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POR

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ABSTRACT

In the first section of this paper, the ordinal space $[1,\Omega]$ is characterised. Then its quotients are completely described in easy terms. In the second section, the ideas of a well ordered sum of an arbirary collection of T_2 spaces and a Chandelier space are introduced. The Chandelier spaces are obtained by taking compact ordinals and forming their interweaves and well ordered sums and one-point compactification of such spaces. Finally it is proved that the Hausdorff quotients of compact ordinals are precisely the Chandelier spaces.

INTRODUCTION

The study of scattered spaces is getting more attention recently. Mrowka, Rajagopalan, and T. Soundararajan showed recently that a compact Hausdorff space X in which every ordered net has a convergent cofinal subnet is scattered and conversely. (TOPO 72 Lecture Notes in Math. #378, Springer-Verlag (1974) - 288-298). Thus chain compact spaces or scattered spaces as they are more commonly known form a natural generalisation of the categroies of sequential spaces and chain net spaces among compact spaces.

The easiest class of chain compact Hausdorff spaces are the spaces of Ordinals. It was natural to conjecture that the class of ordinals form a class of generators for the category of scattered compact Hausdorff spaces by using closed subspaces, - finite products, and Hausdorff quotients. However, Mrowka, - Rajagopalan, and T. Soundararajan showed in their paper that this is not the case. They rasied a conjecture in that paper that the category of strongly scattered spaces is generated by the above process by Compact Ordinals. This conjecture is still open. The first difficulty in solving this problem is to give a characterisation of the quotients of compact ordinals. The present paper deals with exactly that problem.

Notations 1.1. All spaces in this paper are Hausdorff. Let α be any ordinal number. $[1,\alpha)$ or S_α denotes the ordered space of all ordinals $\beta < \alpha$ with the usual order topology. If α is an ordinal, then $cf(\alpha)$ is the least cardinal of a subset which is cofinal with $S(\alpha)$. Ω denotes the first uncountable ordinal. We use $[1,\Omega]$ to mean $S(\omega_1+1)$ sometimes.

The following theorem 1.2 is easy to prove and its proof - is omitted.

THEOREM 1.2. If α is an ordinal and there exists a countable cofinal subset of $S(\alpha)$ then $cf(\alpha)$ is countable and conversely. If α is an ordinal, then $S(\alpha+1) = [1,\alpha]$ and $S(\alpha)$ is compact if and only if α is a successor. The free union countably many ordinals $S(\alpha)$ $(\alpha \epsilon J)$ is again of the form $S(\beta)$ for some ordinal β, provided that $cf(\alpha)$ is countable for every α in J.

Definition 1.3. If X, Y are topological spaces and $\phi : X \to Y$ a map from X onto Y then ϕ is called perfect if it is continuous and closed and $\phi^{-1}(Y)$ is compact for all $y \epsilon Y$.

Definition 1.4. Let X_α be a countable compact ordinal space for each $\alpha \epsilon [1,\Omega)$. Let x_α be the last element of X_α for each $\alpha \epsilon [1,\Omega)$. Let Y be the free union $\oplus X_\alpha$ of the sets X_α where $\alpha \epsilon [1,\Omega)$. We define a topology on Y as follows: If $x \epsilon Y$ then a neighborhood base of x in the space Y is the same as a neighborhood base of x in X_α in the following two cases:

(i) $x \epsilon X_\alpha$ for some non-limit ordinal α.

(ii) $x \epsilon X_\alpha$ for some limit ordinal α and $x \neq x_\alpha$. If $x = x_\alpha$ and α is a limit ordinal, then a neighborhood base at x_α is the collection of all sets of the form $V \cup \bigcup_{\beta \epsilon [\gamma,\alpha)} X_\beta$

for $\alpha \in [1,\Omega)$. Clearly we have that $\bigcup_{\beta<\alpha} X_\alpha = \bigcup_{\beta<\alpha} O_\beta$ for all $\alpha \in [1,\Omega)$. Now, let α be a limit ordinal. Then $\overline{\bigcup_{\beta<\alpha} O_\beta} = \{x_\alpha\} \cup \bigcup_{\beta<\alpha} O_\beta$ is compact. Then an open set W containing x_α in X contains a set of the form $\bigcup_{\delta<\beta<\alpha} O_\beta \cup V$ where δ is an ordinal less than α and V is a compact open set in X_α which contains x_α.

