1}\left(\frac{2}{\log 2} \int_0^t A(f^*(s)\ \phi(s)) \frac{ds}{s}\right). +$$

(2.2.3) THEOREM. Let A be a generalised Young's function such that $A(t)\ t^{-1}\downarrow$ and suppose the following conditions hold,

(i) $\exists\ \theta > 0$ such that $A(st) \leq \theta\, A(s)\, A(t)$, $\forall s, t \geq 0$.

(ii) $\exists\ M > 0$ such that

$$\int_t^\infty A\left(\frac{\log 2}{2} \frac{1}{\phi_{X'}(s)}\right) \phi_Y(s) \frac{ds}{s} \leq \frac{M\ \phi_Y(t)}{A(\phi_X(t))}, \ \forall\, t > 0.$$

Then,

$$\int_0^\infty A(f^{**}(s)\phi_X(s))\ \phi_Y(s) \frac{ds}{s} \approx \int_0^\infty A(f^*(s)\phi_X(s))\ \phi_Y(s) \frac{ds}{s} .$$

Proof. We shall use (2.2.2).

$$A(f^{**}(t)\phi_X(t)) = A\left(\frac{1}{\phi_{X'}(t)} \int_0^t f^*(s)\ s \frac{ds}{s}\right)$$

$$\leq A\left(\frac{\log 2}{\phi_{X'}(t)2}\ A^{-1}\left(\frac{2}{\log 2} \int_0^t A(f^*(s)s) \frac{ds}{s}\right)\right)$$

$$\leq \theta\, A\left(\frac{\log 2}{\phi_{X'}(t)2}\right) \frac{2}{\log 2} \int_0^t A(f^*(s)s)\frac{ds}{s}$$

$$(2.2.4) \qquad \leq \theta^2 A\left(\frac{\log 2}{\phi_{X'}(t)2}\right) \frac{2}{\log 2} \int_0^t A(f^*(s)\phi_X(s))\; A(\phi_{X'}(s))\frac{ds}{s}$$

Let $d\mu(t) = \phi_Y(t)\frac{dt}{t}$, then using (2.2.4) we get

$$\int_0^\infty A(f^{**}(t)\phi_X(t))\; d\mu(t) \leq \frac{2\;\theta^2}{\log 2} \int_0^\infty A(f^*(s)\phi_X(s))\; A(\phi_{X'}(s)) \int_s^\infty A\left(\frac{\log 2}{\phi_{X'}(t)2}\right) d\mu(t)\frac{ds}{s}$$

$$\leq \frac{2\theta^2}{\log 2} \int_0^\infty A(f^*(s)\phi_X(s))\; d\mu(s). +$$

(2.2.5) REMARK. Let us suppose that the conditions of (2.2.1) hold and moreover that ϕ_X , ϕ_Y are such that

$$\frac{d\phi_X(t)}{dt} \approx \phi_X(t)t^{-1} \quad , \quad \frac{d\phi_Y(t)}{dt} \approx \phi_Y(t)t^{-1}$$

then integrating by parts and using (2.2.1) one can show that

$$\phi_{\Lambda(A,\phi_X,\phi_Y)}(t) \approx \phi_X(t)/\; A^{-1}(1\; /\; \phi_Y(t)).$$

The following consequence of (2.2.1) will be of importance to determine the dual spaces $\Lambda(A,\phi_X,\phi_Y)^*$.

(2.2.6) COROLLARY. Suppose that the conditions of (2.2.1) hold then $\Lambda(A,\phi_X,\phi_Y)$ is separable.

Proof. We shall prove that $\Lambda(A,\phi_X,\phi_Y)$ has a.c.n.. Since A satisfies the Δ_2 condition $\exists\, k \geq 1$ such that $A(2x) \leq k\, A(x)$, $\forall\, x \geq 0$. Let $f \in \Lambda(A,\phi_X,\phi_Y)$, $\|f\|^{\sim}_{\Lambda(A,\phi_X,\phi_Y)} = r$, let $\varepsilon > 0$ and choose $n \in N$ such that

$2^n > r.\ \varepsilon^{-1}$. Then, $\forall\ E \subsetneq (0,\infty)$ such that $|E| < \infty$, we shall have

$$\int_0^\infty A(f^*(t)\chi_{(0,|E|)}(t)\ \phi_X(t)\varepsilon^{-1})\phi_Y(t)\frac{dt}{t} \leq k^n < \infty .$$

Therefore since $\lim\limits_{|E| \to 0} A(f^*(t)\chi_{(0,|E|)}(t)) = 0$ a.e., we have by the dominated convergence theorem

$$\lim_{|E| \to 0} \int_0^\infty A(f^*(t)\chi_{(0,|E|)}(t)\phi_X(t)\varepsilon^{-1})\ \phi_Y(t)\frac{dt}{t} = 0 .$$

Hence, if $|E|$ is sufficiently small

$$\int_0^\infty A(f^*(t)\chi_{(0,|E|)}(t)\ \phi_X(t)\varepsilon^{-1})\phi_Y(t)\frac{dt}{t} \leq 1$$

that is $\lim\limits_{|E| \to 0} \| f^*\chi_{(0,|E|)} \|^{\sim}_{\Lambda(A,\phi_X,\phi_Y)} = 0$.

Now,

$$\|f\chi_E\|_{\Lambda(A,\phi_X,\phi_Y)} \leq \text{const}\ \|f.\chi_E\|^{\sim}_{\Lambda(A,\phi_X,\phi_Y)} \quad \text{(by (2.2.1))}$$

$$\leq \text{const}\ \|f^*\chi_{(0,|E|)}\|^{\sim}_{\Lambda(A,\phi_X,\phi_Y)}.$$

Therefore,

$$\lim_{|E| \to 0} \|f\chi_E\|_{\Lambda(A,\phi_X,\phi_Y)} = 0. \quad +$$

(2.2.7) REMARK. It can be proved using similar methods to those of (1.5.1) that if A does not satisfy the Δ_2 condition then $\Lambda(A,\phi_X,\phi_Y)$ is not separable.

(2.2.8) EXAMPLE. Let $\phi_X(t) = t^\alpha$, $\phi_Y(t) = t^\beta$, $0 < \alpha < 1$, $0 < \beta < 1$ and further suppose that $\alpha + \beta < 1$, then for every Young's function A satisfying the Δ_2 condition we have

$$\int_0^\infty A(f^*(t)t^\alpha)t^\beta\ \frac{dt}{t} \approx \int_0^\infty A(f^{**}(t)t^\alpha)t^\beta\ \frac{dt}{t} .$$

(2.2.9) REMARK. Let α, β, and A be as in (2.2.8) it is of interest to point out that the norms of the "compression" operators $(E_s f)(t) = f(ts)$ can be estimated for the spaces $\Lambda(A, t^{\alpha}, t^{\beta})$ as in the case briefly discussed in (1.3.3). In fact one can show that

$$h(s,\Lambda(A,t^{\alpha},t^{\beta}),\Lambda(A,t^{\alpha},t^{\beta})) = s^{\alpha} \sup_{t>0} \{A^{-1}(t) \;/\; A^{-1}(ts^{\beta})\} .$$

2.3. THE OPERATOR $\tilde{P}$.

The results obtained in § 2.2. can be interpreted as sufficient conditions for the integral operator $P(f)(t) = t^{-1} \int_0^t f(s)ds$ to act continuously on $\Lambda(A,\phi_X,\phi_Y)^{\sim}$ spaces. The purpose of this section is to establish similar results for the adjoint operator $\tilde{P}(f)(t) = \int_t^{\infty} f(s) \frac{ds}{s} = \tilde{f}(t)$.

(2.3.1) THEOREM. Let A be a Young's function satisfying the Δ_2 condition and suppose that there exists a constant $\theta \geq 1$ such that

$$\text{(i)} \int_t^{\infty} \phi_X(s) \frac{ds}{s^2} \leq \theta\, \phi_X(t)t^{-1}, \quad \forall\, t > 0.$$

$$\text{(ii)} \int_t^{\infty} \phi_X(\tfrac{1}{u})\phi_Y(\tfrac{1}{u}) \frac{du}{u} \leq \theta\, \phi_X(t^{-1})\phi_Y(t^{-1}), \quad \forall\, t > 0.$$

Then, there exists an absolute constant C such that $\forall f \in \Lambda(A,\phi_X,\phi_Y)^{\sim}$,

$$||\tilde{f^*}||^{\sim}_{\Lambda(A,\phi_X,\phi_Y)} \leq C\, ||f||^{\sim}_{\Lambda(A,\phi_X,\phi_Y)}.$$

Proof. There exists $k \geq 1$ such that $A(2x) \leq k\, A(x)$ $\forall\, x \geq 0$. Let us choose $n \in \mathbb{N}$ such that $2^{-n}\theta \leq 1$ and let $f \in \Lambda(A,\phi_X,\phi_Y)^{\sim}$, $||f||^{\sim}_{\Lambda(A,\phi_X,\phi_Y)} = r > 0$.

Observe that $\tilde{f^*}(t)$ is non-increasing, therefore $(\tilde{f^*})^* = f^*$ a.e.. Let $\varepsilon = r^{-1} k^{-n}$, then

$$I(\varepsilon) = \int_0^\infty A\left(\int_t^\infty f^*(s)\frac{ds}{s}\ \phi_X(t)\varepsilon\right)\phi_Y(t)\frac{dt}{t}$$

$$= \int_0^\infty A\left(\int_{1/t}^\infty f^*(s)\frac{ds}{s}\ \phi_X(t^{-1})\varepsilon\right)\phi_Y(t^{-1})\ \frac{dt}{t}$$

$$= \int_0^\infty A\left(\int_0^t f^*(s^{-1})\phi_X(s^{-1})\phi_{X'}(s^{-1})ds\ \phi_X(t^{-1})\varepsilon\right)\phi_Y(t^{-1})\frac{dt}{t}$$

by simple changes of variables. But since

$$\int_0^t \phi_{X'}(s^{-1})ds = \int_{1/t}^\infty \phi_{X'}(s)\frac{ds}{s^2} \leqslant \theta\ \phi_{X'}(t^{-1})t = \theta/\phi_X(t^{-1})$$

we obtain using Jensen's inequality and Fubini's theorem

$$I(\varepsilon) \leqslant \theta^{-1}\int_0^\infty A(\theta\ \varepsilon\ f^*(s^{-1})\phi_X(s^{-1}))\phi_{X'}(s^{-1})\int_s^\infty \phi_X(t^{-1})\phi_Y(t^{-1})\frac{dt}{t}\ ds$$

$$\leqslant \int_0^\infty A(\theta.\varepsilon\ f^*(s)\phi_X(s))\phi_Y(s)\frac{ds}{s}$$

$$= \int_0^\infty A(2^n(2^{-n}\ \theta\varepsilon\ f^*(s)\phi_X(s)))\phi_Y(s)ds$$

$$\leqslant \int_0^\infty A(f^*(s)\phi_X(s)r^{-1})\phi_Y(s)\frac{ds}{s} \leqslant 1.$$

Therefore,

$$\|\tilde{f^*}\|^\sim_{\Lambda(A,\phi_X,\phi_Y)} \leqslant k^n \|f\|^\sim_{\Lambda(A,\phi_X,\phi_Y)}. \quad +$$

(2.3.2) THEOREM. Let A be a Young's function verifying the Δ_2 condition, and let Y be a r.i. space verifying the δ_2 condition. Then, there exists an absolute constant C such that $\forall\ f \in \Lambda(A,\ 1,\phi_Y)^\sim$

$$\|\tilde{f^*}\|^\sim_{\Lambda(A,1,\phi_Y)} \leqslant C\ \ \|f\|^\sim_{\Lambda(A,1,\phi_Y)}\ .$$

Proof. Since A satisfies the Δ_2 condition there exists $p > 1$ such that $A(t)t^{-p} \downarrow$, therefore $A'(t)t \leqslant p\, A(t)$, $\forall\, t \geqslant 0$.

Let $\theta \geqslant 1$ be a constant such that $\int_0^t \phi_Y(s)\,\frac{ds}{s} \leqslant \theta\,\phi_Y(t)$, and let $f \in \Sigma(\Lambda(A,\ 1,\ \phi_Y)^{\sim})$. Then, $\int_0^\infty A(f^*(t))\,\phi_Y(t)\frac{dt}{t} \leqslant 1$. Moreover, we shall assume first that there exists $r > 0$ such that $f^*(t) = 0$ for $t \geqslant r$.

Let $\alpha = \theta p$, then integrating by parts and using Young's inequality gives

$$I = \int_0^\infty A(\tilde{f}^*(t)/\alpha)\ \phi_Y(t)\frac{dt}{t}$$

$$\leqslant \frac{\theta}{\alpha}\int_0^\infty A'(\tilde{f}^*(t)/\alpha)\ f^*(t)\,\phi_Y(t)\frac{dt}{t}$$

$$\text{(2.3.3)}\qquad \leqslant\ \frac{\theta}{\alpha}\left[\int_0^\infty A(f^*(t))\,\phi_Y(t)\frac{dt}{t} + \int_0^\infty \overline{A}(A'(\tilde{f}^*(t)/\alpha))\,\phi_Y(t)\frac{dt}{t}\right]\ .$$

It is well known that $\overline{A}(A'(t)) = t\,A'(t) - A(t)$ (cf. [64], page 16 or [32] page 37), therefore, combining this result with (2.3.3), we get

$$I \leqslant \frac{\theta}{\alpha}\ \left[1 + p\,I - I\right]$$

$$I \leqslant 1.$$

Therefore,

$$\|\tilde{f}^*\|_{\Lambda(A,1,\phi_Y)}^{\sim} \leqslant\ \theta p\ \|f\|_{\Lambda(A,1,\phi_Y)}^{\sim}\ .$$

We conclude the proof for an arbitrary $f \in \Sigma(\Lambda(A,1,\phi_Y)^{\sim})$ using the monotone convergence theorem. +

An extension to generalised Young's functions A such that $A(t).t^{-1}\downarrow$ is given by

(2.3.4) **THEOREM.** Let A be a generalised Young's function such that $A(t)t^{-1}\downarrow$ and suppose that there exists a constant $\theta \geq 1$ such that

(i) $A(t.s) \leq \theta A(t)A(s), \quad \forall\, t,s \geq 0.$

(ii) $\int_0^t \phi_Y(s)\frac{ds}{s} \leq \theta\, \phi_Y(t), \quad \forall\, t \geq 0.$

Then,

$$\int_0^\infty A(\tilde{f}^*(t)\phi_X(t))\phi_Y(t)\frac{dt}{t} \leq C \int_0^\infty A(f^*(t)\phi_X(t))\phi_Y(t)\frac{dt}{t}$$

where $C = A(2 \log 2)2\theta^2/\log 2.$

Proof. It is proved in [58] that $\forall\, t > 0$,

$$\int_t^\infty f^*(s)\phi_X(s)\frac{ds}{s} \leq 2 \log 2\, A^{-1}\left(\frac{1}{\log 2} \int_{t/2}^\infty A(f^*(s)\phi_X(s))\frac{ds}{s}\right) .$$

Now, since $A(\phi_X(t)\int_t^\infty f^*(s)\frac{ds}{s}) \leq A(\int_t^\infty f^*(s)\phi_X(s)\frac{ds}{s})$, we have

$$A(\phi_X(t)\tilde{f}^*(t)) \leq A(2 \log 2\, A^{-1}(\frac{1}{\log 2} \int_{t/2}^\infty A(f^*(s)\phi_X(s))\frac{ds}{s}))$$

$$\leq \frac{\theta A(2 \log 2)}{\log 2} \int_{t/2}^\infty A(f^*(s)\phi_X(s))\frac{ds}{s} .$$

Therefore,

$$\int_0^\infty A(\tilde{f}^*(t)\phi_X(t))\phi_Y(t)\frac{dt}{t} \leq \frac{2\theta^2 A(2 \log 2)}{\log 2} \int_0^\infty A(f^*(s)\phi_X(s))\phi_Y(s)\frac{ds}{s}. +$$

2.4. DUALITY OF O.C.L.H.Z. SPACES.

The purpose of this section is to characterise the associate spaces and dual spaces of O.C.L.H.Z. spaces.

(2.4.1) **THEOREM.** Let A be a Young's function verifying the

Δ_2 condition, and let X,Y be r.i. spaces, then

(i) if there exists a constant $\theta \geqslant 1$ such that $\forall\, t > 0$

$$\int_t^{\infty} \phi_Y(s)\frac{ds}{s^2} \leqslant \theta\phi_Y(t)t^{-1} \ ,\int_0^t \phi_Y(s)\frac{ds}{s} \leqslant \theta\ \phi_Y(t)$$

then $\Lambda(A,1,\phi_Y)' \cong \Lambda(\bar{A},\phi_{Y'},\phi_Y)$ and $\Lambda(A,1,\phi_Y)^*$ is isometrically isomorphic to $\Lambda(\bar{A},\phi_{Y'},\phi_Y)$.

(ii) If X and X' verify the δ_2 condition and moreover $\exists\, \theta \geqslant 1$ such that $\int_t^{\infty} \phi_X(u)\frac{du}{u^2} \leqslant \theta\, \phi_X(t)t^{-1}$, $\quad \forall\, t > 0$, then

$\Lambda(A,\phi_X,1)' \cong \Lambda(\bar{A},\phi_{X'},1)$ and $\Lambda(A,\phi_X,1)^*$ is isometrically isomorphic to $\Lambda(\bar{A},\phi_{X'},1)$. Moreover if A satisfies the ∇_2 condition $\Lambda(A,\phi_X,1)$ is reflexive.

For the proof of (2.4.1) we shall need some auxiliary results which shall be useful also when computing the associate spaces of more general $\Lambda(A,\phi_X,\phi_Y)$ spaces.

In what follows we shall let ϕ_1 , ϕ_2 be strictly positive continuous functions defined on $(0,\infty)$, and define

$$\psi(t) = t/\phi_1(t)\phi_2(t), \quad d\mu(t) = \phi_2(t)\frac{dt}{t}\ .$$

(2.4.2) LEMMA. Let $f,g \in M(0,\infty)$ be such that $\|f^{**}\phi_1\|_{L_A(d\mu)} < \infty$, $\|g^{**}\psi\|_{L_{\bar{A}}(d\mu)} < \infty$, then

$$\int_0^{\infty}|f(t)g(t)|dt \leqslant 2\ \|f^{**}\phi_1\|_{L_A(d\mu)}\ \|g^{**}\psi\|_{L_{\bar{A}}(d\mu)}.$$

Proof.

$$\int_0^{\infty}|f(t)g(t)|dt \leqslant \int_0^{\infty} f^{**}(t)g^{**}(t)dt$$

$$= \int_0^\infty f^{**}(t)\,\phi_1(t)\,g^{**}(t)\,\psi(t)\,d\mu(t)$$

$$\leq 2\,\|f^{**}\phi_1\|_{L_A(d\mu)}\,\|g^{**}\psi\|_{L_{\bar{A}}(d\mu)}$$

by Hölder's inequality for the Orlicz spaces (cf.[32]). +

(2.4.3) LEMMA. Let $S(A,\phi_1,\phi_2) = \{f : f \geq 0; \int_0^\infty A(\tilde{f}(s)\,\phi_1(s))\,d\mu(s) \leq 1\}$, where $\tilde{f}(t) = \int_t^\infty f(s)\frac{ds}{s}$, $t > 0$. Then, $\forall\, g \in M(0,\infty)$,

$$\|g^{**}\psi\|_{L_{\bar{A}}(d\mu)} \leq \sup\{\int_0^\infty r^*(t)g^*(t)\,dt : r \in S(A,\phi_1,\phi_2)\} \leq 2\,\|g^{**}\psi\|_{L_{\bar{A}}(d\mu)} .$$

Proof. It is proved in [32] that

$$\|g^{**}\psi\|_{L_{\bar{A}}(d\mu)} \leq \sup\{\int_0^\infty f(t)g^{**}(t)\,\psi(t)\,d\mu(t) : f \geq 0, \int_0^\infty A(f(s))\,d\mu(s) \leq 1\} \leq 2\|g^{**}\psi\|_{L_{\bar{A}}(d\mu)}.$$

On the other hand, using Fubini's theorem, we have

$$\begin{aligned}\sup\{\int_0^\infty r^*(t)g^*(t)\,dt : r \in S(A,\phi_1,\phi_2)\} &= \sup\{\int_0^\infty \tilde{f}(t)g^{**}(t)\,dt : f \geq 0, \int_0^\infty A(\tilde{f}(s)\,\phi_1(s))\,d\mu(s) \leq 1\} \\ &= \sup\{\int_0^\infty \tilde{f}(t)\,\phi_1(t)\,g^{**}(t)\,\psi(t)\,d\mu(t) : f \geq 0, \int_0^\infty A(\tilde{f}(s)\,\phi_1(s))\,d\mu(s) \leq 1\} \\ &= \sup\{\int_0^\infty f(t)g^{**}(t)\,\psi(t)\,d\mu(t) : f \geq 0, \int_0^\infty A(f(s))\,d\mu(s) \leq 1\} .\end{aligned}$$

The result follows . +

We are now ready for the proof of (2.4.1)

Proof of (2.4.1).(i) We shall prove first that $\Lambda(A,\,1,\,\phi_Y)' \cong \Lambda(\bar{A},\,\phi_{Y'},\,\phi_Y)$.

Let $\phi_1(t) = 1\ \forall\, t > 0$, and $\phi_2(t) = \phi_Y(t)$, then

$d\mu(t) = \phi_Y(t)\frac{dt}{t}$, $\Psi(t) = \phi_{Y'}(t)$. It follows from (2.4.2) that $\Lambda(\bar{A},\phi_{Y'},\phi_Y)$ is continuously embedded in $\Lambda(A,1,\phi_Y)'$. Let $g \in \Lambda(A,1,\phi_Y)'$, we shall compute $\|g\|_{\Lambda(\bar{A},\phi_{Y'},\phi_Y)}$. By (2.4.3),

$$\|g\|_{\Lambda(\bar{A},\phi_{Y'},\phi_Y)} \leqslant \sup \{ \int_0^\infty r^*(t)g^*(t)dt : r \in S(A,1,\phi_Y)\}$$
$$= \sup\{\int_0^\infty \tilde{f}(t)g^*(t)dt : f \geqslant 0, \int_0^\infty \dot{A}(f(s))d\mu(s) \leqslant 1\} .$$

By the proof of (2.3.2) there exists a constant $C > 0$ such that

$$\|\tilde{f}\|_{L_A(d\mu)} \leqslant C \|f\|_{L_A(d\mu)} \qquad \forall f \geqslant 0, \quad f \in M(0,\infty) .$$

Therefore,

$$\|g\|_{\Lambda(\bar{A},\phi_{Y'},\phi_Y)} \leqslant \sup \{\int_0^\infty \tilde{f}(t)g^*(t)dt : f \geqslant 0, \|\tilde{f}\|_{L_A(d\mu)} \leqslant C\}$$

but since, for $f \geqslant 0$, $\tilde{f}(t) \downarrow$, and by (1.3.2) there exists a constant $C^1 > 0$ such that $\|f\|_{\Lambda(A,1,\phi_Y)} \leqslant C^1 \|f^*\|_{L_A(d\mu)}$, we get successively

$$\|g\|_{\Lambda(\bar{A},\phi_{Y'},\phi_Y)} \leqslant \sup \{\int_0^\infty f^*(t)g^*(t)dt : \|f^*\|_{L_A(d\mu)} \leqslant C\}$$
$$\leqslant \sup\{\int_0^\infty f^*(t)g^*(t)dt : \|f\|_{\Lambda(A,1,\phi_Y)} \leqslant C.C^1 \}$$
$$\leqslant C.C^1 \|g\|_{\Lambda(A,1,\phi_Y)'} .$$

Thus, we have proved that $\Lambda(A,1,\phi_Y)' \cong \Lambda(\bar{A},\phi_{Y'},\phi_Y)$. To complete the proof of (i) let us observe that under our current hypothesis, $\Lambda(A,1,\phi_Y) = A^{-1} (\Lambda(Y))$ and $A^{-1} (\Lambda(Y))$ is separable by (1.5.3). Therefore $\Lambda(A,1,\phi_Y)'$ is isometrically isomorphic to $\Lambda(A,1,\phi_Y)^*$.

(ii) Let $\phi_1(t) = \phi_X(t)$, $\phi_2'(t) = 1 \quad \forall\ t > 0$, then

$d\mu(t) = \frac{dt}{t}$, $\psi(t) = \phi_{X'}(t)$. Then by (2.4.2) it follows that $\Lambda(\bar{A},\phi_{X'},1)$ is continuously embedded in $\Lambda(A,\phi_X,1)'$. The proof that $\Lambda(A,\phi_X,1)'$ is continuously embedded in $\Lambda(\bar{A},\phi_{X'},1)$ is similar to the last part of the proof of (i) above, this time instead of using (2.3.2) and (1.3.2) we use (2.3.1) and (2.2.1). It follows that $\Lambda(A,\phi_X,1)' \cong \Lambda(\bar{A},\phi_{X'},1)$. Since, by (2.2.6), $\Lambda(A,\phi_X,1)$ is separable it follows that $\Lambda(A,\phi_X,1)'$ is isometrically isomorphic to $\Lambda(A,\phi_X,1)^*$.

Finally if A satisfies the ∇_2 condition then, by (2.2.6), $\Lambda(\bar{A},\phi_{X'},1)$ is separable and therefore $\Lambda(\bar{A},\phi_{X'},1)^*$ is isometrically isomorphic to $\Lambda(A,\phi_X,1)$.+

Using the same method of proof we can generalise (2.4.1) as follows

(2.4.4) THEOREM. (i) Suppose that A,ϕ_X,ϕ_Y, satisfy the conditions of (2.2.1) and (2.3.1) and moreover suppose that $\xi(t) = \phi_X(t)\phi_Y(t)$ is concave, then $\Lambda(A,\phi_X,\phi_Y)' \cong \Lambda(\bar{A},\psi,\phi_Y)$, where $\Psi(t) = \phi_{X'}(t)/\phi_Y(t)$, and moreover $\Lambda(A,\phi_X,\phi_Y)^*$ is isometrically isomorphic to $\Lambda(\bar{A},\psi,\phi_Y)$.

(ii) Suppose that the conditions of (i) hold, and moreover A satisfies the ∇_2 condition and there exists a constant $\theta > 1$ such that $\int_t^\infty \phi_{X'}(u)\frac{du}{u^2} \leq \theta\,\phi_{X'}(t)t^{-1}$, then $\Lambda(A,\phi_X,\phi_Y)$ is isometrically isomorphic to $\Lambda(A,\phi_X,\phi_Y)^{**}$. +

2.5. NOTES TO CHAPTER 2.

The O.C.L.H.Z. spaces seem to have been introduced here for the first time. Note however that the $\Lambda(A,\phi_X,1)$ spaces where introduced in [58] .

Hardy's type inequalities have been proved, with different degrees of generality, by many authors. However, the inequalities of § 2.2 and § 2.3 seem to be new. Similar inequalities in the context of other, less general, spaces may be found in O'Neil [43] , Torchinsky [58] , see also [23] , [56].

The results in § 2.4 generalise classical results concerning the representation of the duals of Orlicz spaces and Lorentz spaces. The case of Orlicz spaces is treated in Luxemburg [32] , the $L(p,q)$ theory is given, for example, in Oaklander [42] , Hunt [25] and Butzer - Berens [8] , the $\Lambda(\alpha,p)$ and $M(\alpha,p)$ spaces are treated in Lorentz [31] and Sharpley [55] . The methods used here are a combination of the ones used in the theory of $L(p,q)$ spaces (cf. [42]) and Orlicz spaces.

Using the methods of Halperin [22] and Luxemburg [32] one could obtain also results concerning the uniform convexity of O.C.L.H.Z. spaces. We hope to report on these matters elsewhere.

Finally we should mention that our results are valid, with minor modifications, for arbitrary separable measure spaces (Ω,μ) such that $\mu(\Omega) = \infty$. The case where $\mu(\Omega) < \infty$ can be treated in a similar fashion but requires some additional technical work.

CHAPTER 3

EMBEDDING THEOREMS AND INTERPOLATION

In §3.1 we introduce the spaces $O(X)$ ("the Orlicz spaces associated with a rearrangement invariant space") and use them to compare Lorentz spaces and Orlicz spaces.

In §3.2 we extend the interpolation theorem of Marcinkiewicz-Calderón-Hunt to the context of O.C.L.H.Z. spaces.

In §3.3. we illustrate the results obtained in previous sections by proving an interpolation theorem for Orlicz spaces. Finally in §3.4 the reader will find the references to the literature and comparisons with the results of other mathematicians.

3.1. ORLICZ SPACES ASSOCIATED WITH R.I. SPACES.

Let $X(0,\infty)$ be a r.i. space of Lebesgue measurable functions on $(0,\infty)$, in analogy with the spaces $\Lambda(X)$ and $M(X)$, we introduce the space $0(X)$, the Orlicz space associated with X.

We recall a few facts about fundamental functions. The fundamental function of X is defined on $(0,\infty)$, we extend its domain of definition putting $\phi_X(0) = 0$ then ϕ_X is a generalised Young's function. Its inverse $\phi_X^{-1}(t) = \inf\{s: \phi_X(s) > t\}$ is such that $\phi_X^{-1}(0) = 0$ and ϕ_X^{-1} is right continuous. Therefore if we define $\phi_{(X)}(t) = 1/\phi_X^{-1}(1/t)$, $t \in [0,\infty)$, $\phi_{(X)}$ is a generalised Young's function. Moreover, since $\phi_X(t)t^{-1} \downarrow$ we shall have $\phi_{(X)}(t)t^{-1} \uparrow$. Using the generalised Young's function $\phi_{(X)}$ we construct the Young's function $\phi^0_{(X)}$ defined as follows

$$(3.1.1) \quad \phi^0_{(X)}(t) = \int_0^t \phi_{(X)}(s)\frac{ds}{s} \quad , \quad t \in [0,\infty) .$$

Therefore,

$$(3.1.2) \quad \phi^0_{(X)}(t) \leq \phi_{(X)}(t) \leq \phi^0_{(X)}(2t) \ , \quad \forall\, t > 0 .$$

(3.1.3) DEFINITION. Let $X(0,\infty)$ be a r.i. space, and let $\phi^0_{(X)}$ be the Young's function defined by (3.1.1), then the Orlicz space associated with X, $0(X)$, is defined by $0(X) = \phi^{0\,-1}_{(X)}(L^1) = L_{\phi^0_{(X)}}$.

In the remainder of this section we shall compare the O.C.L.H.Z. spaces introduced in Chapter 2 and the Orlicz spaces associated with them.

(3.1.4) THEOREM. Let $Y(0,\infty)$ be a r.i. space, and let A be a Young's function, then

(i) $\Lambda(A,1,\phi_Y)^{\sim}(0,\infty)$ is continuously embedded in

$$O(\Lambda(A,1,\phi_Y)^{\sim}) = L_{B^0}(0,\infty), \text{ where } B^0(t) = \int_0^t |1/\phi_Y^{-1}(1/A(s))|\frac{ds}{s} .$$

(ii) If C is a Young's function, and $\Lambda(A,1,\phi_Y)^{\sim}(0,\infty)$ is continuously embedded in $L_C(0,\infty)$, then $O(\Lambda(A,1,\phi_Y)^{\sim})$ is continuously embedded in $L_C(0,\infty)$.

Proof. (i) We recall that $\Lambda(A,1,\phi_Y)^{\sim} = A^{-1}(\Lambda(Y))$ (cf. (2.1.1)), therefore $M(\Lambda(A,1,\phi_Y)^{\sim})^{\sim} = A^{-1}(M(\tilde{Y}))$. Observe that

$$\phi_{(\Lambda(A,1,\phi_Y)^{\sim})}(t) = 1/\phi^{-1}_{\Lambda(A,1,\phi_Y)^{\sim}}(1/t) = 1/\phi_Y^{-1}(1/A(t)) = B(t).$$

Let $f \in \Lambda(A,1,\phi_Y)^{\sim}$, $\|f\|^{\sim}_{\Lambda(A,1,\phi_Y)} = r$, then $f^*(t)/r \leq A^{-1}(1/\phi_Y(t))$ $\forall t > 0$, therefore since $\phi^{-1}_{\Lambda(A,1,\phi_Y)^{\sim}}(t)t^{-1} \uparrow$, we get

$$B(f^*(t)/r)t \leq A(f^*(t)/r)\phi_Y(t), \qquad \forall t > 0.$$

Thus if we let $B^0(t) = \int_0^t (1/\phi_Y^{-1}(1/A(s)))ds = \int_0^t B(s)\frac{ds}{s}$ we obtain

$$B^0(f^*(t)/r)t \leq A(f^*(t)/r)\phi_Y(t) \qquad \forall t > 0$$

therefore,

$$\int_0^\infty B^0(f^*(t)/r)t\,\frac{dt}{t} \leq \int_0^\infty A(f^*(t)/r)\phi_Y(t)\frac{dt}{t}$$

$$\leq \quad 1.$$

Hence,

$$\|f\|_{O(\Lambda(A,1,\phi_Y)^{\sim})} \leq \|f\|_{\Lambda(A,1,\phi_Y)^{\sim}} .$$

(ii) Suppose that $\Lambda(A,1,\phi_Y)^{\sim}$ is continuously embedded in $L_C(0,\infty)$, then there exists a constant $\theta > 0$ such that $\phi_{\Lambda(A,1,\phi_Y)^{\sim}}(t) \geq \theta^{-1}\phi_{L_C}(t)$, $\forall$ $t \geq 0$. Therefore,

$$\frac{1}{C^{-1}(t^{-1})} \leqslant \frac{\theta}{A^{-1}(1/\phi_Y(t))} \quad , \quad \forall\ t > 0$$

$$A^{-1}(1/\phi_Y(t^{-1})) \leqslant \theta\ C^{-1}(t)$$

$$\phi^{-1}(\Lambda(A,1,\phi_Y)^{\sim})(t) \leqslant \theta\ C^{-1}(t) \quad , \quad \forall\ t > 0.$$

Thus,

$$\phi^{0-1}(\Lambda(A,1,\phi_Y)^{\sim})(t) \leqslant \frac{\theta}{2}\ C^{-1}(t) \quad , \quad \forall\ t > 0$$

and therefore $0(\Lambda(A,1,\phi_Y)^{\sim}) \subseteq L_C$, with a continuous embedding. +

(3.1.5) THEOREM. Let q be a positive number, $1 \leqslant q < \infty$, and let $A_q(t) = t^q \cdot q^{-1}$, $t \in [0,\infty)$. Let $X(0,\infty)$ be a r.i.space then,

(i) if $\phi_X(t)^q \cdot t^{-1} \downarrow$, then $\Lambda(A_q,\phi_X,1)^{\sim}$ is continuously embedded in $0(X)$.

(ii) Suppose that $\phi_X(t)^q \cdot t^{-1} \uparrow$, then $0(X)$ is continuously embedded in $\Lambda(A_q,\phi_X,1)^{\sim}$.

Proof. (i) Let us consider the functional

$$\|f\| = \left\{\int_0^\infty |f^*(t)\phi_X(t)|^q \frac{dt}{t}\right\}^{1/q} .$$

It is easily seen that $\|f\| \approx \|f\|^{\sim}_{\Lambda(A_q,\phi_X,1)}$, $\forall f \in M(0,\infty)$.

Suppose that $f \in \Lambda(A_q,\phi_X,1)^{\sim}$, $\|f\| = 1$. Let $\theta = 2^{1/q}$, then

$$\int_0^\infty \phi^0{}_{(X)}(f^*(t)/\theta)dt \leq \int_0^\infty \phi_{(X)}(f^*(t)/\theta)dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{2^n}^{2^{n+1}} \phi_{(X)}(f^*(t)/\theta)dt$$

$$\leq \sum_{n=-\infty}^{\infty} \phi_{(X)}(f^*(2^n)/\theta)2^n$$

$$= \sum_{n=-\infty}^{\infty} 2^n/\phi_X^{-1}(\theta/f^*(2^n)). \tag{3.1.6}$$

Since $\|f\|^{\sim}_{M(X)} \leq \|f\|$, (cf. [55] and also Chapter 0) we have

$$f^*(2^n) \leq \theta/\phi_X(2^n) \qquad \forall\, n \in Z.$$

Now, $\phi_X^q(t)t^{-1}$ is non-increasing therefore it is readily seen that $t^q/\phi_X^{-1}(t)$ is non-increasing, thus

$$\frac{(\theta/f^*(2^n))^q}{\phi_X^{-1}(\theta/f^*(2^n))} \leq \frac{\phi_X^q(2^n)}{\phi_X^{-1}(\phi_X(2^n))} \qquad , \forall\, n \in Z$$

$$\leq \phi_X^q(2^n)2^{-n} \qquad , \forall\, n \in Z$$

$$\frac{2^n}{\phi_X^{-1}(\theta/f^*(2^n))} \leq \theta^{-q}\phi_X^q(2^n)f^*(2^n)^q \ , \forall\, n \in Z .$$

Combining (3.1.6) and (3.1.7), we get

$$\int_0^\infty \phi^0_{(X)}(f^*(t)/\theta)dt \leq \theta^{-q} \sum_{n=-\infty}^{\infty} \phi_X^q(2^n)f^*(2^n)^q$$

$$\leq \theta^{-q} \cdot 2 \sum_{n=-\infty}^{\infty} \int_{2^{n-1}}^{2^n} |\phi_X(t)f^*(t)|^q \frac{dt}{t}$$

$$\leq 1.$$

Thus,

$$\|f\|_{O(X)} \leq \Theta \|f\| \approx \|f\|^{\sim}_{\Lambda(Aq,\phi_X,1)} .$$

(ii) Let $f \in O(X)$ and suppose that $\|f\|_{O(X)} = 1$. Then, using the notation of the proof of (i), we have

$$\|f\|^q = \sum_{n=-\infty}^{\infty} \int_{2^n}^{2^{n+1}} [f^*(t)\phi_X(t)]^q \frac{dt}{t}$$

$$\leq \sum_{n=-\infty}^{\infty} [f^*(2^n)\phi_X(2^{n+1})]^q \cdot 2^{-1}$$

(3.1.8) $$\leq \sum_{n=-\infty}^{\infty} [f^*(2^n)\phi_X(2^n)]^q \cdot 2^{q-1} .$$

Now since $O(X)$ is continuously embedded in $\tilde{M}(O(X)) = \tilde{M}(X)$, we have

$$f^*(2^n) \leq 2/\phi_X(2^n) \quad , \quad \forall \; n \in Z .$$

$$2^n \leq \phi_X^{-1}(2/f^*(2^n)) \quad , \quad \forall \; n \in Z$$

and using the fact that $\phi_X^q(t)t^{-1}\uparrow$, we obtain

$$[f^*(2^n)\phi_X(2^n)]^q \leq 2^n \; \phi_{(X)}(f^*(2^n)/2)$$

(3.1.9) $$\leq \int_{2^n}^{2^{n+1}} \phi_{(X)}(f^*(t)/2)dt$$

Combining (3.1.8) and (3.1.9) we get

$$\|f\|^q \leqslant 2^{q-1} \int_0^\infty \phi_{(X)}(f^*(t)/2)dt$$

$$\leqslant 2^{q-1} \int_0^\infty \phi^0_{(X)}(f^*(t))dt$$

$$\leqslant 2^{q-1} \quad . +$$

(3.1.10) EXAMPLE. Let A be a Young's function, consider the Orlicz space $L_A(0,\infty)$. It is easy to see that $A(t).t^{-q} \downarrow$ implies $\phi_{L_A}(t)^q t^{-1} \uparrow$ and $A(t).t^{-q} \uparrow$ implies $\phi_{L_A}(t)^q\ t^{-1} \downarrow$.

Thus if there exists $q \geqslant 1$ such that $A(t)t^{-q} \uparrow$, we have

$$\|f\|_{L_A} \leqslant C \{\int_0^\infty [f^*(t)\phi_{L_A}(t)]^q\ dt\}^{1/q} \quad , \quad \forall\ f \in M(0,\infty),$$

where C is an absolute constant.

Similary if there exists $q \geqslant 1$ such that $A(t)t^{-q} \downarrow$, we have

$$\{\int_0^\infty [f^*(t)\phi_{L_A}(t)]^q\ dt\}^{1/q} \leqslant M\ \|f\|_{L_A}, \qquad \forall\ f \in M(0,\infty)$$

where M is an absolute constant.

3.2. INTERPOLATION AND $\Lambda(A,\phi_X,\phi_Y)$ SPACES. In this section we extend the well known interpolation theorems of Marcinkiewicz-Calderón-Hunt to the context of O.C.L.H.Z. spaces.

We shall consider only r.i. spaces of Lebesgue measurable functions on $(0,\infty)$, but the reader will have no problems to extend our results to more general situations.

Let X_i, Y_i, $i = 1,2$ be r.i. spaces. A linear (sublinear) operator T such that $T:\Lambda(X_i) \to M(Y_i)$, $i = 1,2$, continuously, is said to be of weak types (X_i, Y_i), $i = 1,2$. These operators are majorized by integral operators:

(3.2.1) LEMMA, (Calderón [10], Zippin [63], Sharpley [55]). Let T be a linear (sublinear)operator of weak types (X_i,Y_i), $i = 1,2,$. Moreover suppose that $\min_{i=1,2} \{\phi_{X_i}(0^+)\} = 0$. Then,

(i) $$T(f)^*(t) \leq 2 \max_{i=1,2} \{ \|T\|_i\} \int_0^\infty k(t,s) f^*(s) ds ,$$

$\forall f \in \Lambda(X_1) + \Lambda(X_2)$, where $\|T\|_i$ denotes the norm of the operator T, $T: \Lambda(X_i) \to M(Y_i)$, and $k(t,s) = \frac{d}{ds}\psi(t,s)$, $\psi(t,s) = \min_{i=1,2} \{\phi_{X_i}(s)/\phi_{Y_i}(t)\}$.

(ii) A pair of r.i. spaces (X,Y) is intermediate for the interpolation segment $\sigma = [\Lambda(X_1), M(Y_1) ; \Lambda(X_2), M(Y_2)]$ (i.e. every linear (sublinear) operator of weak types (X_i,Y_i), $i = 1,2$, maps X into Y continuously) if and only if $\forall f \in X$, $S_\sigma(f) = \int_0^\infty k(t,s) f(s) ds$ is well defined and moreover $S_\sigma(f) \in Y$, where $k(t,s)$ is defined as in (i) . +

We shall give sufficient conditions for O.C.L.H.Z. spaces to be intermediate for the interpolation segment $\sigma = [\Lambda(X_1), M(Y_1); \Lambda(X_2), M(Y_2)]$.