Conversely, each such set $V \cup (\bigcup_{\delta<\beta<\alpha} O_\beta) = M$ is open in X. Indeed, O_δ and $X_\alpha - V$ are compact while O_α is open in X. Also, O_α is the disjoint union of M, and $X_\alpha - V$. Hence, M is open in X. So, it follows that Y is homeomorphic to the well ordered sum of the spaces X_α indexed by $[1,\Omega)$.

<u>Remark 1.6.</u> One nice feature about Theorem 1.5 is that one more additional condition ((iv) below) is enough to guarantee that X is homeomorphic to $[1,\Omega)$:

(iv) $\overline{\bigcup_{\alpha<\gamma} O_\alpha}$ is open in X for each limit ordinal.

This follows from Theorem 1.7 below. This characterisation of $[1,\Omega)$ has also been obtained by J. W. Baker [1] independently.

THEOREM 1.7. <u>Let X be a non-compact T_2 space which is locally compact. Then X is homeomorphic to $[1,\Omega)$ if and only if X satisfies the conditions (i), (ii) and (iii) of Theorem 1.5 and also the condition (iv) of Remark 1.6.</u>

<u>Proof</u>: Let X_α be as in Theorem 1.5 $\forall\ \alpha \in [1,\Omega)$. Then X_α is homeomorphic to a compact countable ordinal space $\alpha \in [1,\Omega)$ by a theorem of S. Mazurkiewicz and W. Sierpinski [7]. Since $\{x_\alpha\}$ is open in X_α when α is a limit ordinal, the order in X_α as given by the theorem of S. Mazurkiewicz and W. Sierpinski [7] can be chosen so that x_α is the least element in X_α when α is a limit ordinal. We assume that it is done. Now, define an order $\leq$ in X as follows: If $x, y \in X$, then $x \leq y$ if and only if $x \in X_\alpha$ and $y \in X_\beta$ and $\alpha < \beta$ or $x, y \in X_\alpha$ and $x \leq y$ in X_α. This order $\leq$ in X is easily seen to be a well order in X and also X is order isomorphic to $[1,\Omega)$. An application of the proof of Theorem 1.5 shows that this order isomorphism is also a homeomorphism. This completes the proof of Theorem 1.7.

THEOREM 1.8. <u>Let</u> Y <u>be a well ordered sum of countable ordinals. Let</u> X_α <u>be the countable ordinal space associated with</u> $\alpha \in [1,\Omega)$ <u>in the construction of</u> Y. <u>Let</u> $J = \{\alpha \mid \alpha \in [1,\Omega),\ \{x_\alpha\}$ <u>is open in</u> $X_\alpha\}$. <u>Then</u> Y <u>is homeomorphic to</u> $[1,\Omega)$ <u>if and only if</u> J <u>contains a closed set</u> J_1 <u>cofinal with</u> Ω.

<u>Proof</u>: If J contains a closed set J_1 cofinal with Ω, then it follows that Y is homeomorphic to $[1,\Omega)$ from Theorem 1.5 and Theorem 1.7. Conversely, let Y be homeomorphic to $[1,\Omega)$. Then from Theorem 1.5 and Theorem 1.7 Y can be expressed as a union $\bigcup_{\alpha \in [1,\Omega)} 0_\alpha$ of subsets $0_\alpha \subset Y$ with the following properties:

(i) $0_\alpha \neq \phi$ for each $\alpha \in [1,\Omega)$ and $\bigcup_{\alpha \in \Lambda} 0_\alpha = Y$.

(ii) 0_α is compact, open in Y for each $\alpha \in [1,\Omega)$.

(iii) $0_\alpha \subset 0_\beta$ if $\alpha \leq \beta$ and $\alpha, \beta \in [1,\Omega)$.

(iv) If α is a limit ordinal in $[1,\Omega)$, then

$$\overline{\bigcup_{\beta<\alpha} 0_\beta} - \bigcup_{\beta<\alpha} 0_\beta \text{ is a singleton.}$$

(v) If α is a limit ordinal in $[1,\Omega)$, then $\overline{\bigcup_{\beta<\alpha} 0_\beta}$ is open in Y.

Now, put $A_\alpha = \bigcup_{\beta\leq\alpha} X_\beta$ $\forall\, \alpha \in [1,\Omega)$. Then $\bigcup_{\alpha\in\Omega} A_\alpha = Y$ and $\alpha, \beta \in [1,\Omega)$. Also, each A_α is compact, open in Y. Moreover, if $\alpha < \beta$ and $\alpha, \beta \in [1,\Omega)$, then $X_\beta \cap A_\alpha = \phi$. Now, there exists $\alpha_1 \in [1,\Omega)$ such that $A_1 \subset 0_{\alpha_1}$. Since 0_{α_1} is also compact, there is a $\beta_1 > \alpha_1$ in $[1,\Omega)$ so that $0_{\alpha_1} \subset A_{\beta_1}$. Choosing $\alpha_2 \in [1,\Omega)$ so that $\alpha_2 > \beta_1$ and so that $0_{\alpha_2} \supset A_{\beta_1}$ we get $0_{\alpha_1} \subset 0_{\alpha_2}$. Also $X_\beta \cap 0_{\alpha_1} = \phi$ for $\beta \in [1,\Omega)$ and $\beta \geq \alpha_2$. It is obvious that we can proceed by transfinite induction and get a strictly increasing transfinite sequence of ordinals α_i where $i \in [1,\Omega)$ so that the following hold:

(a) $\bigcup_{i\in\Omega} 0_{\alpha_i} = Y$.

(b) $0_{\alpha_i} \subset 0_{\alpha_j}$ if $i, j \in [1,\Omega)$ and $i \leq j$.

(c) If $i, j \in [1,\Omega)$ and $i < j$ then $0_{\alpha_i} \cap X_{\alpha_j} = \phi$

(d) $\bigcup_{\alpha<\gamma} 0_\alpha = \bigcup_{\alpha<\gamma} A_\alpha$ for all γ which is a limite of a distinct sequence α_i.

Now put $J_1 = \{\alpha_i \mid i \in [1,\Omega)\}$. Let J_1' be the set of limit points of J_1 in $[1,\Omega)$. Then, clearly J_1' is cofinal with Ω, since J_1 is cofinal with Ω. We claim that J_1' is a closed - subset of Ω such that $\{x_\gamma\}$ is open in X_γ for each $\gamma \in J_1'$. For, let $\gamma \in J_1'$. Then, we can choose a strictly increasing - sequence $\alpha_1 < \alpha_2 < \ldots < \alpha_n \ldots$ of element in J_2 sucha that $\lim_{n\to\infty} \alpha_n = \gamma$ in $[1,\Omega)$. Then $x_\gamma \in \overline{\bigcup_{i=1}^{\infty} 0_{\alpha_i}}$. So, $x_\gamma \in \overline{\bigcup_{\alpha<\gamma} 0_\alpha}$. From the definition of topology of Y it follows that if $x \in X_\gamma$ and $x \neq x_\gamma$ then $x \in \overline{\bigcup_{\alpha<\gamma} 0_\alpha}$. Thus $\overline{\bigcup_{\alpha<\gamma} 0_\alpha} = \{x_\gamma\} \cup (\bigcup_{\alpha<\gamma} 0_\alpha) = \{x_\gamma\} \cup (\bigcup_{\alpha<\gamma} A_\alpha)$.

From (v) and the definition of topology of Y it follows that $\{x_\gamma\}$ is open in X_γ. Clearly J_1' is also closed in $[1,\Omega)$ and this completes the proof of the theorem.

Remark 1.9. The above theorem 1.8 also give us a characterisation of $[1,\Omega)$. It is easy to switch back and forth between the characterisations of $[1,\Omega)$ and $[1,\Omega]$. The latter can be characterised as the one-point compactification of the former and the former can be characterised as the space got by removing the unique point of $[1,\Omega]$ where the first axiom of countability - fails.

Definition 1.10. Let π be a partition of $[1,\Omega]$ and $X = \frac{[1,\Omega]}{\pi}$ be the quotient of $[1,\Omega]$ under π. We say that a subset $A \subset [1,\Omega]$ is saturated under π if A is a union of members of π.

THEOREM 1.11. Let π be a partition of $[1,\Omega]$ and $X = \frac{[1,\Omega]}{\pi}$ be the corresponding quotient of $[1,\Omega]$. Let $\phi : [1,\Omega] \to X$ be the quotient map. Let $\{\Omega\} \varepsilon \pi$ and let X be Hausdorff in the quotient topology of $[1,\Omega]$ by ϕ. Then X is homeomorphic to the one point compactification of a well ordered sum of countable ordinals.

We prove this by first giving a definition.