Let X, Y, Z, be r.i. spaces. Let $d\mu(t) = \phi_Y(t)\frac{dt}{t}$, and for

$f \in M(0,\infty)$ let us denote by f^0 the non-increasing rearrangement of f with respect to the measure $d\mu(t)$. Let $\psi(t,s)$ be the function defined in (3.2.1) and define

$$F(s,t) = \psi(t,s) \cdot \phi_Z(t)/\phi_X(s), \quad s,t \in (0,\infty).$$

Using the same methods as in [31], [55], we can prove the following

(3.2.2) LEMMA. Suppose that $\min_{i=1,2} \{\phi_{X_i}(0^+)\} = 0$ and moreover assume that there exists a constant $M > 0$ such that

$$\int_0^\infty F(s,t)d\mu(t) \leq M \qquad \forall\ s \in (0,\infty)$$

$$\int_0^\infty F(s,t)d\mu(s) \leq M \qquad \forall\ t \in (0,\infty).$$

Then $\forall\ r > 0$, we have

$$\int_0^r (S_\sigma(f^*)\phi_Z)^0(t)dt \leq M \int_0^r (f^*\phi_X/\phi_Y)^0(t)d\mu(t). \quad +$$

(3.2.3) THEOREM. Suppose that the conditions of (3.2.2) are satisfied, and let T be a linear (sublinear) operator of weak types (X_i,Y_i), $i = 1,2$. Let A be a Young's function, then there exists an absolute constant $C > 0$, such that

$$\|Tf\|^{\sim}_{\Lambda(A,\phi_Z,\phi_Y)} \leq C\, \|f\|^{\sim}_{\Lambda(A,\frac{\phi_X}{\phi_Y},\ \phi_Y)}$$

$\forall f \in \Lambda(A,\phi_X/\phi_Y\ ,\phi_Y)^{\sim}$.

Proof. Let $f \in \Lambda(A,\phi_X/\phi_Y,\phi_Y)^{\sim}$, then by (3.2.2) we have

$$\|S_\sigma(f^*)\phi_Z\|_{L_A(d\mu)} \leq M \,\|f^*\phi_X/\phi_Y\|_{L_A(d\mu)}$$

since $L_A(d\mu)$ is a r.i. space with respect to $d\mu$.

Therefore by (3.2.1),

$$\|T(f)^*\phi_Z\|_{L_A(d\mu)} \leq 2.\ \max_{i=1,2}\{\|T\|_i\}\ M\ .\ \|f^*\phi_X/\phi_Y\|_{L_A(d\mu)}$$

$$\|T(f)\|_{\tilde{\Lambda}(A,\phi_Z,\phi_Y)} \leq C\,\|f\|_{\tilde{\Lambda}(A,\ \frac{\phi_X}{\phi_Y}\ ,\ \phi_Y)}$$

where $C = 2\ \max_{i=1,2}\{\|T\|_i\}\ M.$ +

Using the results of Chapter 2 we can state and prove a similar result replacing the $\Lambda(A,\phi_X,\phi_Y)^\sim$ spaces by the $\Lambda(A,\phi_X,\phi_Y)$ spaces.

These results when specialised to the case where $\phi_Y(t) \equiv 1$ give the interpolation theorems of [58] and if specialised further to Young's functions defined by powers we obtain the results of [55] .

We point out that similar results hold when we consider generalised Young's functions. The required inequalities are worked out in Chapter 2, but we shall leave the formulation of the results to the interested reader.

3.3 ORLICZ SPACES AS INTERMEDIATE SPACES.

The results in §3.1 and §3.2 can be combined to obtain interpolation theorems that give Lorentz spaces and Orlicz spaces as intermediate spaces of weak interpolation segments.

We shall follow the notation set out in the previous section.

Let us consider a simple, but important, case. Let $X_i = L^{p_i}$, $Y_i = L^{q_i}$, then $\Lambda(X_i) = L(p_i,1)$, $M(Y_i) = L(q_i,\infty)$, $i = 1,2$, where $0 \leqslant \frac{1}{q_i} = \beta_i \leqslant \frac{1}{p_i} = \alpha_i \leqslant 1$, $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$. Define,

$$\varepsilon = (\beta_1 - \beta_2)(\alpha_1 - \alpha_2)^{-1}$$
$$\gamma = \left(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}\right)\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right)^{-1} .$$

Therefore $\gamma = \beta - \varepsilon\alpha_1 = \beta_2 - \varepsilon\alpha_2$, and the function $\psi(t,s)$ can be easily computed

$$\psi(t,s) = \begin{cases} s^{\alpha_1} \; t^{-\beta_1} & \text{if} \quad s \leqslant t^{\varepsilon} \\ s^{\alpha_2} \; t^{-\beta_2} & \text{if} \quad s > t^{\varepsilon} \end{cases}$$

Let $\sigma = [L(p_1,1),\ L(q_1,\infty);\ L(p_2,1),\ L(q_2,\infty)]$, we have the following result:

(3.3.1) THEOREM. Let A and B be Young's functions such that

(i) $B^{-1}(t) = A^{-1}(t^{\varepsilon})t^{\gamma}$

(ii) There exists a constant $\theta > 0$ such that the following inequalities are satisfied $\forall\ t > 0$,

$$\int_0^t (s^{\beta_1}/\phi_{L_B}(s))\,\frac{ds}{s} \leqslant \theta(t^{\beta_1}/\phi_{L_B}(t))$$

$$\int_0^t s^{-\alpha_2}\,\phi_{L_A}(s)\frac{ds}{s} \leqslant \theta\,\phi_{L_A}(t)t^{-\alpha_2}$$

$$\int_t^{\infty} (s^{\beta_2}/\phi_{L_B}(s))\,\frac{ds}{s} \leqslant \theta(t^{\beta_1}/\phi_{L_B}(t))$$

$$\int_t^{\infty} s^{-\alpha_1}\,\phi_{L_A}(s)\frac{ds}{s} \leqslant \theta\; t^{-\alpha_1}\phi_{L_A}(t) .$$

(iii) There exist q,s, such that $1 \leq q \leq s < \infty$, an $A(t)t^{-q} \downarrow$ and $B(t)t^{-s} \uparrow$.

Then $(L_A(0,\infty),\ L_B(0,\infty))$ is intermediate for the interpolation segment σ.

Proof. Let T be an operator of weak types $(p_i,\ q_i)$, $i = 1,2$.

We shall prove that the conditions of (3.2.2) hold. For example,

$$\int_0^\infty \frac{\psi(t,s)\phi_{L_B}(t)}{\phi_{L_A}(s)}\ \frac{ds}{s} = \int_0^{t^\varepsilon} s^{\alpha_1} t^{-\beta_1}[\phi_{L_B}(t)/\phi_{L_A}(s)]\ \frac{ds}{s} +$$

$$\int_{t^\varepsilon}^\infty s^{\alpha_2} t^{-\beta_2}[\phi_{L_B}(t)/\phi_{L_A}(s)]\ \frac{ds}{s}$$

$$= I_1 + I_2 .$$

Now,

$$I_1 \approx \phi_{L_B}(t)t^{-\beta_1} \int_0^t (s^{\beta_1}/\phi_{L_B}(s))\frac{ds}{s}$$

$$\leq \theta \cdot$$

Similarly one can bound I_2 and also $\displaystyle\int_0^\infty \frac{\psi(t,s)\phi_{L_B}(t)}{\phi_{L_A}(s)}\ \frac{dt}{t}$.

Therefore by (3.2.2), we get

$$T\colon \Lambda(A_q,\ \phi_{L_A},\ 1) \to \Lambda(A_q,\ \phi_{L_B},\ 1), \quad 1 \leq q \leq \infty ,$$

continuously, where $A_q(t) = t^q \cdot q^{-1}$ for $1 \leq q \leq \infty$, and

$$A_\infty(t) = \begin{cases} 0 & \text{if} \quad t \leq 1 \\ \infty & \text{if} \quad t > 1 \end{cases}$$

Using the embedding theorems of §3.1 (cf. (3.1.10)) and condition (iii), we get

$T: L_A \to L_B$, continuously . +

3.4. NOTES TO CHAPTER 3. The embedding theorems obtained in §3.1 extend some well known results for Orlicz spaces, $\Lambda(\alpha, p)$ spaces, $L(p,q)$ spaces. The standart references are Lorentz [29], [30], Luxemburg [32] , Hunt [25], O'Neil [43], [45].

Similar results, concerning the posibility of including Orlicz spaces as intermediate spaces for weak interpolation segments have been given recently by Torchinsky [58] using a different approach.

In [38] it is shown how these results can be used to obtain continuity results for the Laplace transform.

PART III

TENSOR PRODUCTS OF FUNCTION SPACES

INTRODUCTION

Let $(\Omega_1,\mu_1),(\Omega_2,\mu_2)$ be measure spaces and denote by $(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$ the product measure space. Let $X(\Omega_1)$, $Y(\Omega_2)$, $Z(\Omega_1 \times \Omega_2)$ be B.F. spaces and denote by $X \otimes Y$ the algebraic tensor product of X and Y; in the vector space $X \otimes Y$ consider the projective cross-norm (cf. [53])

$$\|u\|_{X \underset{\pi}{\otimes} Y} = \inf \{\sum_{i=1}^{n} \|f_i\|_X \|g_i\|_Y : u = \sum_{i=1}^{n} f_i \otimes g_i\}$$

where $(f \otimes g)(x,y) = f(x)\, g(y)$.

It is well known, and easy to see, that the completion of $(X \otimes Y, \|\ \|_{X \underset{\pi}{\otimes} Y})$ which we denote by $X \hat{\underset{\pi}{\otimes}} Y$, is not necessarily a B.F.S.. Therefore we consider the problem of embedding $X \underset{\pi}{\otimes} Y$ into $Z(\Omega_1 \times \Omega_2)$, in other words we are searching for necessary and sufficient conditions for inequalities $\|f \otimes g\|_Z \leq c \|f\|_X \|g\|_Y$ to hold.

Our interest in this problem is justified by the following theorem

THEOREM A. $X \underset{\pi}{\otimes} Y \subseteq Z$ if and only if $\forall k \in Z'$, the integral operator

$$z_k(f)(y) = \int_{\Omega_1} k(x,y)\, f(y)\, d\mu_1(y)$$

defines a bounded linear operator, $z_k : X \to Y'$.

Proof. Similar to the one given in [27] in the context of Orlicz spaces. +

Thus, to a certain extent, the problems of stability or admissibility for integral operators (cf. Corduneanu [13] and the references quoted in this book) can be discussed from the point of view of tensor products of functions spaces. Moreover, in many situations, the study of the problem of embedding the tensor product of two function spaces is much easier to deal with than the corresponding admissibility problem.

Consider the following result (for a proof see (4.3.1) below)

THEOREM B. Let $\zeta_1(X)(\Omega_1)$, $\zeta_2(Y)(\Omega_2)$, $\zeta_3(Z)(\Omega_1 \times \Omega_2)$ be Calderón spaces. Then,

$$\zeta_1(X) \underset{\pi}{\otimes} \zeta_2(Y) \subseteq \zeta_3(Z)$$

whenever the following conditions are satisfied,

(i) $X \underset{\pi}{\otimes} Y \subseteq Z$

(ii) There exists a constant $\theta > 0$, such that

$\zeta_1(x,t)\ \zeta_2(y,s) \leq \theta\ \zeta_3(x,y,t.s)$, $\forall (x,y) \in \Omega_1 \times \Omega_2$,

$0 < s, t < \infty$ • +

Combining the above results we get,

THEOREM C. (O'Neil [43].) Let A,B,C be Young's functions, then

(i) $L_A(0,\infty) \underset{\pi}{\otimes} L_B(0,\infty) \subseteq L_C((0,\infty) \times (0,\infty))$ if and only if there exists a constant $\theta > 0$ such that $A^{-1}(t)B^{-1}(s) \leq \theta\, C^{-1}(t.s)$, $\forall$ $t, s > 0$.

(ii) A necessary and sufficient condition for $z(f,k) = z_k(f)$ to define a bounded bilinear operator, $z: L_A(0,\infty) \times L_C((0,\infty)\times(0,\infty)) \to L_B(0,\infty)$, is the existence of a constant $\theta > 0$ such that $\forall$ $t,s > 0$,

$$C^{-1}(t.s)\; A^{-1}(s) \leq \theta\; s\; B^{-1}(t) .$$

Proof. (i) Notice that $L_A = \zeta_1(L^1)$, $L_B = \zeta_2(L^1)$, $L_C = \zeta_3(L^1)$ with $\zeta_1(x,t) = A^{-1}(t)$, $\zeta_2(y,s) = B^{-1}(s)$, and $\zeta_3(x,y,r) = C^{-1}(r)$. Therefore, since $L^1 \underset{\pi}{\otimes} L^1 \subseteq L^1$, the sufficiency part follows from Theorem B. The necessity of the condition on the inverses of the Young's functions follows from (4.1.2) below.

(ii) Follows directly from (i), Theorem A, and the duality theory for Orlicz spaces. +

In the above formulation Theorem C is due to O'Neil [43], and it is closely related to previous work by Andô [1] who considered less general Orlicz spaces of measurable functions on finite measure spaces. The present simple proof seems new and in fact shows that the theory of embeddings of tensor products of Orlicz spaces is a simple consequence of the corresponding one for L^p spaces.

In (4.2.1) below it is proved that

$\Lambda(X)(0,\infty) \underset{\pi}{\otimes} \Lambda(Y)(0,\infty) \subseteq \Lambda(Z)(0,\infty)^2$ if and only if

$\phi_Z(t,s) \leq \theta \ \phi_X(t)\ \phi_Y(s)$, $\forall t,s > 0$, where θ is an absolute constant.

Therefore using Theorem B we obtain the following generalisation of Theorem C.

THEOREM D. Let A, B, C be Young's functions.

Then $A^{-1}(\Lambda(X))(0,\infty) \underset{\pi}{\otimes} B^{-1}(\Lambda(Y))(0,\infty) \subseteq C^{-1}(\Lambda(Z))(0,\infty)^2$, whenever the following conditions are satisfied,

(i) There exists $\theta > 0$ such that

$$\phi_Z(t.s) \leq \theta \ \phi_X(t)\ \phi_Y(s)\ , \quad \forall t,s > 0 .$$

(ii) There exists $M > 0$ such that

$$A^{-1}(t)\ B^{-1}(s) \leq M.C^{-1}(t.s), \ \forall t,s > 0 . \ +$$

In particular we get the following result of O'Neil [43] (see (4.3.3) below).

COROLLARY. $L(p,q)(0,\infty) \underset{\pi}{\otimes} L(p,q)(0,\infty) \subseteq L(p,q)(0,\infty)^2$.

whenever $1 \leq q \leq p < \infty$. +

The conditions of Theorem C can be formulated in terms of the fundamental functions of the Orlicz spaces involved, therefore we are led to ask if a similar condition on the fundamental functions of X, Y and Z gives a necessary and sufficient condition for $X \underset{\pi}{\otimes} Y \subseteq Z$ to hold. From O'Neil's results for the L(p,q) spaces (see the above corollary) we know that the answer is no ! . We insist in another direction: what properties of the factor spaces are needed in order to generalise Theorem C to "arbitrary" r.i. spaces ? In this direction we prove (see (4.2.4) below)

THEOREM E. $\Lambda(X) \underset{\pi}{\otimes} Y \subseteq Z$ if and only if there exists a constant $\theta > 0$ such that

$$\|E_{1/s}\|_{Y \to \hat{Z}} \leq \theta\, \phi_X(s) \quad , \quad \forall s > 0 \,,$$

where $E_{1/s}$ is the compression operator and X is assumed to have the δ_2 property. +

The above results are generalised for arbitrary "tensor product operators"(see (4.4.1) below) using the following interpolation theorem (cf. (4.4.4) below).

THEOREM F. Let T be a tensor product operator.

Then,

$$T(f,g)^{**}(t) \leq \int_0^\infty f^{**}(t/s)\, g^*(s) \frac{ds}{s} \quad . \quad +$$

Using Theorem F we obtain sufficient conditions for tensor product operators to be continuous on O.C.L.H.Z. spaces. Among the interesting consequences of Theorem F we point out the following

COROLLARY. $L(p,\infty)(0,1) \underset{\pi}{\otimes} L(p,\infty)(0,1) \subseteq M(Z)(0,1)^2$, where $\phi_Z(t) = - t^{1-1/p} \log t$, $t \in (0,1)$. +

In 4.5 we consider the analogous problem of embedding the space $X(Y) = \{f \in M(\Omega_1 \times \Omega_2): \| \| f(x,\cdot)\|_Y \|_X = \|f\|_{X(Y)} < \infty\}$ into Z.

THEOREM G. Suppose that there exists $M > 0$ such that

$$\|\phi_{Y'}(|u|)\|_{X'} \leq M \, \phi_{Z'}(\|u\|_{L^1}), \qquad \forall\, u \in L^1(0,\infty).$$

Then, $X(M(Y))((0,\infty)\times(0,\infty))$ is continuously embedded in $M(Z)((0,\infty)\times(0,\infty))$.

Proof. See (4.5.2) below. +

The method of Theorem B can be modified to obtain embedding theorems for Calderón spaces with mixed double norms. Therefore extending the above results.

In Chapter 5 we consider the continuity of product operators and convolution operators on r.i. spaces. These results are analogues of the results obtained in Chapter 4 for projective tensor products of r.i. spaces as Banach modules.

In particular the classical theorems on Fractional integration are generalised to the context of O.C.L.H.Z. spaces.

In the last part of the thesis we give an application of our results to the theory of admissibility of integral operators. To do so, we need to introduce still another class of B.F. spaces suitable for our purposes. The main tools here are Theorem A and the observation that continuity results for integral operators acting on r.i. spaces also give corresponding results for r.i. spaces "with weights". These results are of interest since, due to our generalised setting, they can be applied to obtain existence, uniqueness and asymptotic behaviour of solutions of integral equations with strong non-linearities.

CHAPTER 4

EMBEDDINGS OF TENSOR PRODUCTS OF BANACH FUNCTION SPACES

Let $X(\Omega_1)$, $Y(\Omega_2)$, $Z(\Omega_1 \times \Omega_2)$ be B.F.spaces. In this chapter we give necessary and sufficient conditions for $X \otimes_\pi Y$ to be continuously embedded in Z.

In §4.2 we consider the case where one of the factor spaces is a Λ space, in §4.3 we consider tensor products of Calderón spaces, in §4.4. we obtain estimates for generalised operators which behave like $\otimes$ and use these results to obtain embeddings of tensor products of O.C.L.H.Z. spaces.

In §4.5 we consider the problem of embedding X(Y) into Z.

These results are of special interest in the theory of non linear parabolic initial value problems (cf. [16]) and of course in the theory of integral operators and equations.

In §4.6 and §4.7 the reader will find some more technical material that could be omitted in a first reading.

Finally §4.8 contains our usual notes and comments on the results of this Chapter and a detailed bibliographical review.

4.1. PROJECTIVE TENSOR PRODUCTS OF B.F. SPACES.

In this section we establish some general properties of the operation $\otimes_\pi$, which are valid for a wide range of spaces.

In what follows we shall let $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty) \times (0,\infty))$ be r.i. spaces of Lebesgue measurable functions on the half line or the cross product $(0,\infty) \times (0,\infty)$.

(4.1.1) LEMMA. Suppose that $X \otimes Y \subseteq Z$, then

(i) $\exists\, \theta > 0$ such that $\|f \otimes g\|_Z \leq \theta \|f\|_X \|g\|_Y$, $\forall f \otimes g \in X \otimes Y$.

(ii) $X \otimes_\pi Y$ is continuously embedded in Z.

Proof. (i) Consider the bilinear operator $\otimes(f, g) = f \otimes g$, $\otimes: X \times Y \to Z$. By the Uniform Boundedness Theorem we only need to show that $\otimes(,)$ is separately continuous, and this can be easily proved using the Closed Graph Theorem. Indeed let $f \in X$ be fixed and let $T_f(g) = f \otimes g$, $T_f: Y \to Z$, and suppose $g_n \to g$ in Y and $T_f(g_n) \to h$ in Z. Then there exists a subsequence $\{g_{n_j}\}$ such that $g_{n_j} \to g$ a.e and $T_f(g_{n_j}) \to h$ a.e., therefore $T_f(g) = h$ a.e., and the result follows.

(ii) Let $u \in X \underset{\pi}{\otimes} Y$, then for any representation $u = \sum_{i=1}^{n} f_i \otimes g_i$, we have $\|u\|_Z \leq \sum_{i=1}^{n} \|f_i \otimes g_i\|_Z \leq \theta \sum_{i=1}^{n} \|f_i\|_X \|g_i\|_Y$. Therefore $\|u\|_Z \leq \theta \|u\|_{X \underset{\pi}{\otimes} Y}$. +

In what follows we shall use the symbol $\subseteq$ to denote a continuous embedding, unless otherwise indicated.