Definition 1.12. Let π be a partition of $[1,\Omega]$ so that $\frac{[1,\Omega]}{\pi}$ is T_2 in the quotient topology. Let $\{\Omega\} \varepsilon \pi$. Let $\phi : [1,\Omega] \to \frac{[1,\Omega]}{\pi}$ be the canonical map. For each $\alpha \,\varepsilon\, [1,\Omega)$, put $\phi_\alpha = \sup(\phi^{-1}(\phi([1,\alpha])))$. For each $\alpha \,\varepsilon [1,\Omega)$, put $\lambda^1 \alpha = \phi_\gamma$ where $\gamma = \phi_\alpha$. If $n > 1$ is an integer and $\gamma^{n-1}\alpha$ has been defined for some $\alpha \,\varepsilon\, [1,\Omega)$, put $\lambda^n \alpha = \lambda \ (\lambda^{n-1}\alpha)$. Finally put $\ell_\alpha = \text{lub}\,\{\lambda^n \alpha \mid n \,\varepsilon\, N\}$ for all countable ordinals α. Note that $\ell_\alpha \neq \Omega$ for all $\alpha \,\varepsilon\, [1,\Omega)$.

Proof of Theorem 1.11: We observe that $\phi([1,\ell_\alpha))$ is open in X for all $\alpha \,\varepsilon\, [1,\Omega)$. If we put $B_\alpha = \phi([1,\ell_\alpha))$ for each $\alpha \,\varepsilon\, [1,\Omega)$, then $\bigcup_{\alpha \varepsilon [1,\Omega)} B_\alpha = X - \{\phi(\Omega)\}$. Since $\phi([1,\ell_\alpha])$ is compact and hence closed in X it follows that $\overline{B}_\alpha$ is metrisable for each $\alpha \in [1,\Omega)$. Thus $\overline{B}_\alpha$ is homeomorphic to some countable ordinal space (see [7]) and so the first axiom of countability is satisfied at each point of $X - \{\phi(\Omega)\}$. Now, choose an increasing cofinal set of ordinals (α_i) $i \in [1,\Omega)$ in $[1,\Omega)$ so that $B_{\alpha_i} \neq B_{\alpha_j}$ if $\alpha_i < \alpha_j$ and $\alpha_i, \alpha_j \in [1,\Omega)$. Then $\bigcup_{i\in[1,\Omega)} B_{\alpha_i} = X - \{\phi(\Omega)\}$. For each $\alpha_i, \alpha_j \in [1,\Omega)$ we have that $\phi(\ell_{\alpha_i}) \in B_{\alpha_j}$ if $\alpha_j > \alpha_i$. Put $J = \{\alpha_i \mid i \in [1,\Omega)\}$. Given $i \in [1,\Omega)$, put K_{α_i} to be a compact open set containing ℓ_{α_i} but $K_{\alpha_i} \subset B_{\alpha_{i+1}}$ if $\phi(\ell_{\alpha_i}) \notin B_{\alpha_i}$, and to be ϕ if $\phi(\ell_{\alpha_i}) \in B_{\alpha_i}$. Finally put $X_{\alpha_i} = B_{\alpha_i} \cup K_{\alpha_i}$ for all $\alpha_i \in J$. Then the collection $\{X_{\alpha_i} \mid \alpha_i \in J\}$ satisfies the conditions of Theorem 1.5. Thus we get the Theorem 1.11.

Remark 1.13. It was wrongly announced in [6] that the quotient space X of Theorem 1.11 is again homeomorphic to $[1,\Omega]$.

THEOREM 1.14. Let π be a partition of $[1,\Omega]$ and X the quotient set $\frac{[1,\Omega]}{\pi}$ and $\phi : [1,\Omega] \to X$ the canonical map. Let X be Hausdorff in the quotient topology and let there be an $\alpha_0 \in [1,\Omega)$ so that if $\alpha > \alpha_0$ and $\alpha \in [1,\Omega)$ then α and Ω do not lie in the same member of π. Then X is homeomorphic to the one point compactification of a free union $A \oplus B$ of a countable ordinal space A and a space B homeomorphic to a well ordered sum of countable ordinals.

Proof. Let π_1 be the restriction of π to $[1,\Omega)$. Then the partition class of an element $x \in [1,\Omega)$ under π_1 is countable. Thus imitating the construction of ℓ_α for each ordinal $\alpha \in [1,\Omega)$ as in definition 1.9 we get an ordinal β in $[1,\Omega)$ so that $[1,\beta)$ is saturated under π_1 and $[\beta,\Omega)$ is saturated under both π and π_1. Then we can find a compact open set W in $\phi([1,\Omega))$ so that $\phi(\beta) \in W$. Then taking $M = [1,\beta) \bigcup \phi^{-1}(W)$ and $N = [1,\Omega) - M$ we get M to be a compact, open set which is saturated under π_1 in $[1,\Omega)$ and we get N to be an open set in $[1,\Omega)$ which is closed under π as well as π_1. Then we get that $X = \phi(M) \bigcup \phi(N)$ and $\phi(M) \bigcap \phi(N) = \phi$. Put $A = \phi(M) - \{\phi(\Omega)\}$ and $B = \phi(N)$, Then $\phi(M)$ and $\phi(N)$ are disjoint open sets of X and X is the one point compactification of the free union $A \oplus B$. Now the restriction of ϕ to N also gives the quotient topology on $\phi(N)$ So by an easy application of theorem 1.8 we get that $\phi(N)$ is homeomorphic to a well ordered sum of countable ordinals. Now the map $\phi|M:M \to \phi(M)$ also gives the quotient topology on $\phi(M)$. So $\phi(M)$ in homeomorphic to a compact, countable, ordinal space. So $\phi(M) - \{\phi(\Omega)\} = A$ is homeomorphic to a countable ordinal space. Thus we have the theorem.

THEOREM 1.15. <u>Let π be a partition of $[1,\Omega)$ and let X be the quotient set $\frac{[1,\Omega]}{\pi}$ and ϕ: $[1,\Omega] \to X$ the quotient map. Let X be given the quotient topology and let X be Hausdorff in this topology. Let the partition class of Ω under π be uncountable. Suppose further that there is no member in π which contains a closed ray of the form $[\alpha,\Omega]$ for some $\alpha \in [1,\Omega)$. Then X is</u>

homeomorphic to the one point compactification of a free union of uncountably many countable ordinal spaces.

Proof: Let D be the partition class of Ω under π. For each $\alpha \in [1,\Omega)$, put $X_\alpha = ([1,\Omega)-D) \cap [1,\alpha]$. Now, it is clear that if $\alpha \in [1,\Omega)-D$, then its partition class under π is contained in $[1,\Omega)$ and is countable. Given $\alpha \in [1,\Omega)$, put $d_\alpha = \text{lub } \phi^{-1}(\phi(X_\alpha))$, and let Y_α be the least ordinal in D so that $Y_\alpha \geq d_\alpha$ and $\mu^1\alpha = Y_\alpha$. Given $\alpha \in [1,\Omega)$ and an integer $n \geq 1$ for which $\mu^n\alpha$ is defined put $\mu^{n+1}\alpha = \mu^1(\mu^n\alpha)$ and let $\delta_\alpha = \text{lub } \{\mu^n\alpha \mid n \in N\}$. Then $[1,\delta_\alpha) - D$ is saturated under π, and $\delta_\alpha \in D$ and $\alpha \leq \delta_\alpha$ for all $\alpha \in [1,\Omega)$.

Let $S = \overline{\{\delta_\alpha \mid \alpha \in [1,\Omega) - D\}}$. Then it is clear that if $x \in S$ and x^+ is the successor of x in S then $[x,x^+) - D$ is saturated under π. Let x_o be the least element in S. Then $[1,\Omega) = [1,x_o) \bigcup \{\bigcup_{\substack{x\in S\\ x\geq x_o}} [x,x^+)\}$. Moreover, it is clear that

$$\phi([1,x_o)) \cap \phi([x,x^+)) = \phi([x,x^+)) \cap \phi([Y,Y^+]) = \{\phi(\Omega)\}$$

for all $x, y \in S$ and $x \neq Y$ and $x > x_o$. Now it is clear that $\phi([1,x_o))$ is homeomorphic to a countable ordinal $[1,\gamma_{x_o}]$ where γ_{x_o} can be chosen to correspond to $\phi(\Omega)$. Likewise $\phi([x,x^+))$ is homeomorphic to a countable ordinal space of the form $[1,\gamma_x]$ where γ_x corresponds to $\phi(\Omega)$ for each $x \in S$. Since $[x,x^+) - D$ is open $[1,\Omega)$ and saturated under π it follows that $[1,\gamma_x)$ is open in X for all $x \varepsilon S$ and $[1,\gamma_{x_o})$ is - open in X.

Thus X is the one point compactification of a free union of an uncountable collection of countable ordinal spaces.

SECTION 2. Invariance properties of perfect images of $S(\alpha)$.