(4.1.2) THEOREM. The following conditions are necessary for $X \underset{\pi}{\otimes} Y \subseteq Z$,

(i) $\exists\, \theta > 0$ such that $\phi_Z(t.s) \leq \theta\, \phi_X(t)\phi_Y(s)$, $\forall\, t,s > 0$.

(ii) $X \subseteq \hat{Z}$, $Y \subseteq \hat{Z}$, where $\hat{Z}$ is the Luxemburg representation of Z.

Proof. (i) Let $t,s > 0$, and define $f = \chi_{(0,t)}$, $g = \chi_{(0,s)}$. Then $\|f \otimes g\|_Z = \|(f \otimes g)^*\|_{\hat{Z}} = \phi_{\hat{Z}}(t.s) = \phi_Z(t.s)$. On the other hand by (4.1.1) there exists a constant $\theta > 0$ such that $\|f \otimes g\|_Z \leq \theta \|f\|_X \|g\|_Y$, therefore

$$\phi_Z(t.s) \leq \theta\, \phi_X(t)\, \phi_Y(s).$$

(ii) Suppose for example that $X \not\subseteq \hat{Z}$, then there exists $f \in X$ such that $f \notin \hat{Z}$. Let $h = f \otimes \chi_{(0,1)}$, then $h \in Z$ by hypothesis, however $h^* = f^*$ which implies $f^* \in \hat{Z}$, a contradiction. +

4.2. TENSOR PRODUCTS WITH Λ SPACES. In this section we consider the problem of embedding the tensor product of r.i. spaces when one of the factor spaces is a Lorentz Λ space.

In order to simplify the formulation of our results we shall assume throughout this section that $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty)\times(0,\infty))$ are r.i. spaces verifying the δ_2 condition.

(4.2.1) THEOREM. The following statements are equivalent

(i) $\Lambda(X) \underset{\pi}{\otimes} \Lambda(Y) \subseteq \Lambda(Z)$

(ii) $\Lambda(X) \underset{\pi}{\otimes} M(Y) \subseteq M(Z)$

(iii) $\Lambda(X) \underset{\pi}{\otimes} \tilde{M}(Y) \subseteq \tilde{M}(Z)$

(iv) $\exists\, \theta > 0$ such that $\phi_Z(t.s) \leq \theta\, \phi_X(t)\phi_Y(s)$, $\forall\, t,s > 0$.

Proof. It is easy to see that if one of the statements (i), (ii), (iii) holds then (iv) holds. For example if (i) holds, then since X, Y and Z satisfy the δ_2 condition, we have $\phi_{\Lambda(X)}(t) \approx \phi_X(t)$, $\phi_{\Lambda(Y)}(t) \approx \phi_Y(t)$, $\phi_{\Lambda(Z)}(t) \approx \phi_Z(t)$, and therefore (iv) holds by (4.1.2).

We shall prove that (iv) implies (i), (ii) and (iii).

[(iv) => (i)]. Let $f \in \Lambda(X)$, and $g = c\chi_E$, $|E| < \infty$.

Then,

$$\|f \circledast g\|_{\Lambda(Z)} = \int_0^\infty (f \circledast g)^*(t)\phi_Z(t)\frac{dt}{t}$$

$$\leqslant \int_0^\infty |c|\ f^*(t/|E|)\phi_Z(t)\frac{dt}{t}$$

$$\leqslant |c|\int_0^\infty f^*(u)\phi_Z(u.|E|)\frac{du}{u}$$

$$\leqslant \theta\ |c|\phi_Y(|E|)\int_0^\infty f^*(u)\phi_X(u)\frac{du}{u}$$

$$\leqslant C\ \|g\|_{\Lambda(Y)}\ \|f\|_{\Lambda(X)}\ . \qquad (4.2.2)$$

where C is an absolute constant.

Let g be a simple function, then using the Riesz Lemma (cf. [33]), we can write $g = \sum_{i=1}^{n} g_i$, where each g_i, $1 \leqslant i \leqslant n$, is a simple function taking only one value and $g^* = \sum_{i=1}^{n} g_i^*$. Hence,

$\|g\|_{\Lambda(Y)} = \sum_{i=1}^{n} \|g_i\|_{\Lambda(Y)}$ and

$$\|f \circledast g\|_{\Lambda(Z)} \leqslant \sum_{i=1}^{n} \|f \circledast g_i\|_{\Lambda(Z)}$$

$$\leqslant C\ \|f\|_{\Lambda(X)}\ \|g\|_{\Lambda(Y)} \quad \text{(by (4.2.2))}.$$

Finally one can extend the above inequality for an arbitrary $g \in \Lambda(Y)$ using the monotone convergence theorem.

[(iv) $\Rightarrow$ (ii)]. Let $f \in M(Y)$, and $g = c\chi_E \in \Lambda(X)$, then

$$\|f \circledast g\|_{M(Z)} = \sup_{t>0} \{(f \circledast g)^{**}(t)\phi_Z(t)\}$$

$$\leq \sup_{t>0} \{|c| f^{**}(t/|E|)\phi_Z(t)\}$$

$$\leq \theta |c| \phi_X(|E|) \sup_{t>0} \{f^{**}(t)\phi_Y(t)\}$$

$$\leq C \|g\|_{\Lambda(X)} \|f\|_{M(Y)}$$

where C is absolute constant. The above inequality can be extended using Riesz's Lemma and the monotone convergence theorem.

The proof of the implication [(iv) ⇒ (iii)] is similar and will be omitted . +

The same method used above also yields the following

(4.2.3) THEOREM. Let A be a Young's function and suppose that there exists a constant $\theta > 0$ such that $\phi_Z(t.s) \leq \theta\, \phi_X(t)\phi_Y(s)$, $\forall\ t,s > 0$. Then,

(i) $\Lambda(X) \circledast_{\pi} \Lambda(A,\phi_Y,1) \subseteq \Lambda(A,\phi_Z,1)$

(ii) $\Lambda(X) \circledast_{\pi} \Lambda(A,\phi_Y,1)^{\sim} \subseteq \Lambda(A,\phi_Z,1)^{\sim}$. +

The above results suggest the following

(4.2.4) THEOREM. $\Lambda(X) \underset{\pi}{\otimes} Y \subseteq Z$ if and only if there exists a constant $\theta > 0$ such that $\|E_{1/s}\|_{Y \to \hat{Z}} \leq \theta\, \dot{\phi}_X(s)$, $\forall\, s > 0$, where $(E_{1/s}(f))(t) = f(t/s)$ is the compression operator.

Proof. Let $f \in Y$, $g = c\chi_E \in \Lambda(X)$, then

$$(f \otimes g)^*(t) = |c|\, f^*(t/|E|) = |c|\, E_{1/|E|}(f^*)(t)$$

Therefore,

$$\|f \otimes g\|_Z = \|(f \otimes g)^*\|_{\hat{Z}} = |c|\, \|E_{1/|E|}(f^*)\|_{\hat{Z}}$$

$$\leq |c|\, \|E_{1/|E|}\|_{Y \to \hat{Z}}\, \|f\|_Y$$

$$\leq \theta\, |c|\, \phi_X(|E|)\, \|f\|_Y$$

$$\leq C\, \|g\|_{\Lambda(X)}\, \|f\|_Y$$

where C is an absolute constant. It follows readily, using the methods of (4.2.1), that $\Lambda(X) \underset{\pi}{\otimes} Y \subseteq Z$.

Suppose that the condition on the compression operator is not satisfied: then there exists a sequence of positive numbers $\{s_n\}$ such that

$$\|E_{1/s_n}\|_{Y \to \hat{Z}} > 2^{4n}\, \phi_X(s_n), \quad n = 1, \ldots\ldots\ldots.$$

Then there exists a sequence $\{f_n\}$ such that $\|f_n\|_Y \leq 1$, $f_n \downarrow$, $n = 1, \ldots,$ and

$$\|E_{1/s_n}(f_n)\|_{\hat{Z}} \geq 2^{4n}\, \phi_X(s_n)\ , \quad n = 1,\ \ldots\ldots\ .$$

For $n \in N$, let $g_n = [\phi_X(s_n)]^{-1}\chi_{(0,s_n)}$, then

$$(f_n \circledast g_n)^*(t) = \frac{1}{\phi_X(s_n)} E_{1/s_n}(f_n) , \quad n = 1 \ldots\ldots\ldots$$

Thus,

$$\|(f_n \circledast g_n)^*\|_{\hat{Z}} \geq 2^{4n} , \quad n = 1 \ldots\ldots\ldots$$

Define $f = \sum_{n=1}^{\infty} 2^{-n} f_n$, $g = \sum_{n=1}^{\infty} 2^{-n} g_n$, then $f \in Y$, $g \in \Lambda(X)$, and

$$\|(f \circledast g)^*\|_{\hat{Z}} \geq 2^{2n} , \quad n = 1 \ldots\ldots\ldots$$

Therefore $f \circledast g \notin Z$. +

(4.2.5) COROLLARY. Let A, B, C be Young's functions, then

(i) $\Lambda(\Lambda(A, 1, t^{\alpha})^{\sim}) \circledast_{\pi} \Lambda(B, 1, t^{\alpha})^{\sim} \subseteq \Lambda(C, 1, t^{\alpha})^{\sim}$, where $0 < \alpha \leq 1$, if and only if there exists a constant $\theta > 0$ such that

$$(4.2.6) \quad A^{-1}(t)\, B^{-1}(s) \leq \theta\, C^{-1}(s.t) , \quad \forall\, t,s > 0.$$

(ii) Suppose that C satisfies the Δ_2 condition, and $0 < \alpha + \beta < 1$, $0 \leq \alpha$, $\beta \leq 1$, then $\Lambda(\Lambda(A,t^{\alpha},t^{\beta})) \circledast_{\pi} \Lambda(B,t^{\alpha},t^{\beta}) \subseteq \Lambda(C,t^{\alpha},t^{\beta})$ if and only if there exists a constant $\theta > 0$ such that (4.2.6) holds.

Proof. (i) The necessity of (4.2.6) follows from (4.1.2), the sufficiency follows readily from (4.2.4) and (1.3.3).

The proof of (ii) is similar and we omit the details (see (2.2.9)). +

4.3 TENSOR PRODUCTS OF CALDERON SPACES.

In this section we show the import of the Calderón spaces in the theory of embeddings of tensor products of B.F. spaces.

(4.3.1) THEOREM. Let $\zeta_1(X)(0,\infty)$, $\zeta_2(Y)(0,\infty)$, $\zeta_3(Z)((0,\infty)\times(0,\infty))$ be Calderón spaces and suppose that $X \underset{\pi}{\otimes} Y \subseteq Z$. Then,

$$\zeta_1(X) \underset{\pi}{\otimes} \zeta_2(Y) \subseteq \zeta_3(Z)$$

whenever there exists a constant $\theta > 0$ such that

$$\zeta_1(x,t)\ \zeta_2(y,s) \leqslant \theta\, \zeta_3(x,y,t.s), \qquad \forall\, x,y,t,s \in (0,\infty).$$

Proof. Let $f \in \Sigma(\zeta_1(X))$, $g \in \Sigma(\zeta_2(Y))$, then there exist $\lambda_i > 0$, $i = 1,2$, $\bar{f} \in \Sigma(X)$, $\bar{g} \in \Sigma(Y)$ such that

$$|f(x)| \leqslant \lambda_1\ \zeta_1(x,|\bar{f}(x)|) \qquad \forall\, x \in (0,\infty)$$

$$|g(y)| \leqslant \lambda_2\ \zeta_2(y,|\bar{g}(y)|) \qquad \forall\, y \in (0,\infty).$$

By hypothesis there exists an absolute constant $M \geqslant 1$ such that $\|\bar{f} \otimes \bar{g}\|_Z \leqslant M\,\|\bar{f}\|_X\,\|\bar{g}\|_Y$. Therefore

$$|(f \otimes g)(x,y)| \leqslant \lambda_1\ \lambda_2\ \theta\, \zeta_3(x,y,|\bar{f} \otimes \bar{g}(x,y)|)$$

$$\leqslant \lambda_1\lambda_2\ \theta\, M\, \zeta_3(x,y,|\bar{f} \otimes \bar{g}(x,y)|/M).$$

Thus,

$$\|f \otimes g\|_{\zeta_3(Z)} \leq \theta . M \|f\|_{\zeta_1(X)} \|g\|_{\zeta_2(Y)} \; . \quad +$$

(4.3.2) COROLLARY. Let A, B, C be Young's functions, then,

(i) $L_A(0,\infty) \underset{\pi}{\otimes} L_B(0,\infty) \subseteq L_C((0,\infty) \times (0,\infty))$ if and only if there exists a constant $\theta > 0$ such that

$$A^{-1}(t)\; B^{-1}(s) \leq \theta\, C^{-1}(s.t), \qquad \forall\; t,s > 0.$$

(ii) $A^{-1}(\Lambda(X)) \underset{\pi}{\otimes} B^{-1}(\Lambda(Y)) \subseteq C^{-1}(\Lambda(Z))$, whenever there exists a constant $\theta > 0$ such that the condition of (i) is satisfied and moreover

$$\phi_Z(t.s) \leq \theta\; \phi_X(t)\; \phi_Y(s)\;, \qquad \forall\; t,s > 0.$$

(iii) Let $q \in [1,\infty)$ and $A_q(t) = t^q .\, q^{-1}$, and suppose the following conditions are satisfied

(a) $\phi_Z^q(t) \;.\; t^{-1} \downarrow$

(b) $\exists\; \theta > 0$, such that, $\phi_Z(t.s) \leq \theta\; \phi_X(t)\; \phi_Y(s)$, $\forall\; t,s > 0$.

Then,

$$\Lambda(A_q,\phi_X,1)^{\sim} \underset{\pi}{\otimes} \Lambda(A_q,\phi_Y,1)^{\sim} \subseteq \Lambda(A_q,\phi_Z,1)^{\sim} .$$

Proof. (i) Since $L_A = A^{-1}(L^1)$, $L_B = B^{-1}(L^1)$, $L_C = C^{-1}(L^1)$ and $L^1 \underset{\pi}{\otimes} L^1 \subseteq L^1$, the sufficiency part follows from (4.3.1). The necessity follows by (4.1.2).

(ii) Follows from (4.2.1) and (4.3.1).

(iii) Can be proved directly or using the embedding theorems of §3 . +

(4.3.3) COROLLARY. (i) Let $q \in [1,\infty)$, $p \in (1,\infty)$ and suppose that $q \leq p$, then

$$L(p,q) \underset{\pi}{\otimes} L(p,q) \subseteq L(p,q) .$$

(ii) Let $\alpha \in (0,1]$, and let A, B, C be Young's functions, then

$$M(A^{-1}(\Lambda(L^{1/\alpha})))^{\sim} \underset{\pi}{\otimes} B^{-1}(\Lambda(L^{1/\alpha})) \subseteq M(C^{-1}(\Lambda(L^{1/\alpha})))^{\sim}$$

if and only if there exists a constant $\theta > 0$ such that

$$A^{-1}(t)\ B^{-1}(s) \leq \theta\ C^{-1}(t.s) , \qquad \forall\ t,s > 0.$$

Proof. (i) Follows from (4.3.2) (ii). (Compare with O'Neil [43] .)

(ii) Observe that $M(A^{-1}(\Lambda(L^{1/\alpha})))^{\sim} = A^{-1}(M(L^{1/\alpha})^{\sim})$, $M(C^{-1}(\Lambda(L^{1/\alpha})))^{\sim} = C^{-1}(M(L^{1/\alpha})^{\sim})$. Therefore if the condition on the Young's functions is satisfied we have by (4.2.1),

$$M(L^{1/\alpha})^{\sim} \underset{\pi}{\otimes} \Lambda(L^{1/\alpha}) \subseteq M(L^{1/\alpha})^{\sim}$$

and by (4.3.2)

$$A^{-1}(M(L^{1/\alpha})^{\sim}) \underset{\pi}{\otimes} B^{-1}(\Lambda(L^{1/\alpha})) \subseteq C^{-1}(M(L^{1/\alpha})^{\sim}) .$$

The necessity of the condition follows from (4.1.2). +

(4.3.4) REMARK. Notice that the same proof given in (4.3.3) (ii) yields, for $0 < \alpha < 1$,

$$A^{-1}(M(L^{1/\alpha})) \underset{\pi}{\otimes} B^{-1}(\Lambda(L^{r/\alpha})) \subseteq C^{-1}(M(L^{1/\alpha}))$$

if and only if the condition of (4.3.3) (ii) holds. However our argument does not work, if we remove the tildes, for the case where $\alpha = 1$. Indeed in this case $A^{-1}(M(L^1)) \cong L_A$ which is, in general, different from $M(A^{-1}(L^1)) = M(L_A)$.

(4.3.5) DEFINITION. Let (Ω,μ) be a measure space, $X_1(\Omega)$, $X_2(\Omega)$ be B.F. spaces. For $0 < t < 1$, define

$$X_1^t X_2^{1-t}(\Omega) = \{f \in M(\Omega): |f(x)| \leq \lambda |f_1(x)|^t |f_2(x)|^{1-t},$$

$\forall x \in \Omega$, where $f_i \in \Sigma(X_i)$, $i = 1,2$ and $\lambda > 0\}$.

$$\|f\|_{X_1^t X_2^{1-t}} = \inf\{\lambda > 0: \exists f_i \in \Sigma(X_i),\ i = 1,2 \text{ such that } |f(x)| \leq \lambda |f_1(x)|^t |f_2(x)|^{1-t}\ \forall x \in \Omega\}.$$

It can be proved (cf. [9]) that $X_1^t X_2^{1-t}$ is a B.F.S. .

Using the same arguments of (4.3.1) we obtain,

(4.3.6) THEOREM. Let $X_i(\Omega)$, $Y_i(\Omega)$, $Z_i(\Omega \times \Omega)$, be B.F. spaces, $i = 1,2$. Suppose that $X_i \underset{\pi}{\otimes} Y_i \subseteq Z_i$, $i = 1,2$, then

$$X_1^t X_2^{1-t} \underset{\pi}{\otimes} Y_1^t Y_2^{1-t} \subseteq Z_1^t Z_2^{1-t} \quad . \qquad +$$

4.4. TENSOR PRODUCT OPERATORS. In this section we obtain estimates for the maximal rearrangement of operators which behave like $\otimes$, in some technical sense to be specified below. These results are applied to obtain sufficient conditions for embeddings of tensor products of O.C.L.H.Z. spaces.

(4.4.1) DEFINITION. Let (Ω_1,μ_1), (Ω_2,μ_2), (Ω_3,μ_3) be measure spaces. A bilinear operator is said to be a tensor product operator (t.p.o.) if

$$\text{(i)}\ \|T(f,g)\|_{L^1(\Omega_3)} \leq \|f\|_{L^1(\Omega_1)} \|g\|_{L^1(\Omega_2)}$$

$$\text{(ii)}\ \|T(f,g)\|_{L^\infty(\Omega_3)} \leq \|f\|_{L^\infty(\Omega_1)} \|g\|_{L^\infty(\Omega_2)} .$$

(4.4.2) EXAMPLES. (i) If we let $\Omega_3 = \Omega_1 \times \Omega_2$, then $T(f,g) = f \otimes g$ is a t.p.o..

(ii) Let $\Omega_1 = \Omega_2 = \Omega_3 = [0,1] \times [0,1]$, and define

$$T(f,g)(x,y) = \int_0^1 \int_0^1 f(x,t)\, g(s,y)\, dt\, ds ,$$

then T is a t.p.o. .