In this section we describe the quotients of $S(\alpha + 1)$ where α is any ordinal whatsoever. We introduce the idea of a Chandelier space. We like to point out that the essential ideas of the proofs of this section were already given in section 1.

Definition 2.1. Let α be a limit ordinal. Let $(X_\gamma)_{\gamma \varepsilon S(\alpha)}$ be a family of T_1 spaces indexed by $S(\alpha)$. Let $L \subset S(\alpha)$ be the set of all limit ordinals $\gamma < \alpha$. Let $\phi: L \to \bigcup_{\gamma \varepsilon S(\alpha)} X_\gamma$ be a function such that $\phi(\gamma) \varepsilon X_\gamma$ for all $\gamma \varepsilon L$. We define the - well ordered sum of the spaces $(X_\gamma)_{\gamma \varepsilon S(\alpha)}$ with respect to the choice function ϕ as follows: The underlying set of the well ordered sum is the disjoint union $Y = \bigoplus_{\gamma \varepsilon S(\alpha)} X_\gamma$ of the given - family. We define neighbourhoods in Y as follows:

If $Y_o \varepsilon Y$ and Y_o is not a value of the function ϕ and $Y_o \varepsilon X_{\gamma_o}$ for a $\gamma_o < \alpha$ then the neighbourhoods of Y_o in X_{γ_o} form a neighbourhood base of Y_o in Y. If $Y_o = \phi(\gamma_1)$ where $\gamma_1 \varepsilon L$ then a neighbourhood base of Y_o consists of sets of the form $W_1 \cup W_2$ where $W_1 \subset X_{\gamma_1}$ and is a neighbourhood of Y_o in X_{γ_1} and $W_2 = \bigcup_{\delta < i < \gamma_1} X_i$ where δ is a non-limit ordinal -

strictly less than γ_1.

Definition 2.2. Let X, Y be compact Hausdorff spaces. Let $\phi : X \to Y$ be a continuous map from X onto Y. ϕ is called an almost homeomorphism or a finite to one map if $\phi^{-1}(\{a\})$ is finite for all $a \in Y$. In this case we say that Y is almost homeomorphic to X. Let $X_1, X_2, \ldots, X_n, \ldots$ be locally compact Hausdorff spaces. A compact Hausdorff space Y is called an interweave of the family $(X_n)_{n=1,2,3,\ldots}$ if it is homeomorphic to the one point compactification of a locally compact space W satisfying the following conditions:

i) there is a map $\phi : \overset{\infty}{\underset{n=1}{+}} X_n \to W$ shich is a quotient map.

ii) $\phi^{-1}(\{a\})$ is finite for all $a \in W$.

iii) $\phi^{-1}(\{a\})$ X_n is empty or singleton for each $a \in W$ and $n=1,2,3,\ldots$.

Notation 2.3. We denote the category of all compact ordinals $S(\alpha + 1)$ by . We denote the category of all perfect Hausdorff images of by Γ. We note that the Hausdorff continuous images of is same as Γ.

THEOREM 2.4. *A compact Hausdorff space which is almost homeomorphic to a member of Γ is again in Γ. The one point compactification of a free union of members in Γ is again in Γ. The one point compactification of a well ordered sum of members of Γ is again in Γ.*