(4.4.3) LEMMA. Let T be a t.p.o., then $\forall\, f \in L^1(\Omega_1) + L^\infty(\Omega_1)$, $\forall\, E \subseteq \Omega_2$, μ_2 measurable, $\mu_2(E) < \infty$, and $\forall\, c \in R$, we have

$$T(f, c\chi_E)^{**}(t) \leq |c|\, f^{**}(t/\mu_2(E)).$$

Proof. Let r be an arbitrary, but fixed positive number, and write $f = f_r + f^r$, where

$$f_r(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| \leq r \\ r \operatorname{sign}(f(x)) & \text{if} \quad |f(x)| > r \end{cases}$$

and $f^r = f - f_r$. Then

$$h = T(f, c\chi_E) = T(f_r, c\chi_E) + T(f^r, c\chi_E) = h_1 + h_2.$$

Using the conditions of (4.4.1) we get

$$\|h_1\|_{L^1(\Omega_3)} \leq \|f^r\|_{L^1(\Omega_1)} \|c\, \chi_E\|_{L^1(\Omega_2)}$$

$$\leq |c|\, \mu_2(E) \int_r^\infty \lambda_f(u)\,du.$$

$$\|h_2\|_{L^\infty(\Omega_3)} \leq \|f_r\|_{L^\infty(\Omega_1)} \|c\chi_E\|_{L^\infty(\Omega_2)} \leq |c|\ r .$$

Therefore,

$$t\ h^{**} \leq \|h_1\|_{L^1(\Omega_1)} + t\, \|h_2\|_{L^\infty(\Omega_2)}$$

$$\leq |c|\ \mu_2(E) \int_r^\infty \lambda_f(u)\,du + |c|\ r\ t .$$

Now, if we choose $r = f^*(t/\mu_2(E))$, $\ell = t/\mu_2(E)$, we get

$$t\ h^{**}(t) \leq |c|\ \mu_2(E) \int_{f^*(\ell)}^\infty \lambda_f(u)\,du + t|c|f^*(\ell)$$

$$\leq |c|\ \mu_2(E)\ \left[\frac{t}{\mu_2(E)}\,(f^{**}(\ell) - f^*(\ell))\right] + t.|c|f^*(\ell)$$

$$\leq |c|\ t\ f^{**}(\ell).$$

Thus,

$$h^{**}(t) \leq |c|f^{**}(t/\mu_2(E)) . \quad +$$

(4.4.4) THEOREM. Let T be a t.p.o., and let $f \in M(\Omega_1)$, $g \in M(\Omega_2)$ be such that $T(f,g)$ is well defined. Then

$$T(f,g)^{**}(t) \leqslant \int_0^\infty f^{**}(t/s)g^*(s)\frac{ds}{s} .$$

Proof. Suppose first that $g = c\chi_E$, $\mu_2(E) = u < \infty$, then by (4.4.3)

$$T(f,g)^{**}(t) \leqslant |c|\ f^{**}(t/u).$$

But $f^{**}(t/s)s^{-1}$ decreases with s, therefore

$$|c|\ f^{**}(t/u) \leqslant |c|\int_0^u f^{**}(t/s)\frac{ds}{s}$$

$$\leqslant \int_0^\infty f^{**}(t/s)g^*(s)\frac{ds}{s} .$$

Therefore combining the above inequalities we get

$$T(f,g)^{**}(t) \leqslant \int_0^\infty f^{**}(t/s)g^*(s)\frac{ds}{s} .$$

Let g be a simple function, then we can write $g = \sum_{i=1}^n g_i$, where each g_i, $1 \leqslant i \leqslant n$, is a simple function taking one value only, and $g^* = \sum_{i=1}^n g_i^*$. Then using the bilinearity of T,

$$T(f,g) = \sum_{i=1}^n T(f,g_i)$$

$$T(f,g)^{**}(t) \leqslant \sum_{i=1}^n T(f,g_i)^{**}(t)$$

$$\leqslant \sum_{i=1}^n \int_0^\infty f^{**}(t/s)g_i^*(s)\frac{ds}{s}$$

$$\leqslant \int_0^\infty f^{**}(t/s)g^*(s)\frac{ds}{s} .$$

Finally we extend the above inequality to an arbitrary g using the monotone convergence theorem. +

(4.4.5) **REMARK.** The result in (4.4.4) was suggested to the author by R. Sharpley.

(4.4.6) **THEOREM.** Let A,B,C be Young's functions, and let $X(0,\infty)$, $Y(0,\infty)$, $Z(0,\infty)^2$ be r.i.spaces. Then every t.p.o. T defines a bounded bilinear mapping,

$$T: \Lambda(A,\phi_X,1) \times \Lambda(B,\phi_Y,1) \to \Lambda(C,\phi_Z,1)$$

whenever there exists a constant $\theta > 0$ such that

(i) $\phi_Z(t.s) \leq \theta\ \phi_X(t)\phi_Y(s)$, $\forall\, t,s > 0.$

(ii) $A^{-1}(t)\ B^{-1}(t) \leq \theta\ \ t\ C^{-1}(t), \forall\, t > 0.$

Proof. We shall denote by $\circledast$ the convolution product on $(R^+, \frac{dt}{t})$, that is

$$(f \circledast g)(t) = \int_0^\infty f(t/s)\ g(s)ds.$$

It follows from [44], that

$$L_A(\tfrac{dt}{t}) \circledast L_B(\tfrac{dt}{t}) \subseteq L_C(\tfrac{dt}{t})$$

whenever condition (ii) is satisfied.

Let $f \in \Sigma(\Lambda(A,\phi_X,1))$, $g \in \Sigma(\Lambda(B,\phi_Y,1))$, then by (4.4.4),

$$T(f,g)^{**}(t) \leqslant \int_0^\infty f^{**}(t/s)g^*(s)\frac{ds}{s}$$

$$T(f,g)^{**}(t)\phi_Z(t) \leqslant \int_0^\infty f^{**}(t/s)g^*(s)\phi_Z(t)\frac{ds}{s}$$

$$\leqslant \theta\int_0^\infty f^{**}(t/s)\phi_X(t/s)g^*(s)\phi_Y(s)\frac{ds}{s}$$

$$\leqslant \theta(f^{**}\phi_X \circledast g^*\phi_Y).$$

Therefore,

$$\|T(f,g)^{**}\phi_Z\|_{L_C(\frac{dt}{t})} \leqslant \theta \, \|f^{**}\phi_X \circledast g^*\phi_Y\|_{L_C(\frac{dt}{t})}$$

$$\leqslant \theta \, .M$$

where M is an absolute constant. +

(4.4.7) THEOREM. Let $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty)\times(0,\infty))$ be r.i.spaces, and suppose that there exists a constant $\theta > 0$ such that

$$\int_0^\infty \frac{1}{\phi_X(t/s)} \, \frac{1}{\phi_Y(s)} \, \frac{ds}{s} \leqslant \theta/\phi_Z(t), \quad \forall \; t > 0.$$

Then every t.p.o. T defines a bounded bilinear operator,

$$T: M(X) \times M(Y) \to M(Z).$$

Proof. Follows readily from (4.4.4). +

4.5. EMBEDDINGS OF B.F.SPACES WITH MIXED NORMS.

Let

(Ω_1,μ_1), (Ω_2,μ_2) be measure spaces and let $X(\Omega_1)$, $Y(\Omega_2)$, $Z(\Omega_1 x \Omega_2)$ be B.F.Spaces. Define

$$X(Y)(\Omega_1 x \Omega_2) = \{f \in M(\Omega_1 x \Omega_2): \| \| f(x,\cdot)\|_Y \|_X = \|f\|_{X(Y)} < \infty\}.$$

It is easily seen that $X(Y)$ is a B.F.S. and, moreover, that $X \otimes_\pi Y \subseteq X(Y)$. Therefore we may ask: when is $X(Y) \subseteq Z$?.

Consider first the case of the L^p spaces, here the situation is entirely trivial,

(4.5.1) THEOREM. (i) $L^p(0,\infty) \otimes_\pi L^q(0,\infty) \subseteq L^s((0,\infty) \times (0,\infty))$, if and only if $p = q = s$.

(ii) $L^p(0,\infty)(L^q(0,\infty)) \subseteq L^s((0,\infty) \times (0,\infty))$ if and only if $p = q = s$.

(iii) $L^p(0,1) \otimes_\pi L^q(0,1) \subseteq L^s((0,1) \times (0,1))$ if and only if $s \leq p$ and $q \leq s$.

(iv) $L^p(0,1)(L^q(0,1)) \subseteq L^s((0,1)x(0,1))$ if and only of $s \leq p$, $q \leq s$.

Proof. (i) The necessity of the condition follows by (4.1.2), the sufficiency follows by (4.3.2).

(ii) Suppose that $L^p(0,\infty)(L^q(0,\infty)) \subseteq L^s((0,\infty)x(0,\infty))$ then

$L^p(0,\infty) \underset{\pi}{\otimes} L^q(0,\infty) \subseteq L^s((0,\infty) \times (0,\infty))$, therefore by (i), $p = q = s$. The sufficiency part follows readily.

(iii) and (iv) can be proved in a similar fashion. +

This state of affairs changes rather dramatically if we consider more complicated B.F. spaces.

(4.5.2) THEOREM. Let $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty) \times (0,\infty))$ be r.i.spaces. Suppose that there exists a constant $M > 0$ such that $\forall\ u \in L^1(0,\infty)$ we have

$$\| \phi_{Y'}(|u|) \|_{X'} \leq M\, \phi_{Z'}(\| u\|_1)$$

where $\|u\|_1 = \|u\|_{L^1}$. Then,

$$X(M(Y)) \subseteq M(Z) .$$

Proof. Let $f \in X(M(Y))$, we compute $\|f\|_{M(Z)}$.

$$\|f\|_{M(Z)} = \sup_{t>0} \{\phi_Z(t) f^{**}(t)\}$$

$$= \sup_{t>0} \phi_Z(t) \{ \sup_{\substack{E \subseteq (0,\infty)^2 \\ |E| = t}} t^{-1} \iint_E |f(x,y)|\,dx\,dy\}$$

$$\leq \sup_{t>0} \phi_Z(t)\{ \sup_{\substack{E \subseteq (0,\infty)^2 \\ |E| = t}} t^{-1} \int_0^\infty \int_{P_x(E)} |f(x,y)|\,dy\,dx\}$$

$$\leq \sup_{\substack{t>0 \\ E \subseteq (0,\infty)^2 \\ |E| = t}} t^{-1}\phi_Z(t) \int_0^\infty \int_0^\infty |f(x,y)| \chi_{P_x(E)}(y)\,dx\,dy$$

where $P_x(E) = \{y \in (0,\infty) : (x,y) \in E\}$.

Therefore using Hölder's inequality we get

$$\|f\|_{M(Z)} \leq \sup_{\substack{t>0 \\ E \subseteq (0,\infty)^2 \\ |E| = t}} t^{-1}\phi_Z(t) \int_0^\infty \|f(x,\cdot)\|_{M(Y)} \|\chi_{P_x(E)}\|_{M(Y)'}\,dx$$

$$\leq \sup_{\substack{t>0 \\ E \subseteq (0,\infty)^2 \\ |E| = t}} t^{-1}\phi_Z(t) \int_0^\infty \|f(x,\cdot)\|_{M(Y)} \phi_{Y'}(|P_x(E)|)\,dx$$

$$\leq \sup_{\substack{t>0 \\ E \subseteq (0,\infty)^2 \\ |E| = t}} t^{-1}\phi_Z(t) \| \, \|f(x,\cdot)\|_{M(Y)} \|_X \|\phi_{Y'}(|P_x(E)|)\|_{X'} .$$

Now, let $u(x) = |P_x(E)|$ then by Fubini's theorem we have

$$\|u\|_1 = \int_0^\infty |u(x)|\,dx = |E| = t.$$

Therefore using (ii) we get

$$\|f\|_{M(Z)} \leq M \sup_{t>0}\{t^{-1}\phi_Z(t)\phi_{Z'}(t)\} \| \, \|f(x,\cdot)\|_{M(Y)} \|_X$$

$$\leq M. \| \, \|f(x,\cdot)\|_{M(Y)} \|_X . \quad +$$

We shall collect some consequences of the above result. In fact the complicated condition of (4.5.2) takes a particular simple form when applied in concrete situations.

(4.5.3) THEOREM. Let A,B,C be Young's functions. Then the following statements are equivalent,

(i) $L_A(0,\infty)(M(L_B)(0,\infty)) \subseteq M(L_C)((0,\infty)\times(0,\infty))$

(ii) $L_A(0,\infty) \underset{\pi}{\otimes} M(L_B)(0,\infty) \subseteq M(L_C)((0,\infty)\times(0,\infty))$

(iii) There exists a constant $\theta > 0$ such that

$$A^{-1}(t)\; B^{-1}(s) \leqslant \theta\; C^{-1}(t.s), \quad \forall\; t,s > 0.$$

Proof. It is easy to see that (i) => (ii) => (iii). We shall prove that (iii) => (i). We apply (4.5.2) .

Some computations show that (iii) implies

$$\phi_{L_{\bar{B}}}(s)/\phi_{L_{\bar{C}}}(t) \leqslant \beta.\; \bar{A}^{-1}(s/t), \quad \forall\; s,t > 0,$$

where β is an absolute constant. Therefore,

$$\bar{A}(\phi_{L_{\bar{B}}}(s)/\beta\phi_{L_{\bar{C}}}(t)) \leqslant s/t, \quad \forall\, s,t > 0.$$

Let $u \in L^1(0,\infty)$, $||u||_1 = t$ and let $|u(x)| = s$, then

$$\bar{A}(\phi_{L_{\bar{B}}}(|u(x)|)/\beta\phi_{L_{\bar{C}}}(||u||_1)) \leqslant |u(x)|/\; ||u||_1$$

$$\bar{A}(\phi_{L_{\bar{B}}}(|u(x)|)/\beta\ \phi_{L_{\bar{C}}}(\|u\|_1)) \leqslant |u(x)|/\|u\|_1$$

$$\int_0^\infty \bar{A}(\phi_{L_{\bar{B}}}(|u(x)|)/\beta\ \phi_{L_{\bar{C}}}(\|u\|_1))dx \leq 1$$

which implies

$$\|\phi_{L_{\bar{B}}}(|u|)\|_{L_{\bar{A}}} \leqslant \beta\ \phi_{L_{\bar{C}}}(\|u\|_1).$$

Therefore (i) holds by (4.5.2). +

(4.5.4) THEOREM. Let $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty)x(0,\infty))$ be r.i.spaces. Then,

(i) If X and X' satisfy the δ_2 condition, then
$\Lambda(X)[M(Y)] \subseteq M(Z) \iff \Lambda(X) \underset{\pi}{\otimes} M(Y) \subseteq M(Z)$.

(ii) If X satisfies the conditions of (i) then, $M(X)(M(Y)) \subseteq M(Z)$, whenever there exists a constant $\theta > 0$ such that

$$\int_0^\infty \phi_{Y'}(s/t)\ \phi_{X'}(t)\frac{dt}{t} \leqslant \theta\ \phi_{Z'}(s), \qquad \forall\, s > 0.$$

Proof. (i) One implication is trivial. Suppose that $\Lambda(X) \underset{\pi}{\otimes} M(Y) \subseteq M(Z)$, then by (4.2.1) there exists a constant $\theta > 0$ such that

$$\phi_Z(t.s) \leqslant \theta\ \phi_X(t)\phi_Y(s) \qquad \forall\, t,s > 0.$$

Some computations show that

$$\phi_{Y'}(s/t)\phi_{X'}(t) \leqslant \theta\ \phi_{Z'}(s) \qquad \forall\ t,s > 0.$$

We shall use (4.5.2). Let $u \in L^1(0,\infty)$, $\|u\|_1 = s$, then $\phi_{Y'}(|u(t)|)$ is equimeasurable with $\phi_{Y'}(u^*(t))$, thus

$$\|\phi_{Y'}(|u|)\|_{\Lambda(X)'} = \|\phi_{Y'}(|u|)\|_{M(X')} \approx \|\phi_{Y'}(|u|)\|_{\tilde{M}(X')}$$

$$\|\phi_{Y'}(|u|)\|_{\tilde{M}(X')} = \sup_{t>0} \{ \phi_{Y'}(u^*(t))\phi_{Y'}(t)\} \quad .$$

But, $u^*(t) \leqslant st^{-1}$, $\forall\ t > 0$, so that

$$\sup_{t>0} \{\phi_{Y'}(u^*(t))\phi_{X'}(t)\} \leqslant \sup_{t>0} \{ \phi_{Y'}(s/t)\phi_{X'}(t)\}$$

$$\leqslant \theta\ \phi_{Z'}(s) = \theta\ \phi_{Z'}(\|u\|_1) \quad .$$

The result follows by (4.5.2).

(ii) The proof is similar to the one given in (i). The reader should compare this result with (4.4.7). +

We consider Calderón spaces with mixed norms.

(4.5.5) THEOREM. Let $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty)\times(0,\infty))$ be B.F.spaces, and let A,B,C be Young's functions. Suppose that the following conditions hold,

(i) $X(Y) \subseteq Z$.

(ii) There exists a constant $\theta > 0$ such that

$$A^{-1}(t)\ B^{-1}(s) \leq \theta\ C^{-1}(t.s), \quad \forall\ t,s > 0.$$

Then,

$$A^{-1}(X)\left[B^{-1}(Y)\right] \subseteq C^{-1}(Z).$$

Proof. Let $f \neq 0$, $f \in A^{-1}(X)\left[B^{-1}(Y)\right]$, $h(x) = \|f(x,\cdot)\|_{B^{-1}(Y)}$

and $r = \|\ \|f(x,\cdot)\|_{B^{-1}(Y)}\|_{A^{-1}(X)}$.

Observe that $h(x) = 0 \iff f(x,y) = 0$ for a.e.y.

Therefore if we let $E = \{x \in (0,\infty): |h(x)| > 0\}$, we get using (ii)

$$C(|f(x,y)|/r.\theta) = C\left(\frac{1}{\theta}\ \frac{|f(x,y)|}{h(x)}\ \chi_E(x)\ \frac{h(x)}{r}\right) \quad \text{a.e.}x,y$$

$$\leq A\left(\frac{h(x)}{r}\right)\ B\left(\frac{|f(x,y)|}{h(x)}\ \chi_E(x)\right) \quad \text{a.e.}x,y$$

$$\|C(|f(x,y)|/r.\theta)\|_Z \leq \left\|A\left(\frac{h(x)}{r}\right)\ B\left(\frac{|f(x,y)|}{h(x)}\ \chi_E(x)\right)\right\|_{X(Y)} (\text{by(i)})$$

$$\leq \left\|A\left(\frac{h(x)}{r}\right)\right\|_X \leq 1.$$

Therefore,

$$\|f\|_{C^{-1}(Z)} \leq \theta\ \|f\|_{A^{-1}(X)(B^{-1}(Y))} \quad . \quad +$$

(4.5.6) COROLLARY. Let A,B,C, be Young's functions. Then, the following statements are equivalent

(i) $L_A(0,\infty) \underset{\pi}{\otimes} L_B(0,\infty) \subseteq L_C((0,\infty)\times(0,\infty))$

(ii) $L_A(0,\infty)(L_B(0,\infty)) \subseteq L_C((0,\infty)\times(0,\infty))$

(iii) There exists a constant $\theta > 0$ such that

$$A^{-1}(t)\ B^{-1}(s) \leq \theta\ \ C^{-1}(t.s) \quad \forall\ t,s > 0.$$

Proof. We only need to show that (iii) => (ii). This follows readily from (4.5.5) since $L^1(L^1) \subseteq L^1$. +

(4.5.7) COROLLARY. Let $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty)\times(0,\infty))$ be r.i.spaces, and let $q \in [1,\infty)$. Then,

(i) $\Lambda(A_q,\phi_X,1)^{\sim}(\Lambda(A_q,\phi_Y,1)^{\sim}) \subseteq O(Z)$, whenever there exists a constant $\theta > 0$ such that $\phi_Z(t.s) \leq \theta\ \phi_X(t)\phi_Y(s)$, $\forall$ t,s > 0 and moreover $\phi_X^q(t).t^{-1}$, $\phi_Y^q(t)t^{-1}$ are non-increasing.

(ii) $\Lambda(A_q,\phi_X,1)^{\sim}(\Lambda(A_q,\phi_Y,1)^{\sim}) \subseteq \Lambda(A_r,\phi_Z,1)^{\sim}$, where $r \in [1,\infty)$, whenever, (a) there exists a constant $\theta > 0$ such that $\phi_Z(t.s) \leq \theta\ \phi_X(t)\phi_Y(s)$ $\forall$ t,s > 0, (b) $\phi_X^q(t).t^{-1}$, $\phi_Y^q(t).t^{-1}$ are non-increasing and $\phi_Z^r(t).t^{-1}$ is non decreasing.

Proof. (i) Follows by (4.5.6) and (3.1.5)(i); (ii) follows from (i) and (3.1.5) (ii). +

(4.5.8) REMARK. When specialized to the $L(p,q)$ spaces the above results are sharp, in fact one can show that if $1 < p < \infty$, $1 \leq q_i \leq \infty$, $i = 1,2,3$, then

$$L(p,q_1)(L(p,q_2)) \subseteq L(p,q_3) \quad \text{if and only if} \quad q_3 = \infty \quad \text{and}$$

$q_1 \leqslant p$ or $q_3 < \infty$ and $\max\{q_1, q_2\} \leqslant p \leqslant q_3$.

This result has been announced in [40] and will be published elsewhere.

4.6. SOME REMARKS CONCERNING THE δ_2 CONDITION.

In this chapter we have often considered r.i.spaces X satisfying the δ_2 condition. In some cases this assumption was made in order to insure that $\phi_X(t) \approx \phi_{\Lambda(X)}(t)$. This problem can be overcame if we consider the spaces

$$k(X)(\Omega) = \{f \in M(\Omega) : \|f\|_{k(X)} < \infty\}$$

$$\|f\|_{k(X)} = \int_0^\infty \phi_X(\lambda_f(t))dt.$$

It is not difficult to show that $\Lambda(X) \cong k(X)$ whenever X satisfies the δ_2 condition, and clearly

$$\phi_{K(X)}(t) = \phi_X(t).$$

The results proved in previous sections for $\Lambda(X)$ spaces can be generalised for $k(X)$ spaces. This follows from the fact that

$$\lambda_{f\blacksquare g}(t) = \int_0^\infty \lambda_f(t/g^*(s))ds.$$

See [43] .

4.7. Λ SPACES WITH MIXED NORMS. It is natural to ask if a similar result to (4.5.5) holds for the Lorentz Λ spaces. The answer is, surprisingly, no.

Some care is needed since the class of Λ spaces has a non-trivial intersection with the class of Orlicz spaces.

(4.7.1) THEOREM. Let $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty)x(0,\infty))$ be r.i. spaces. Suppose that the following conditions hold

(i) There exists a constant $\varepsilon > 0$ such that

$$\int_1^\infty \frac{x^{-\varepsilon}}{\phi_X(x)}\, dx = \infty .$$

(ii) $\int_0^1 \phi_X(x)\frac{dx}{x} < \infty$, and Y satisfies the δ_2 condition.

(iii) There exists a constant $\theta > 0$ such that

$$\phi_Z(t.s) = \theta\, \phi_X(t)\, \phi_Y(s) \quad \forall\ t,s > 0.$$

Then,

$$\Lambda(X)\ (\Lambda(Y)) \not\subset \Lambda(Z).$$

Proof. Let $F(x)$ be defined as follows

$$F(x) = \begin{cases} 1/\phi_Y(e^x) & \text{if} \quad x \in [0,1) \\ (\phi_Y(e^x)\phi_X(x)x^{\varepsilon})^{-1} & \text{if} \quad x \in [1,\infty). \end{cases}$$

Let

$$f(x,y) = \chi_{(0,e^x)}(y)\ F(x), \quad x,y \in (0,\infty).$$

We shall prove that $f \in \Lambda(X)(\Lambda(Y))$ but $f \notin \Lambda(Z)$. Observe that $F(x)$ is non-increasing, therefore

$$\lambda_f(t) = |\{(x,y):\ 0 \leq y \leq e^x\ ,\ F(x) \geq t\}|$$

$$= |\{(x,y):\ 0 \leq y \leq e^x\ ,\ 0 \leq x \leq \lambda_F(t)\}|$$

$$= \int_0^{\lambda_F(t)} \{\int_0^{e^x} dy\}\ dx = e^{\lambda_F(t)} - 1.$$

Thus,

$$f^*(t) = F^*(\log(t+1)).$$

We compute $\|f\|_{\Lambda(Z)}$:

$$\|f\|_{\Lambda(Z)} = \int_0^\infty F^*(\log(t+1))\phi_Z(t)\frac{dt}{t}$$

$$= \int_0^\infty F^*(t)\phi_Z(e^t-1).(e^t-1)^{-1}e^t\ dt$$

$$\geq \int_1^\infty \frac{1}{\phi_Y(e^t)}\ \frac{1}{\phi_X(t)}\ t^{-\varepsilon}\ \phi_Z(e^t-1)(e^t-1)^{-1}e^t\ dt$$

$$\geq \theta\int_1^\infty \frac{t^{-1}}{\phi_X(t)}\ dt = \infty\ .$$

We compute $\|f\|_{\Lambda(X)(\Lambda(Y))}$:

$$\|f(x,\cdot)\|_{\Lambda(Y)} = \|\chi_{(0,e^x)}(\cdot)\ F(x)\|_{\Lambda(Y)} \approx F(x)\phi_Y(e^x)\ .$$

Observe that $F(x)\phi_Y(e^x)$ is non-increasing, therefore

$$\|\,\|f(x,\cdot)\|_{\Lambda(Y)}\|_{\Lambda(X)} = \int_0^\infty F(x)\phi_Y(e^x)\phi_X(x)\frac{dx}{x}$$

$$= \int_0^1 F(x)\phi_Y(e^x)\phi_X(x)\frac{dx}{x} + \int_1^\infty F(x)\phi_Y(e^x)\phi_X(x)\frac{dx}{x}$$

$$= \int_0^1 \phi_X(x)\frac{dx}{x} + \int_1^\infty \frac{dx}{x^{1+\varepsilon}} < \infty. \quad +$$

4.8. NOTES TO CHAPTER 4. The embedding theorems obtained in this chapter generalise work by O'Neil [43] , for the Orlicz spaces and L(p,q) spaces.

In the particular case of the Orlicz spaces we have stronger results (See (4.5.6),(4.5.3)) than the ones obtained by O'Neil. Moreover, our techniques are different and allow us to treat a wide class of spaces in a unified manner.

The result (4.5.2) originated in work by Walsh [59], who obtained the corresponding result for the L(p,q) spaces. The results in §4.5 show that our abstract version of his result is far from being trivial.

As far as the author is aware the embedding theorems for r.i. spaces with mixed norms are new. For related work for the Orlicz spaces see

Donaldson [16].

In the case of r.i.spaces based on finite measure spaces one is able to prove a result closely related to (4.5.1).

THEOREM. Let $X(0,1)$, $Y(0,1)$, $Z((0,1)\times(0,1))$ be r.i.spaces, then $M(X)(M(Y)) \subseteq \Lambda(Z)$, whenever $\bar{\gamma}_X < \underline{\gamma}_Z$, $\bar{\gamma}_Y < \underline{\gamma}_Z$. (for the definition and properties of the indices of r.i.spaces we refer the reader to [63] an [55]).

Proof. Let p be such that $p > 1$, and $\bar{\gamma}_X < \frac{1}{p} < \underline{\gamma}_Z$, $\bar{\gamma}_Y < \frac{1}{p} < \underline{\gamma}_Z$, then

$$M(X)(M(Y)) \subseteq L^p(L^p) \subseteq L^p \subseteq L(p\infty) \subseteq \Lambda(Z). \quad +$$

COROLLARY. Suppose that the conditions of the above theorem hold, then

(i) $\Lambda(X)(\Lambda(Y)) \subseteq \Lambda(Z)$

(ii) $X(Y) \subseteq Z$

(iii) $X \hat{\otimes}_\pi Y \subseteq Z$. +

In the above results we cannot replace $<$ by $\leq$ in the hypothesis on the indices. Indeed, Shimogaki has constructed r. i. spaces X such that $\underline{\gamma}_X = \bar{\gamma}_X = \frac{1}{2}$, but $X \not\subseteq L^2$, $L^2 \not\subseteq X$.

The result in §4.7 extends previous work by Cwikel [14] for the $L(p,1)$ spaces, see also Milman [40].

We have not considered in this chapter embedding theorems for the $\Lambda(A,\phi_X,\phi_Y)$ spaces. Results in this direction can be derived using the same methods of section §4.4 and the convolution theorems of Chapter 5.

It would be of considerable interest to obtain embedding theorems of the type studied in this Chapter, for other cross-norms. We remark that embedding theorems for the "ε" cross-norm can be readily obtained using duality and our results for the "π" cross norm.

In the same vein the results of §4.5 give by duality embedding theorems of r.i.spaces into r.i.spaces with mixed norms.

These results could find applications in the theory of ideals of operators (cf.(5.4.3)).

We point out that our method to obtain embedding theorems for $A^{-1}(X)$ spaces can be easily generalised for $GA^{-1}(X)$ spaces.

THEOREM. Let A,B,C be generalised Young's functions, and $X(\Omega_1)$, $Y(\Omega_2)$, $Z(\Omega_1 \times \Omega_2)$ be B.F.spaces. Suppose that the following conditions hold

(i) $\|f \otimes g\|_Z \leq \|f\|_X \|g\|_Y \quad \forall (f,g) \in X \times Y.$

(ii) There exist θ, $\alpha > 0$ such that

$$A^{-1}(t)\, B^{-1}(s) \lesssim \theta\, C^{-1}(\alpha.t.s) \qquad \forall\, t,s > 0.$$

Then,

$$\|f \otimes g\|_{GC^{-1}(Z)} \leq \max\{\theta,\alpha\} \; \|f\|_{GA^{-1}(X)} \|g\|_{GB^{-1}(X)} \; .$$

For example, since $L^1 \underset{\pi}{\otimes} L(1,\infty) \subseteq L(1,\infty)$ we have, under the assumption that (ii) above holds,

$$GA^{-1}(L^1) \underset{\pi}{\otimes} GB^{-1}(L(1,\infty)) \subseteq GC^{-1}(L(1,\infty)).$$

Compare with O'Neil [43], where the spaces $GB^{-1}(L(1,\infty))$ are denoted by W_B.

Finally we should point out that (4.2.4) can be used to obtain a general theorem for tensor products of r.i.spaces. Let ϕ be a decreasing function on $(0,\infty)$, the Lorentz space $\Lambda_\phi(0,\infty)$ is the r.i.space of all measurable functions f which satisfy

$$\|f\|_{\Lambda_\phi} = \int_0^\infty f^*(t)\phi(s)ds < \infty \; .$$

Then, for any r.i.space $X(0,\infty)$ we have

$$\|f\|_X = \sup_{\|g\|_{X'} \leq 1} \|f\|_{\Lambda_{g^*}}$$

Notice that $\phi_{\Lambda_{g^*}}(t) = \int_0^t g^*(s)ds$, therefore, if $Y(0,\infty)$, $Z((0,\infty)\times(0,\infty))$ are r.i.spaces, we have $\Lambda_{g^*} \underset{\pi}{\otimes} Y \subseteq Z$ whenever there exists a constant $\theta > 0$ such that

$$s^{-1}\|E_{1/s}\|_{Y \to \hat{Z}} \leq \theta \; g^{**}(s) \; , \quad \forall \; s > 0.$$

Thus, if there exists $\theta > 0$ such that the above inequality holds for every $0 \neq g \in S(X')$, we get

$$X \underset{\pi}{\otimes} Y \subseteq Z.$$

CHAPTER 5

TENSOR PRODUCTS OF R.I. SPACES AS BANACH MODULES.

Let $X(-\infty,\infty)$, $Y(-\infty,\infty)$, $Z(-\infty,\infty)$ be r.i. spaces. In this chapter we look at the problem of finding necessary and sufficient conditions for convolution operators and product operators to act continuously on O.C.L.H.Z. spaces.

In §5.1 we treat product operators and in §5.2. we deal with convolution operators, in §5.3 we give several applications of these results and outline generalizations to the setting of non-commutative r.i. spaces. These results lead to non-commutative versions of some classical theorems in analysis.

5.1. ESTIMATES FOR PRODUCT OPERATORS. Let $X(0,\infty)$, $Y(o,\infty)$, $Z(0,\infty)$ be r.i. spaces, in this section we obtain necessary and sufficient conditions for $f(t)\,g(t) \in Z$, whenever $f \in X$, $g \in Y$.

These results imply embedding theorems of projective tensor products of r.i. spaces as L^{∞}modules (cf. [24]).

Our results are based on estimates for the maximal function associated with a product operator and the properties of the Calderón functors. The result (5.1.2) has been stated, without proof, in O'Neil [45].

The following definition is the analogue of (4.4.1) for product operators.

(5.1.1) DEFINITION. Let (Ω_1,μ_1), (Ω_2,μ_2), (Ω_3,μ_3) be measure spaces. A bilinear operator Π is called a product operator (p.o.) if it verifies the following conditions,

(i) $\|\Pi(f,g)\|_{L^{\infty}(\Omega_3)} \leq \|f\|_{L^{\infty}(\Omega_1)} \|g\|_{L^{\infty}(\Omega_2)}$

(ii) $\|\Pi(f,g)\|_{L^{1}(\Omega_3)} \leq \|f\|_{L^{1}(\Omega_1)} \|g\|_{L^{\infty}(\Omega_2)}$

(iii) $\|\Pi(f,g)\|_{L^{1}(\Omega_3)} \leq \|f\|_{L^{\infty}(\Omega_1)} \|g\|_{L^{1}(\Omega_2)}$.

(5.1.3) **LEMMA.** Let Π be a p.o. then for any r.i. space $X(0,\infty)$, we have

(i) $\quad \|\Pi(f,g)\|_1 \leq \|f\|_X \|g\|_{X'}$

(ii) $\quad \|\Pi(f,g)\|_X \leq \|f\|_\infty \|g\|_X$

(iii) $\quad \|\Pi(f,g)\|_X \leq \|f\|_X \|g\|_\infty$

where $\| \ \|_1$, $\| \ \|_\infty$ denote respectively the $L^1(0,\infty)$, $L^\infty(0,\infty)$ norms .

Proof. (i) Let $f \in X$, $g \in X'$, then by (5.1.2)

$$t\Pi(f,g)^{**}(t) \leq \int_0^t f^*(u)g^*(u)\,du$$

$$\leq \|f\|_X \|g\|_{X'} .$$

Therefore,

$$\|\Pi(f,g)\|_1 = \lim_{t \to \infty} t\, P(f,g)^{**}(t) \leq \|f\|_X \|g\|_{X'} .$$

(ii) Let $0 \neq f \in L^\infty$ and consider the linear operator $\Pi_f(g) = \dfrac{\Pi(f,g)}{\|f\|_\infty}$, then Π_f defines a bounded linear operator,

$$\Pi_f : \begin{cases} L^\infty \to L^\infty \\ L^1 \to L^1 \end{cases}$$

with norm less or equal to one, thus by the interpolation theorem of Calderón [10],

$\Pi_f : X \to X$, continuously for any r.i.space X, and moreover $\| \Pi_f \| \leq 1$.

The proof of (iii) is the same as the one given in (ii). +

(5.1.4) THEOREM. Let A, B, C be Young's functions, and let X_1, X_2, X_3, Y be r.i.spaces. Then every p.o. Π defines a bounded bilinear operator,

$$\Pi : \Lambda(A,\phi_{X_1},\phi_Y) \times \Lambda(B,\phi_{X_2},\phi_Y) \to \Lambda(C,\phi_{X_3},\phi_Y)$$

whenever the following conditions are satisfied,

(i) $\exists\ \theta > 0$ such that $\phi_{X_3}(t) \leq \theta\, \phi_{X_1}(t)\, \phi_{X_2}(t)$, $\forall\ t > 0$.

(ii) $\exists\, M > 0$ such that $A^{-1}(t)\, B^{-1}(t) \leq MC^{-1}(t)$, $\forall\, t > 0$.

(iii) The conditions of (2.2.1) are satisfied by C, ϕ_{X_3}, ϕ_Y.

Proof. Let $f \in \Sigma(\Lambda(A,\phi_{X_1},\phi_Y))$, $g \in \Sigma(\Lambda(B,\phi_{X_2},\phi_Y))$ then by (5.1.2)

$$\Pi(f,g)^{**}(t) \leq \frac{1}{t} \int_0^t f^*(u) g^*(u)\, du$$

therefore by (2.2.1),

$$\|\Pi(f,g)\|_{\Lambda(C,\phi_{X_3},\phi_Y)} \leq D\, \|f^* g^* \phi_{X_3} \theta^{-1}\|_{L_C(\phi_Y(t)\frac{dt}{t})}$$

where D is an absolute constant. Thus, by (i)

(5.1.3) **LEMMA.** Let Π be a p.o. then for any r.i. space $X(0,\infty)$, we have

$$\text{(i)} \qquad \| \Pi(f,g)\|_1 \leq \|f\|_X \; \|g\|_{X'}$$

$$\text{(ii)} \qquad \| \Pi(f,g)\|_X \leq \|f\|_\infty \; \|g\|_X$$

$$\text{(iii)} \qquad \| \Pi(f,g)\|_X \leq \|f\|_X \; \|g\|_\infty$$

where $\| \;\; \|_1$, $\| \;\; \|_\infty$ denote respectively the $L^1(0,\infty)$, $L^\infty(0,\infty)$ norms .

Proof. (i) Let $f \in X$, $g \in X'$, then by (5.1.2)

$$t\Pi(f,g)^{**}(t) \leq \int_0^t f^*(u)g^*(u)\,du$$

$$\leq \|f\|_X \; \|g\|_{X'} \; .$$

Therefore,

$$\|\Pi(f,g)\|_1 = \lim_{t\to\infty} t\; P(f,g)^{**}(t) \leq \|f\|_X \; \|g\|_{X'} \; .$$

(ii) Let $0 \neq f \in L^\infty$ and consider the linear operator $\Pi_f(g) = \dfrac{\Pi(f,g)}{\|f\|_\infty}$, then Π_f defines a bounded linear operator,

$$\Pi_f: \begin{cases} L^\infty \to L^\infty \\ L^1 \to L^1 \end{cases}$$

with norm less or equal to one, thus by the interpolation theorem of Calderón [10],

$$\|\Pi(f,g)\|_{\Lambda(C,\phi_{X_3},\phi_Y)} \leq D\|(f^*\phi_{X_1})(g^*\phi_{X_2})\|_{L_C(\phi_Y(t)\frac{dt}{t})}$$

and the result follows by O'Neil's theorem for products of Orlicz spaces (cf. [44]). +

A similar result can be obtained for products of O.C.L.H.Z. spaces "with tildes". Notice that if $\Pi(f,g)(t) = f(t)\, g(t)$, then $\Pi(f,g)^*(2t) \leq f^*(t)g^*(t)$.

It is not difficult to prove that the conditions on the fundamental functions are necessary, in fact we have the following

(5.1.5) THEOREM. Let X, Y, Z be r.i. spaces, and let $\Pi(f,g)(t) = f(t)g(t)$, then a necessary condition for Π to define a bounded bilinear operator, $\pi: X \times Y \to Z$, is the existence of $\theta > 0$ such that

$$(5.1.6) \qquad \phi_Z(t) \leq \theta\phi_X(t)\,\phi_Y(t), \qquad \forall\, t > 0.$$

Proof. Suppose that (5.1.6) does not hold, then there exists a sequence of positive numbers $\{t_n\}_{n \in N}$ such that

$$\phi_Z(t_n) \geq 2^{4n}\,\phi_X(t_n)\phi_Y(t_n) \qquad n = 1,\ldots\ldots\ldots\ldots$$

Let $f_n = \chi_{(0,t_n)} \cdot [\phi_X(t_n)]^{-1}$, $g_n = \chi_{(0,t_n)} \cdot [\phi_Y(t_n)]^{-1}$, then

$$f_n\, g_n \geq 2^{4n}\,\chi_{(0,t_n)} \cdot [\phi_Z(t_n)]^{-1}, \qquad n = 1 \ldots\ldots\ldots\ldots .$$

Therefore if we let $f = \sum_{n=1}^{\infty} \frac{f_n}{2^n}$, $g = \sum_{n=1}^{\infty} \frac{g_n}{2^n}$, we easily verify that $f \in X$, $g \in Y$, but $fg \notin Z$. +

We shall now consider products of Calderón spaces.

(5.1.7) THEOREM. Let A, B, C, be Young's functions, and suppose that there exists $\theta > 0$ such that

$$A^{-1}(t)\ B^{-1}(t) \leq \theta\ C^{-1}(t) \quad , \quad \forall\ t > 0.$$

Moreover suppose that $\|f + g\|_Z \leq \|f\|_X + \|g\|_Y$, $\forall\ (f,g) \in X \times Y$.

Then, if $f \in A^{-1}(X)$, $g \in B^{-1}(Y)$, $f.g \in C^{-1}(Z)$ and

$$\|fg\|_{C^{-1}(Z)} \leq 2\theta\ \|f\|_{A^{-1}(X)}\ \|g\|_{B^{-1}(Y)}.$$

Proof. It is easy to see (cf [44] , p. 303) that the condition on the Young's functions implies

$$C(t.s/\theta) \leq A(t) + B(s) \quad , \quad \forall\ t,s \geq 0.$$

The result follows readily from the above inequality. +

A similar result can be proved for our generalised Calderón spaces.

(5.1.8) THEOREM. Let A, B, C be generalised Young's functions, and let X, Y, Z be r.i spaces verifying the condition of (5.1.7). Then,

(i) if there exist constants $\theta > 0$, $\alpha > 0$ such that

$$(5.1.9)\quad A^{-1}(t)\, B^{-1}(t) \leqslant \theta\, C^{-1}(\alpha . t), \qquad \forall\, t > 0$$

then $f \in GA^{-1}(X)$, $g \in GB^{-1}(Y)$ implies that $fg \in GC^{-1}(Z)$ and moreover

$$\|f\, g\|_{GC^{-1}(Z)} \leqslant \max\{2\alpha, \theta\}\, \|f\|_{GA^{-1}(X)}\, \|g\|_{GB^{-1}(Y)} \quad .$$

(ii) If $X = Y = Z = L^1$ and (5.1.9) does not hold then there exist $f \in GA^{-1}(L^1)$, $g \in GB^{-1}(L^1)$ such that $fg \notin GC^{-1}(L^1)$.

Proof. (i) It is easy to see that (5.1.9) implies

$$C(t.s/\theta) \leqslant \alpha(A(t) + B(s)), \qquad \forall\, t,s \geqslant 0.$$

Let $f \in GA^{-1}(X)$, $g \in GB^{-1}(Y)$, $\|f\|_{GA^{-1}(X)} = r$, $\|g\|_{GB^{-1}(Y)} = s$, then,

$$\|C(\tfrac{2\alpha}{\theta} \;.\; \tfrac{f}{r} \,.\, \tfrac{g}{s} \,/\, 2\alpha)\|_{GC^{-1}(Z)} \leqslant 2\alpha\; r.s$$

thus,

$$\|\tfrac{2\alpha}{\theta} \,.\, f.g\|_{GC^{-1}(Z)} \leqslant 2\alpha\, \|f\|_{GA^{-1}(X)}\, \|g\|_{GB^{-1}(Y)} \quad .$$

From the above inequality we get

$$\|fg\|_{GC^{-1}(Z)} \leqslant \max\{2\alpha, \theta\}\, \|f\|_{GA^{-1}(X)}\, \|g\|_{GB^{-1}(Y)} \quad .$$

(ii) Suppose that (5.1.9) does not hold, then there exists a sequence of positive numbers $\{t_n\}_{n \in N}$ such that

$$\infty > A^{-1}(t_n)\, B^{-1}(t_n) > 8^n \,.\, C^{-1}(t_n.8^n), \quad n = 1,\ldots\ldots .$$

Let $\{E_n\}_{n \in N}$ be a sequence of disjoint measurable sets such that $|E_n| = 2^{-n} t_n^{-1}$. For $n \in N$ define $f_n = 2^{-n} A^{-1}(t_n)\chi_{E_n}$, $g_n = 2^{-n} B^{-1}(t_n)\chi_{E_n}$ and $f = \sum_{n=1}^{\infty} f_n$, $g = \sum_{n=1}^{\infty} g_n$. It can be easily verified that $f \in GA^{-1}(L^1)$, $g \in GB^{-1}(L^1)$ but $fg \notin GC^{-1}(L^1)$. +

5.2. ESTIMATES FOR CONVOLUTION OPERATORS.

In this section we give an estimate for the maximal rearrangement of a convolution operator from which we derive convolution theorems for the O.C.L.H.Z. spaces. The inequality referred above is due essentially to O'Neil [45] for the case of "positive" convolution operators. The problem of extending these estimates to more general operators is not entirely trivial (cf[4], [5]).