Proof: The first two statements are easy to prove. Now we shall prove the third statement. Let α be a limit ordinal and $(X_\gamma)_{\gamma \varepsilon S(\alpha)}$ be a family of members of Γ. Let $x_\gamma \varepsilon X_\gamma$ be an element chosen in X_γ already where $\gamma \varepsilon S(\alpha)$ and γ is a limit ordinal. For each $i \varepsilon S(\alpha)$, let δ_i be an ordinal and let $\phi_i : S(\delta_i + 1) \to X_i$ be a continuous map from $S(\delta_i + 1)$ onto X_i. For every limit ordinal $j \varepsilon S(\alpha)$ let e_j be an element which does not belong to $S(j)$ and put $Y_j = \{e_j\} \cup S(\delta_j + 1)$. Then Y_j is well ordered when we take the usual order between two elements in $S(\delta_j + 1)$ and put $e_j < x$ for all $x \varepsilon S(\delta_j + 1)$. Now take the free union B of the family $(Y_j)_{j \varepsilon S(\alpha)}$ where we take $Y_j = S(j + 1)$ for all non-limit ordinals $j \varepsilon S(\alpha)$. Put $x < Y$ if $x \varepsilon Y_{\gamma_1}$ and $y \varepsilon Y_{\gamma_2}$ and $\gamma_1, \gamma_2 \varepsilon S(\alpha)$ and $\gamma_1 < \gamma_2$. If $\gamma \varepsilon S(\alpha)$ and $x, Y \varepsilon Y_\gamma$ and put $x \leq Y$ if and only if they are in that relation in Y_γ. Then B is well - ordered. Let ∞ be a symbol that is defined to be larger than all elements in B. Let $C = B \cup \{\infty\}$. Then C is a compact ordinal space. For each Y_γ let $\Psi_\gamma : Y_\gamma \to X_\gamma$ be the map ϕ_γ if γ is a non-limit ordinal in $S(\alpha)$. If $\gamma \varepsilon S(\alpha)$ is a limit ordinal put $\Psi_\gamma(e_\gamma) = x_\gamma$ and $\Psi_\gamma(x) = \phi_\gamma(x)$ for all $x \varepsilon S(\gamma + 1)$. Let X be the well ordered direct sum of the family (X_γ). Let $f : B \to X$ be the map whose restriction to Y_γ is Ψ_γ for all $\gamma \varepsilon S(\alpha)$. Finally let $X \cup \{(\alpha + 1)\} = M$ be the one point compactification of X. Let $\lambda : C \to M$ be the map that maps $\alpha + 1$ to ∞ and coincides with f on B. Then M is the quotient of the compact ordinal C. Thus we get the theorem.

Definition 2.5. A compact ordinal $S(\alpha + 1)$ is called a - chandelier space of order 1. We call α to be the suspension - point of $S(\alpha + 1)$. (Note that the last point of $S(\alpha + 1)$ is α). Suppose that δ is an ordinal and we have defined a chandelier space of order γ and also its suspension point for all ordinals $\gamma < \delta$. A Hausdorff space X is called a chandelier space of order $\leq \delta$ if it can be either obtained as the one-point compactification of an interweave of a sequence of spaces $X_1 - \{a_1\}$, $x_2 - \{a_2\}, \ldots, X_n - - \{a_n\}, \ldots$ where each X_n is a chandelier space of order $\gamma_n < \delta$ and a_n is the supension point of X_n or X is homeomorphic to the one point compactification of a free union of chandelier spaces $(Y_i)_{i \in J}$ with the order of each Y_i to be $< \delta$ or X is homeomorphic to the one-point compactification of a well ordered sum of such spaces $(X_\gamma)_{\gamma \in S(t)}$ with the proviso that $t \leq \delta$ and the element x_γ chosen in X_γ in forming the well ordered sum of the (X_γ) is the suspension point of X_γ for each limit ordinal $\alpha \leq t$. In any case the point at ∞ of X is called its suspension point. Furthur X is said to be of order β if it is of order $\leq \beta$ and is not of order γ for any $\gamma < \beta$.

THEOREM 2.6. *A compact Hausdorff space S is a quotient and hence a perfect image of a compact ordinal $S(\alpha + 1)$ if and only if it* is a chandelier space of some order γ where $\gamma \leq \alpha$. Moreover every Hausdorff quotient Y of a compact ordinal $S(\beta + 1)$ can be obtained as a chandelier space with the image of β as its suspension point.

Proof: Theorem 2.4 gives us that a chandelier space of any order γ is a quotient of a compact ordinal. So we have to only prove that a Hausdorff quotient of a compact ordinal $S(\beta + 1)$ is a chandelier space with the image of β as its suspension point. The theorems 1.11 and 1.12 of the previous section say essentially that the Hausdorff quotients of $[1,\Omega] = S(w_1 + 1)$ are chandelier spaces with the image of Ω as a suspension point.

We are going to prove our assertion by transfinite induction. For this we assume that α is a given ordinal and that every Hausdorff quotient X of $S(\gamma + 1)$ is a chandelier space with the image of γ as its suspension point for all limit ordinals $\gamma < \alpha$. If α is not a limit ordinal then there is a largest limit ordinal $\gamma_o < \alpha$. It is clear that $S(\alpha + 1)$ is homeomorphic to $S(\gamma_o + 1)$. (We are interested in infinite ordinals only). Then $S(\gamma_o + 1)$ is homeomorphic to $S(\alpha + 1)$ and γ_o is a limit ordinal. So by induction assumption the quotient of $S(\alpha + 1)$ is a chandelier space. An easy argument also gives that it is possible to represent X in such a way that the image of α is the suspension point. We have to consider two cases:

Case (a): The cofinality $cf(\alpha)$ of α is not countable and $\phi^{-1}(\{\phi(\alpha))\}) = \{\alpha\}$ where $\phi : S(\alpha + 1) \to X$ is the quotient map.