The results in this section are partially based on some joint work with R. Sharpley (cf. [41]).

(5.2.1) DEFINITION. A bilinear operator T is called a convolution operator if the following conditions are satisfied,

(5.2.2) $\|T(f,g)\|_\infty \leq \|f\|_1 \|g\|_\infty$

(5.2.3) $\|T(f,g)\|_\infty \leq \|f\|_\infty \|g\|_1$

(5.2.4) $\|T(f,g)\|_1 \leq \|f\|_1 \|g\|_1$.

When we write T(f,g) we assume that its existence is forced by the conditions above.(Notice that T(f,g) may not be defined for arbitrary pairs of functions (f,g).)

(5.2.5) EXAMPLES. (i) $T(f,g)(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds$ is a convolution operator.

(ii) More generally $T(f,g)(x) = \int_G f(y)g(y^{-1}x)dy$ is a convolution operator, where G is a locally compact abelian group.

(ii) Let $h \in L^1 \cap L^\infty$ and define $T(f,g) = \int_{-\infty}^{\infty} P(f,h)(s)g(t-s)ds$, where P is a product operator, then T is a convolution operator.

Notice that in general $T(f,g) \neq T(g,f)$, and moreover T is not, in general, translation invariant.

(5.2.6) THEOREM. (O'Neil [44]). Let T be a convolution operator (c.o.), and suppose that $\int_0^\infty f^*(s)g^*(s)ds < \infty$, then $T(f,g)$ is well defined and

$$\|T(f,g)\|_\infty \leq \int_0^\infty f^*(s)g^*(s)ds. \quad +$$

(5.2.7) THEOREM. Let T be a c.o., and suppose that for some $0 < t < \infty$, $tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)ds$ is finite, then $T(f,g)$ is well defined and

$$T(f,g)^{**}(t) \leq tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)ds.$$

Proof. Let $f_1(s) = [|f(s)| \wedge f^*(t)]\,\mathrm{sgn}(f(s))$,

$g_1(s) = [|g(s)| \wedge g^*(t)]\,\mathrm{sgn}(g(s))$, and $f_2 = f - f_1$, $g_2 = g - g_1$, then

$$(5.2.8)\quad T(f_2, g_2)^{**}(t) = h_1^{**}(t) \leq \frac{1}{t}\|h_1\|_1 \leq \frac{1}{t}\|f_2\|_1 \|g_2\|_1$$
$$= \frac{1}{t}\int_{f^*(t)}^{\infty}\lambda_f(s)ds \int_{g^*(t)}^{\infty}\lambda_g(s)ds$$

by (5.2.4),

$$(5.2.9)\quad T(f_2, g_1)^{**}(t) = h_2^{**}(t) \leq \|h_2\|_\infty \leq \int_{f^*(t)}^{\infty}\lambda_f(s)ds g^*(t)$$

By (5.2.2),

$$(5.2.10)\quad T(f_1, g_2)^{**}(t) = h_3^{**}(t) \leq \|h_3\|_\infty \leq f^*(t)\int_{g^*(t)}^{\infty}\lambda_g(s)ds$$

by (5.2.3), and

$$(5.2.11)\quad T(f_1, g_1)^{**}(t) = h_4^{**}(t) \leq \|h_4\|_\infty \leq \int_0^{\infty} f_1^*(s)g_1^*(s)ds$$
$$= tf^*(t)g^*(t) + \int_t^{\infty} f^*(s)g^*(s)ds$$

by (5.2.6). Combining inequalities (5.2.8) and (5.2.10) we have

$$(5.2.12)\quad T(f, g_2)^{**}(t) \leq h_1^{**}(t) + h_3^{**}(t)$$
$$\leq \frac{1}{t}\int_{f^*(t)}^{\infty}\lambda_f(s)ds\int_{g^*(t)}^{\infty}\lambda_g(s)ds + f^*(t)\int_{g^*(t)}^{\infty}\lambda_g(s)ds$$
$$= \left[\frac{1}{t}\int_{f^*(t)}^{\infty}\lambda_f(s)ds + f^*(t)\right]\int_{g^*(t)}^{\infty}\lambda_g(s)ds$$
$$= f^{**}(t)\int_{g^*(t)}^{\infty}\lambda_g(s)ds$$

on the other hand (5.2.9) and (5.2.11) yield,

$$(5.2.13)\quad T(f,g_1)^{**}(t) \leq h_2^{**}(t) + h_4^{**}(t)$$

$$\leq tf^{**}(t)g^*(t) + \int_t^\infty f^*(s)g^*(s)ds .$$

Therefore combining (5.2.12) and (5.2.13) we get

$$T(f,g)^{**}(t) \leq T(f,g_1)^{**}(t) + T(f,g_2)^{**}(t)$$

$$\leq f^{**}(t)\left[\int_{g^*(t)}^\infty \lambda_g(s)ds + tg^*(t)\right] + \int_t^\infty f^*(s)g^*(s)ds$$

$$= f^{**}(t)g^{**}(t)t + \int_t^\infty f^*(s)g^*(s)ds. \quad +$$

Using (5.2.7) and Hardy type inequalities we can easily derive continuity results for convolution operators acting on O.C.L.H.Z. spaces.

We shall consider first the abstract version of the celebrated Fractional Integration Theorem due to Hardy-Littlewood-Sobolev- O'Neil.

(5.2.14) THEOREM. Let X_i, $i = 1,2,3$, Y be r.i. spaces and let T be a convolution operator. Moreover let A be a generalised Young's function, then

(i) if there exists a constant $\theta > 0$ such that

$$(5.2.15)\quad \phi_{X_3}(t)t \leq \theta\, \phi_{X_1}(t)\, \phi_{X_2}(t), \qquad \forall\, t > 0$$

and A is a Young's function such that A, ϕ_{X_3}, ϕ_Y verify the conditions of (2.3.1), then there exists an absolute constant $C > 0$ such that

$$\|T(f,g)\|_{\Lambda(A,\phi_{X_3},\phi_Y)} \leq C\, \|f\|_{M(X_1)}\, \|g\|_{\Lambda(A,\phi_{X_2},\phi_Y)} .$$

(ii) If (5.2.15) holds and A is such that A, ϕ_{X_3}, ϕ_Y satisfy the conditions of (2.3.4), then there exists an absolute constant C > 0 such that

$$\int_0^\infty A(T(f,g)^{**}(t)\phi_{X_3}(t))\phi_Y(t)\frac{dt}{t} \leq C \|f\|_{M(X_1)} \int_0^\infty A(f^*(t)\phi_{X_3}(t))\phi_Y(t)\frac{dt}{t} .$$

Proof. We shall prove only (i) the proof of (ii) being similar.

Let $f \in \Sigma(M(X_1))$, $g \in \Sigma(\Lambda(A,\phi_{X_2},\phi_Y))$, then

$$T(f,g)^{**}(t)\phi_{X_3}(t)\theta^{-1} \leq g^{**}(t)\phi_{X_2}(t) + \phi_{X_3}(t)\theta^{-1}\int_t^\infty \frac{g^*(u)}{\phi_{X_1}(u)} \frac{du}{u} .$$

Let $d\mu(t) = \phi_Y(t)\frac{dt}{t}$, then

$$\theta \|T(f,g)^{**}\phi_{X_3}\|_{L_A(d\mu)} \leq \|g^{**}\phi_{X_2}\|_{L_A(d\mu)} + \|\tilde{P}(\frac{g^*\ \theta^{-1}}{\phi_{X_1}})\|_{\tilde{\Lambda}(A,\phi_{X_3},\phi_Y)}$$

but,

$$\|\tilde{P}(\frac{g^*.\theta^{-1}}{\phi_{X_1}})\|_{\tilde{\Lambda}(A,\phi_{X_3},\phi_Y)} \leq C\|\frac{g^*.\theta^{-1}}{\phi_{X_1}} . \phi_{X_3}\|_{L_A(d\mu)}$$

$$\leq C\|g\|_{\tilde{\Lambda}(A,\phi_{X_2},\phi_Y)} . \quad +$$

On a similar vein we can prove a general convolution theorem for O.C.L.H.Z. spaces. We shall formulate the result for Young's functions although a suitable generalisation holds for concave generalised Young's functions.

(5.2.16) THEOREM. Let T be a convolution operator, let X_i, Y, i = 1,2,3 be r.i.spaces and let A,B,C be Young's functions. Suppose that the following conditions are satisfied,

(i) $\exists\ \theta > 0$ such that $\phi_{X_3}(t)t \leq \theta\ \phi_{X_1}(t)\phi_{X_2}(t)$, $\forall\ t > 0$.

(ii) $\exists\ M > 0$ such that $A^{-1}(t)B^{-1}(t) \leq M\ C^{-1}(t)$, $\forall\ t > 0$.

(iii) C, ϕ_{X_3} and ϕ_Y satisfy the conditions of (2.3.1).

Then, there exists an absolute constant $r > 0$ such that

$$\|T(f,g)\|_{\Lambda(C,\phi_{X_3},\phi_Y)} \leq r.\ \|f\|_{\Lambda(A,\phi_{X_1},\phi_Y)} \cdot \|g\|_{\Lambda(B,\phi_{X_2},\phi_Y)}\ .+$$

We shall show that conditions given above are best possible.

(5.2.17) THEOREM. Let $X(-\infty,\infty)$, $Y(-\infty,\infty)$, $Z(-\infty,\infty)$ be r.i.spaces, let T be the convolution operator given by

$$T(f,g)(t) = f * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds.$$

Then,

(i) a necessary condition for T to define a bounded bilinear operator, $T: X \times Y \to Z$, is the existence of $\theta > 0$ such that

$$\phi_Z(t).t \leq \theta\ \phi_X(t)\phi_Y(t), \qquad \forall\ t > 0.$$

(ii) Suppose that $t\ \phi_Z(t) \geq \phi_X(t)\phi_Y(t)$, $\forall\ t > 0$. then a necessary condition for T to define a bounded bilinear operator, $T: \Lambda(A_{q_1},\phi_X,1)^{\sim} \times \Lambda(A_{q_2},\phi_Y,1)^{\sim} \to \Lambda(A_{q_3},\phi_Z,1)^{\sim}$ is that $\frac{1}{q_3} \leq \frac{1}{q_1} + \frac{1}{q_2}$, where $A_{q_i}(t) = \frac{t^{q_i}}{q_i}$, $0 < q_i \leq \infty$, $i = 1,2,3$, with the convention that $A_\infty(t) = \chi_{[0,1]}(t) + \infty\chi_{(1,\infty)}(t)$.

Proof. We shall omit the simple proof of (i) (cf. [41]).

Suppose that T defines a bounded bilinear operator,
$T: \Lambda(A_{q_1},\phi_X,1)^\sim \times \Lambda(A_{q_2},\phi_Y,1)^\sim \to \Lambda(A_{q_3},\phi_Z,1)^\sim$ and that
$\phi_Z(t)t \geqslant \phi_X(t)\phi_Y(t)$, $\forall$ $t > 0$, but to the contrary $\frac{1}{q_1}+\frac{1}{q_2}<\frac{1}{q_3}$. Let ε be such that $\frac{1}{q_1}+\frac{1}{q_2}+2\varepsilon = \frac{1}{q_3}$. Then define

$$f(t) = \begin{cases} (\log|2t|)^{-(\frac{1}{q_1}+\varepsilon)}/\phi_X(|2t|) & \text{if } |t| \geqslant e \\ (\log|2e|)^{-(\frac{1}{q_1}+\varepsilon)}/\phi_X(|2e|) & \text{if } |t| < e \end{cases}$$

$$g(t) = \begin{cases} (\log|2t|)^{-(\frac{1}{q_2}+\varepsilon)}/\phi_X(|2t|) & \text{if } |t| \geqslant e \\ (\log|2e|)^{-(\frac{1}{q_2}+\varepsilon)}/\phi_Y(|2e|) & \text{if } |t| < e \end{cases}$$

It follows that f and g are symmetrically decreasing functions and moreover

$$\int_0^\infty [f^*(t)\phi_X(t)]^{q_1}\frac{dt}{t} \leqslant \text{const}\int_0^\infty [f(t/2)\phi_X(t)]^{q_1}\frac{dt}{t}$$
$$\leqslant \text{const}\int_e^\infty (\log t)^{-1-\varepsilon q_1}\frac{dt}{t} < \infty$$

i.e. $f \in \Lambda(A_{q_1},\phi_X,1)^\sim$ and similarly $g \in \Lambda(A_{q_2},\phi_Y,1)^\sim$.

However, for $|t| \geqslant e$, we have

$$(f*g)(t) \geqslant \int_{2t}^\infty f(s)g(t-s)ds = \int_{2t}^\infty f(s)g(s-t)ds$$
$$\geqslant \int_{2t}^\infty f(s)g(s)ds = \int_{2t}^\infty [\phi_X(2s)\phi_Y(2s)]^{-1}(\log 2s)^{-1/q_3}ds$$
$$\geqslant \int_{2t}^\infty [\phi_Z(2s)]^{-1}(\log 2s)^{-1/q_3}ds$$

$$\geq \int_{2t}^{3t} [\phi_Z(2s)]^{-1} (\log 2s)^{-1/q_3} \, ds$$

$$\geq \frac{1}{\phi_Z(6t)} (\log 6t)^{-1/q_3} .$$

Notice that $(f*g)(t) \geq 2.e\, f(e)g(e)$ for $0 < t < e$, therefore

$(f*g)^*(t) \geq \text{const.}\ (\log 6t)^{-1/q_3} / \phi_Z(6t)$, for $t > e$

hence,

$$\int_0^\infty [(f*g)^*(t)\phi_Z(t)]^{q_3} \frac{dt}{t} \geq \text{const} \int_e^\infty (\log 6t)^{-1} \frac{dt}{t}$$

$$\geq \infty .$$

Thus, $f * g \notin \Lambda(A_{q_3}, \phi_Z, 1)^\sim$, a contradiction. +

5.3. APPLICATIONS. In this section we give several examples illustrating the results obtained in this Chapter and outline several generalisations.

(5.3.1) FRACTIONAL INTEGRALS. Let $0 < \alpha < 1$, and define $I_\alpha f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n(1-\alpha)}} dy$, the Riesz fractional integral of f of order α. It is well known and easy to see that $r(x) = 1/|x|^{n(1-\alpha)}$ $\in L(\frac{1}{\alpha}, \infty)(R^n)$, thus since $I_\alpha f = f * r$ we get from (5.2.17)

THEOREM. Let $X(R^n)$, $Y(R^n)$ be r.i. spaces, then a necessary condition for I_α to define a bounded linear operator $I_\alpha : X \to Y$, is the existence of $\theta > 0$ such that $\phi_Y(t)t \leq \theta\, \phi_X(t)t^\alpha$. +

Sufficient conditions for I_α to act continuously on

O.C.L.H.Z. spaces are given in (5.2.14). It should be noticed that our results allow us to handle more general operators as well.

(5.3.2) SOBOLEV SPACES. Let $X(R^n)$ be a r.i. space, we may construct "range spaces" for operators acting on X. For example, let $\alpha > 0$, and consider

$$X_\alpha(R^n) = \{f: \exists\, g \in X \text{ such that } G_\alpha * g = f\}$$

with,

$$\|f\|_{X_\alpha(R^n)} = \|g\|_X$$

where,

$$\hat{G}_\alpha(x) = (1 + |x|^2)^{-\alpha/2}$$

or explicitly

$$G_\alpha(x) = c_{\alpha,n} \int_0^\infty e^{-\pi^2|x|^2}\, t^{-n/2 + \alpha/2 - 1} \,.\, e^{-t}\, dt$$

observe that $G_\alpha \in L^1(R^n)$.

Using our results for product operators we can obtain embedding theorems for the X_α spaces. We shall consider here the case where $X = L(p,q)$.

Let $1 < p < \frac{n}{\alpha}$, then $L(\frac{n}{n-\alpha}, \infty) * L(p,s) \subseteq L(q,s)$, where $\frac{1}{q} = \frac{n-\alpha}{n} + \frac{1}{p} - 1 = \frac{1}{p} - \frac{\alpha}{n}$, $0 < s \leq \infty$, and $L(\frac{n}{\alpha}, r_1) \,.\, L(q,r_2) \subseteq L(p,r_3)$, where $\frac{1}{r_1} \leq \frac{1}{r_2} + \frac{1}{r_3}$, $0 < r_i \leq \infty$, $i = 1,2,3$, in particular $L(\frac{n}{\alpha},\infty).L(q,s) \subseteq L(p,s)$, thus we have the following

THEOREM. Let $1 < p < \frac{n}{\alpha}$, $0 < s \leq \infty$, and suppose that

$f \in L(\frac{n}{\alpha},\infty)$, $g \in L_\alpha(p,s)$, then

$$\|fg\|_{L(p,s)} \leq C \|f\|_{L(\frac{n}{\alpha},\infty)} \|g\|_{L_\alpha(p,s)}$$

where C is an absolute constant.

Proof. Use the previous remarks together with the fact that $G_\alpha \in L(\frac{n}{n-\alpha},\infty)$. +

In the particular case where s = p, the above result is known as the "Strichartz inequality" (cf. [57] and [18])

(5.3.3) Let G be a locally compact abelian group, and let $X(\hat{G})$ be a r.i.space, consider

$$X_\Lambda(G) = \{f \in L^1(G) : \hat{f} \in X(\hat{G})\}$$

$$\|f\|_{X_\Lambda} = \max \{\|f\|_1 , \|\hat{f}\|_{X(\hat{G})}\}$$

where $\hat{f}$ denotes the Fourier transform of f. One could easily formulate theorems concerning the continuity of t.p.o. or p.o. on these spaces using our results. See [11] .

(5.3.4) NON-COMMUTATIVE R.I. SPACES. Let R be a von Neumann algebra on a Hilbert space H, m be a faithful semifinite normal trace on R, and let F denote the ideal of elementary operators, L the algebra of locally measurable operators. (See [60] for notation and background information).

We shall say that $A \in K$ if $A \in L$ and there exists a projection

$E \in F$ such that $A(I - E) \in R$.

We denote by $\|\ \|_\infty$ the usual operator norm on R and for $A \in L$ we denote by $|A|$ the operator $(A^*A)^{1/2}$ with spectral resolution

$$|A| = \int_0^\infty \lambda \, d\, E(\lambda).$$

Let $A \in K$ then we define the rearrangement of A by

$$A_*(s) = \inf\{\lambda \in [0,\infty) : m(I - E(\lambda)) \leqslant s\} \quad , s \in [0, \infty).$$

It can be shown that

$$A_*(s) = \inf\{\|A(I - E)\|_\infty : E \in F, \ m(E) \leqslant s\} .$$

Let $X(0,\infty)$ be a r.i.space, then we define a non-commutative r.i.space of operators as follows: we say that $A \in K$ belongs to $X(R)$ if and only if $A_* \in X(0,\infty)$, and we define

$$\|A\|_{X(R)} = \|A_*\|_X .$$

In this generalised setting it is possible to obtain a parallel theory to the one presented in this thesis for the commutative case.

Consider for example the estimates that hold for a convolution operator, i.e. let T be a bilinear operator such that

$$\|T(A,B)\|_1 \leqslant \|A\|_1 \ \|B\|_1$$

$$\|T(A,B)\|_\infty \leqslant \|A\|_1 \ \|B\|_\infty$$

$$\|T(A,B)\|_\infty \leqslant \|A\|_\infty \ \|B\|_1$$

then, if we denote

$f \in L(\frac{n}{\alpha},\infty)$, $g \in L_\alpha(p,s)$, then

$$\|fg\|_{L(p,s)} \leqslant C \|f\|_{L(\frac{n}{\alpha},\infty)} \|g\|_{L_\alpha(p,s)}$$

where C is an absolute constant.

Proof. Use the previous remarks together with the fact that $G_\alpha \in L(\frac{n}{n-\alpha},\infty)$. +

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$$\|f\|_{X_\Lambda} = \max \{\|f\|_1 , \|\hat{f}\|_{X(\hat{G})}\}$$

where $\hat{f}$ denotes the Fourier transform of f. One could easily formulate theorems concerning the continuity of t.p.o. or p.o. on these spaces using our results. See [11] .

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We shall say that $A \in K$ if $A \in L$ and there exists a projection

$E \varepsilon F$ such that $A(I - E) \varepsilon R$.

We denote by $\| \|_\infty$ the usual operator norm on R and for $A \varepsilon L$ we denote by $|A|$ the operator $(A^*A)^{1/2}$ with spectral resolution

$$|A| = \int_0^\infty \lambda \, d\, E(\lambda).$$

Let $A \varepsilon K$ then we define the rearrangement of A by

$$A_*(s) = \inf \{\lambda \varepsilon [0,\infty): m(I - E(\lambda)) \leqslant s\} \quad , s \varepsilon [0, \infty).$$

It can be shown that

$$A_*(s) = \inf\{\| A(I - E)\|_\infty : E \varepsilon F, \ m(E) \leqslant s\} .$$

Let $X(0,\infty)$ be a r.i.space, then we define a non-commutative r.i.space of operators as follows: we say that $A \varepsilon K$ belongs to $X(R)$ if and only if $A_* \varepsilon X(0,\infty)$, and we define

$$\|A\|_{X(R)} = \|A_*\|_X \ .$$

In this generalised setting it is possible to obtain a parallel theory to the one presented in this thesis for the commutative case.

Consider for example the estimates that hold for a convolution operator, i.e. let T be a bilinear operator such that

$$\|T(A,B)\|_1 \leqslant \|A\|_1 \ \|B\|_1$$
$$\|T(A,B)\|_\infty \leqslant \|A\|_1 \ \|B\|_\infty$$
$$\|T(A,B)\|_\infty \leqslant \|A\|_\infty \ \|B\|_1$$

then, if we denote

$$K(T(A,B),t,\ L^1(R),\ L^\infty(R)) = K(T(A,B),t)$$

$$K(A,t,L^1(R),\ L^\infty(R)) = K(A,t)$$

$$K(B,t,L^1(R),\ L^\infty(R)) = K(B,t)$$

where K is the K-functional of Peetre [47] , we get

$$K(T(A,B),t) \leq \int_t^\infty K(A,u)\ K(B,u)\frac{du}{u^2}$$

and one can show (cf.[47]) that

$$K(A,t) = \int_0^t A_*(s)ds.$$

Therefore,

$$T(A,B)_{**}(t) \leq \int_t^\infty A_{**}(u)\ B_{**}(u)du$$

where $A_{**}(u) = \frac{1}{u} \int_0^u A_*(s)ds$, etc.

From the above inequality we can derive convolution theorems for non-commutative r.i.spaces.

We can treat similary the case of product operators, indeed let Π be a product operator, then we get

$$\Pi(A,B)_{**}(t) \leq \frac{1}{t} \int_0^t A_{**}(u)\ B_{**}(u)du$$

using the K-method of interpolation . Moreover, we can prove the sharper inequalities (5.2.7) and (5.1.2) using suitable modifications of the proofs given in this Chapter. (See [60] where the case of product operators is considered.)

5.4. NOTES TO CHAPTER 5. The central idea of this chapter (i.e. to obtain inequalities for the maximal rearrangement of bilinear operators from their behaviour in the "extreme spaces") seems to have originated in the work of O'Neil [45] by suggestion of E.Stein.

Several results in this chapter can be also obtained using interpolation theory. (See (5.3.4).).

Consider for example the case of convolution operators. Let T be a c.o. and let $X(-\infty,\infty)$ be a r.i.space then by Calderón's interpolation theorem T defines a bounded bilinear operator

$$T: \begin{cases} M(X) \times \Lambda(X') \to L^{\infty} \\ M(X) \times L^{1} \to M(X) \end{cases}$$

Let $f \in M(X)$, and consider the operator

$$T_f(g) = T(f,g)$$

then applying the interpolation theorem (3.2.3) we can get a different proof of (5.2.14).

Similarly we can get a different proof of (5.2.16), if all the Young's functions are powers, using complex interpolation. (The theory of complex interpolation for O.C.L.H.Z. spaces follows from the inequalities proved in Chapter 2 and the general theory derived in Calderón [9].)

It could be of some theoretical interest to find representation theorems for p.o. and t.p.o., (In [4] it is shown that if T is a c.o. and T is translation invariant then T is essentially the usual convolution operator).

CHAPTER 6

APPLICATIONS TO THE THEORY OF INTEGRAL OPERATORS

6.1. BOUNDED OUTPUTS. We consider the following admissibility or stability problems: we are given two B.F.spaces $X(\Omega)$, $Y(\Omega)$ (the input space and the output space) and are asked to characterize

$$A(X,Y) = \{K \in M(\Omega \times \Omega): z_k(f)(y) = \int_\Omega k(x,y)f(x)dx$$

defines a bounded linear operator, $z_k: X \to Y\}$.

In general no simple characterization of the set $A(X,Y)$ is known even when the given spaces are L^P spaces. However, in some special cases which are of interest in applied mathematics, positive results are known. For example $A(L^P,L^\infty)$ has been completely characterized for $1 \leqslant p \leqslant \infty$. (See Corduneanu [13].)

In the applications one usually considers B.F. spaces with weights. Naturally, when we consider B.F.spaces of vector valued functions the problem of considering a suitable type of weighting scheme becomes complicated.

In this section we use ideas of [19] to contruct weighted B.F.spaces, of vector valued functions, which are suitable for the applications.

Let $\bar{X}(0,\infty)$ be a B.F.S. of Lebesgue measurable functions on $(0,\infty)$, from it we can construct a B.F.S. X of Lebesgue measurable functions, defined on $(0,\infty)$ with values in R^n , as follows

$$X((0,\infty), R^n) = \{f: |f| \in \bar{X}\}$$

$$\|f\|_X = \| |f| \|_{\bar{X}}$$

where $|f(t)|$ is the norm of $f(t)$ in R^n. In what follows we shall only consider B.F.spaces of vector valued measurable function constructed in this fashion.

We shall now explain the method we use to construct weighted B.F.spaces.

Let $G: (0,\infty) \to \text{Hom}(R^n,R^n)$ and assume that $t \to \|G(t)\|$ is uniformly bounded on compact subsets of $(0,\infty)$, where $\|G(t)\|$ denotes the norm of $G(t)$ in $\text{Hom}(R^n,R^n)$. Let $R(t)$ and $N(t)$ denote, respectively, the range and the null space of $G(t)$ and let $P_G(t): R^n \to R(t)$ be the orthogonal projection onto $R(t)$ and define

$$G_{-1}(t): R^n \to N(t)^{\perp}, \text{ by}$$

$$G_{-1}(t)x = \begin{cases} G^{-1}(t)x & \text{if} \quad x \in R(t) \\ 0 & \text{if} \quad x \in R(t)^{\perp} \end{cases}$$

The operator $G_{-1}(t)$ will play in our theory the rôle of a weight.

Observe that if G is measurable, an assumption that we shall make in what follows, then G_{-1} is also measurable (cf.[19]).

(6.1.1) **EXAMPLE.** Let $g_i: (0,\infty) \to (0,\infty)$ be continuous functions, $1 \leq i \leq n$, let $G(t) = \text{diag}(g_1(t),\dots,g_n(t))$, then $G_{-1}(t) = \text{diag}\left(\frac{1}{g_1(t)},\dots,\frac{1}{g_n(t)}\right)$.

(E.1.2) DEFINITION. Let $X((0,\infty),R^n)$ be a B.F.S. and let $G: (0,\infty) \to \text{Hom}\,(R^n,R^n)$ be a mapping verifying the above conditions. A measurable function $f: (0,\infty) \to R^n$ is said to be in X_G if and only if the following conditions are satisfied

(i) $P_G(t)f(t) = f(t)$ a.e. $t \in (0,\infty)$

(ii) $\|G_{-1}(t)f(t)\|_X < \infty$.

It is not difficult to prove the following

(E.1.3) LEMMA. X_G is a Banach space.

Proof. It is clear that X_G is a linear space and $\|f\|_{X_G} = \|G_{-1}(t)f(t)\|_X$ is a seminorm. Suppose that $\|f\|_{X_G} = 0$, then $G_{-1}(t)f(t) = 0$ a.e., and since $f(t) = G(t)G_{-1}(t)f(t)$ a.e. we obtain $f(t) = 0$ a.e. . Therefore $\| \; \|_{X_G}$ is a norm.

Let $\{f_n\}$ be a Cauchy sequence in X_G, then $h_n(t) = G_{-1}(t)f_n(t)$ is a Cauchy sequence in X , whence there exists $h \in X$ such that $\lim\limits_{n\to\infty} \|h_n - h\|_X = 0$. Define f by $f(t) = G(t)h(t)$. Observe that $h_n(t) \in N(t)^{\perp}$ a.e. $\forall\, n \in N$, therefore a simple argument shows that $h(t) \in N(t)^{\perp}$ a.e., thus $P_G(t)f(t) = f(t)$ a.e.. Finally we observe that $f_n \to f$ in X_G is a simple consequence of the fact that $h_n \to h$ in X. +

We shall need some definitions which are standart in the theory of stability of feedback systems.

(6.1.4) DEFINITION. Let $X((0,\infty),R^n)$ be a B.F.S. and $Y((0,\infty),R^n)$ be a Banach space of continuous functions, moreover let T be a linear mapping, $T: X \to Y$. We say that (X,Y,T) is a "linear system"; if T is bounded we shall say that the linear system (X,Y,T) is "stable", and if in addition for every bounded set B in X and every $t_0 \varepsilon (0,\infty)$ we have that TB is equicontinuous at t_0 we say that (X,Y,T) is "strongly stable".

(6.1.5) DEFINITION. Let G be a mapping $G: (0,\infty) \to \text{Hom}(R^n,R^n)$, verifying the conditions set out at the beginning of this section. We shall say that a continuous function $x: (0,\infty) \to R^n$ belongs to $C_G((0,\infty),R^n)$ if and only if

(i) $P_G(t)x(t) = x(t) \quad \forall\, t \;\varepsilon\; (0,\infty).$

(ii) $\|x\|_G = \sup_{t \;\varepsilon\; (0,\infty)} |G_{-1}(t)x(t)| < \infty$

where $|\ |$ denotes the euclidean norm of R^n .

Let $k: (0,\infty)x(0,\infty) \to R^{nxn}$ be measurable and consider the integral operator $K(x)(t) = \int_0^\infty k(t,s)x(s)ds$.

(6.1.6) THEOREM. Let $X((0,\infty),R^n)$ be a B.F.S. such that $\bar{X}$ has a.c.n.. Then a necessary and sufficient condition for (X,C_G,K) to be a strongly stable linear system is that the following conditions hold:

(i) $P_G(t)\ \{\int_0^\infty k(t,s)f(s)ds\} = \int_t^\infty k(t,s)f(s)ds, \quad \forall\, f \;\varepsilon\; X, \quad t \;\varepsilon\; (0,\infty).$

(ii) $\|G_{-1}(t)k(t,\cdot)\|_{X'} = b(t) \;\varepsilon\; L^\infty$

(iii) $\lim_{t \to t_0} \|k(t,\cdot) - k(t_0,\cdot)\|_{X'} = 0 \quad \forall t_0 \in (0,\infty)$.

Proof. Let $f \in X$, and assume that conditions (i), (ii), (iii) are verified. Then

$$|G_{-1}(t)K(f)(t)| \leq \int_0^\infty |G_{-1}(t)k(t,s)|\ |f(s)|ds$$

$$\leq \|b\|_\infty \|f\|_X. \text{ (by Hölder's inequality)}$$

Moreover, $\forall t_0 \in (0,\infty)$ we have

$$|K(f)(t) - K(f)(t_0)| \leq \|k(t,\cdot) - k(t_0,\cdot)\|_{X'} \|f\|_X$$

$$\to 0 \text{ as } t \to t_0 .$$

Thus the system is strongly stable.

We shall now prove that conditions (i), (ii), (iii) are necessary for (X,C_G,K) to be strongly stable.

Suppose that (X,C_G,K) is strongly stable, let $S(f)(t) = \int_0^\infty G_{-1}(t)k(t,s)f(s)ds = G_{-1}(t)K(f)(t)$, then by our assumption there exists $M > 0$ such that

$$\|S(f)\|_\infty = \|K(f)\|_G \leq M\|f\|_X .$$

Let $\{e_1,......,e_n\}$ be the standart basis of R^n, $<, >$ the standart inner product in R^n, and define

$$S_{ij}(f)(t) = \int_0^\infty \langle G_{-1}(t)k(t,s)e_j, e_i\rangle \ f(s)ds$$

where $f \in \bar{X}$, $1 \leq i, j \leq n$. Then,

$$S_{ij}(f)(t) = \langle S(f.\bar{e}_j)(t), e_i \rangle$$

where $\bar{e}_j(t) = e_j \ \forall t \in (0,\infty)$. Therefore by the Cauchy-Schwartz inequality

$$|S_{ij}(f)(t)| \leq |S(f.\bar{e}_j)(t)|$$

$$\|S_{ij}(f)\|_\infty \leq \sup_{t \in (0,\infty)} |S(f.e_j)(t)| \leq M \|f\|_{\bar{X}} \ .$$

It follows that each S_{ij} defines a bounded linear operator from $\bar{X}$ into L^∞.

Let $t \in (0,\infty)$ be fixed, then $S_{ij}(\cdot)(t)$ defines a linear operator $S_{ij}(\cdot)(t):\bar{X} \to R$ for each $(i,j) \in \{1,......n\} \times \{1,.......n\}$, and,

$$\sup_{t \in (0,\infty)} |S_{ij}(f)(t)| = \|S_{ij}(f)\|_\infty \leq M \|f\|_{\bar{X}} \leq \infty , \ \forall f \in \bar{X}$$

By the Banach-Steinhauss theorem we obtain constants M_{ij} such that

$$\sup_{t \in (0,\infty)} |||S_{ij}(\cdot)(t)||| \leq M_{ij}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n, \text{ where}$$

$|||S_{ij}(\cdot)(t)|||$ denotes the norm of the operator $S_{ij}(\cdot)(t)$.

Since $\bar{X}$ has a.c.n., it follows that

$a^t_{ij}(s) = \langle G_{-1}(t)k(t,s)e_j, e_i \rangle \in \bar{X}'$ for each $t \in (0,\infty)$, $1 \leq i \leq n$, $1 \leq j \leq n$, and moreover

$$|||S_{ij}(\cdot)(t)||| = \|a^t_{ij}\|_{\bar{X}'} \ .$$

Now we estimate $\|G_{-1}(t)k(t,\cdot)\|_{X'}$:

$$|G_{-1}(t)k(t,s)| \leq C \sum_{i,j} | \langle G_{-1}(t)k(t,s)e_j \ , e_i \rangle|$$

$$\|G_{-1}(t)k(t,\cdot)\|_{X'} \leq C \sum_{i,j} \|a^t_{ij}\|_{X'} \leq C \sum_{i,j} M_{ij} < \infty$$

where C is an absolute constant.

Therefore, $\|b\|_\infty < \infty$, and thus (ii) holds.

It remains to prove that (iii) holds. Let $\varepsilon > 0$, and $t_0 \in (0,\infty)$, then there exists $\delta > 0$ such that

$$|K(f)(t) - K(f)(t_0)| < \varepsilon \quad \forall f \in \Sigma(X), \ |t - t_0| < \delta .$$

Let t be fixed such that $|t - t_0| < \delta$, and define $\phi_i: X \to R$ by $\phi_i(f) = < K(f)(t) - K(f)(t_0), e_i >$, $1 \leq i \leq n$, then $\phi_i \in X^*$, $\|\phi_i\| \leq \varepsilon$ and using the a.c.n. property of $\bar{X}$ as above we get the necessity of (iii). +

Similarly we obtain the following

(6.1.7) THEOREM. Let G and H be mappings, $G,H: (0,\infty) \to Hom(R^n , R^n)$ verifying the usual conditions, and let $X((0,\infty),R^n)$ be a B.F.S. such that $\bar{X}$ has a.c.n., then (X_H,C_G,K) is strongly stable if and only if the following conditions hold:

(i) $P_G(t)\{\int_0^\infty k(t,s)f(s)ds\} = \int_0^\infty k(t,s)f(s)ds, \quad f \in X_H, \ \forall t \in (0,\infty).$

(ii) $\|G_{-1}(t)k(t,\cdot)H(\cdot)\|_{X'} \in L^\infty$

(iii) $\lim_{t \to t_0} \|H(\cdot)\{K(t,\cdot) - K(t_0,\cdot)\}\|_{X'} = 0.$ +

(6.1.8) REMARK. The condition (i) in (6.1.7) is easily seen to be equivalent to $P_G(t)k(t,s)H(s) = k(t,s)H(s)$.

The above results show that stability problems involving B.F. spaces of vector valued functions can be reduced to stability problems of B.F.spaces of scalar valued functions.

The problem of characterizing $A(X,L^\infty)$, where X is a B.F.S., is simple because $X \hat{\otimes}_\pi (L^\infty)' = L^1(X)$, therefore (see §2.5 below) $A(X,L^\infty) = L^\infty(X')$, whenever $\bar{X}$ has a.c.n. (Compare with Corduneanu [13].)

For positive kernels we have the following dual result,

(6.1.9) THEOREM. Suppose that $k:(0,\infty)x(0,\infty) \to R$, and sign $k(t,s)$ is a functions of s only, then $(L^\infty(0,\infty),X(0,\infty),K)$ is stable if and only if $k \in X(L^1)$. +

6.2. STABILITY RESULTS AND TENSOR PRODUCTS OF FUNCTION SPACES.

In view of the results obtained in the previous section we shall restrict ourselves to consider B.F.spaces $X(0,\infty)$, $Y(0,\infty)$, $Z((0,\infty)x(0,\infty))$ of real valued functions.

(6.2.1) DEFINITION. We shall say that (X,Y,Z) is stable if and only if $\forall k \in Z$ the integral operator $z_k(f)(x) = \int_0^\infty k(x,y)f(y)dy$ defines a bounded linear operator, $z_k: X \to Y$.

Similarly we shall say that (X,Y,Z) is "strongly stable" if

$z_k: X \to Y$ is compact, $\forall k \in Z$.

(6.2.2) LEMMA. (X,Y,Z) is stable if and only if $\exists C > 0$ such that

(6.2.3) $\|z_k\| \leqslant C \|k\|_Z$, $\forall k \in Z$,

where $\|z_k\| = \sup \{\|z_k(f)\|_Y : f \in \Sigma(X)\}$.

Proof. Suppose that (6.2.3) is not verified, then there exists a sequence $\{k_n\}$, $k_n \in Z$, $n = 1\ldots\ldots$, $k_n \geqslant 0$, $n = 1 \ldots\ldots\ldots\ldots$ and $\|z_{k_n}\| \geqslant n^3 \|k_n\|_Z$. Let $k = \sum_{n=1}^{\infty} \frac{k_n}{n^2 \|k_n\|_Z}$, then $k \in Z$ but $\|z_k\| = \infty$, thus (X,Y,Z) is not stable. +

Combining (6.2.2) with the Arzelà - Ascoli theorem we obtain the following

(6.2.4) THEOREM. Suppose that Y and Z are separable B.F. spaces, then (X,Y,Z) is stable if and only if (X,Y,Z) is strongly stable. +

The admissibility results can be now derived from the following consequence of Holder's inequality ("direct" and "inverse")

(6.2.5) THEOREM. (X,Y,Z) is stable if and only if $X \otimes_\pi Y'$ is continuously embedded in Z' . +

Combining (6.2.4) and (6.2.5) we obtain the following generalisation of classical results concerning Hilbert-Schmidt operators acting on L^p spaces (cf. [13]).

(6.2.6) **THEOREM.** Suppose that X' and Y have a.c.norms, then (X,Y, Y(X')) is strongly stable. +

We look now at the case where the spaces involved are rearrangement invariant (of course these results can be then applied to obtain admissibility theorems for r.i.spaces with weights). From (6.2.5) and (4.1.2) we get

(6.2.7) **LEMMA.** A necessary condition for (X,Y,Z) to be stable is the existence of $\theta > 0$ such that

(6.2.8) $s\phi_Y(t) \leqslant \theta\,\phi_Z(t.s)\phi_X(s), \quad \forall\, t,s > 0$. +

Positive results can be now derived using the results of chapters 4 and 2. For example,

(6.2.9) **THEOREM.** $(\Lambda(X), M(Y), M(Z))$ is stable if and only if (6.2.8) holds.

Proof. Follows from (6.2.5) and (4.2.1). +

The reader will have no problems to state and prove the admissibility results for O.C.L.H.Z.spaces that follow from our previous work.

6.3 NOTES TO CHAPTER 6. The reader will find a complete set of references for the theory of admissibility of integral operators in [13]. In [13] it is shown how these results can be applied to obtain existence and uniqueness of solutions of integral equations, and moreover to study the properties of the solutions of non linear integral equations.

The results concerning the stability of triples of function spaces generalise the important work of O'Neil [43].

It is interesting to point out that for bilinear operators which behave on the "extreme" spaces in a similar way as ordinary integral operators we can obtain a parallel theory. Let T be a bilinear operator such that

$$\|T(k,f)\|_{\infty} \leq \|k\|_{\infty} \|f\|_{1}$$

$$\|T(k,f)\|_{1} \leq \|k\|_{1} \|f\|_{\infty}$$

then, (cf. [43])

$$T(k,f)^{**}(t) \leq \int_0^{\infty} k^*(t.s)f^*(s)ds.$$

Using the above inequality one can obtain continuity results.

The general setting of O.C.L.H.Z. spaces is of particular interest since we can prove results that hold simultaneously for L(p,q) spaces and Orlicz spaces with weights.

LIST OF NOTATION

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