Following the ideas of the proof of theorem 1.8 we see that for every ordinal $\delta < \alpha$ we can associate an ordinal $\beta_\delta < \alpha$ so that $\delta \leq \beta_\delta$ and $\phi^{-1}(\phi([1,\beta_\delta))) = [1,\beta_\delta)$ where $\phi : S(\alpha + 1) \to X$ is the given quotient map. We can also assume without loss of generality that $\beta_i < \beta_j$ if $i, j \in S(\alpha)$ and $i < j$. Let π be the partition of $S(\alpha + 1)$ which is induced by ϕ. Then it is clear that if $x \in S(\alpha)$ then the partition class π_x to which x belongs is compact and $\subset S(\alpha)$.

Let m_x be the maximum element of π_x. Then we claim that there is a compact, open set V_x which contains π_x and which is saturated under π. For if $y \in \pi_x$ there is an open and closed interval $[\ell_Y, Y]$ $S(\alpha)$ so that no element $z \in [\ell_Y, Y]$ can be in the same member of π as any element $u > m_x$ and $u \in S(\alpha)$. (This is because π_x has m_x for its maximum.) Put $W = \bigcup_{Y \in \pi_x} [\ell_Y, Y]$.

Then W is an open set containing π_x and m_x is the maximum point of W. Since the map $\phi : S(\alpha + 1) \to x$ is a closed map, it follows that there exists an open set V W so that $\pi_x \subset V$ and $\phi(V)$ is an open set in X which contains $\phi(\pi_x)$. Since X is scattered by [4] we get that there is a compact open set $L \subset \phi(V)$ and $\phi(\pi_x) \in L$. Putting $\phi^{-1}(L) = V_x$ we get a compact open set of $S(\alpha)$ so that V_x is saturated under π and V_x having m_x as its maximum element. Now let us look at the set $\{\beta_j \mid j \in s(\alpha)\}$ where we assume that β_j has been chosen so that $j \leq \beta_j < \alpha$ and $\beta_{j_1} < \beta_{j_2}$ if $j_1 < j_2$ and $j_1, j_2 \in s(\alpha)$ and $[1, \beta_j)$ is saturated under π for all $j \in s(\alpha)$. For each j let us choose a compact open set V_j so that $\beta_j \in V_j$ and $V_j \subset [1, \beta_{j+1})$ for all $j \in S(\alpha)$ and V_j is saturated under π. Put $W_j = [1,\beta_j) \cup W_j$ for all $j \in S(\alpha)$. Finally $X_o = \phi(W_1)$ and $X_i = \phi(W_i) - \phi(W_{i-1})$ if i is a non-limit ordinal such that $1 < i < \alpha$. If i is a limit ordinal $< \alpha$, put $X_i = \phi(W_i) - \phi(\bigcup_{j<i} W_j)$. Then it follows that each X_i is a chandelier space of order at most β_{i+1} which is $< \alpha$ and X is the one-point compactification of the well ordered sum of the $(X_i)_{i\in S(\alpha)}$ with $\phi(\alpha)$ as the point ∞. Thus $\phi(\alpha)$ is the suspension point of X. Thus we have the theorem in this case.

Case (b): The cofinality $cf(\alpha)$ is not countable and the given quotient map $\phi : S(\alpha) \to X$ is such that $\phi^{-1}(\{\phi(\alpha)\}) - \{\alpha\}$ has a maximum $\gamma_o < \alpha$.

In this case we write $S(\alpha + 1) = [1,\gamma_o] \cup [\gamma_o + 1,\alpha]$. Putting $\phi([1,\gamma_o]) = X_o$ and $\phi([\gamma_o + 1,\alpha]) = X_1$ we get from case (a) that X_o and X_1 are chandelier spaces and X is almost homeomorphic to the free union of X_o and X_1 with $\phi(\alpha)$ as the point at ∞. Thus we get the theorem in this case also.

Case (c): ϕ, X are as in case (b) and $cf(\alpha)$ is uncountable and $\phi^{-1}(\{\phi(\alpha)\}) - \{\alpha\}$ is cofinal with $S(\alpha)$ and does not contain a ray of the form $[\beta,\alpha)$ for any $\beta < \alpha$.

In this case we use the argument used in the proof of theorem 1.12 with the obvious small modifications. Then we get that X is the one point compactification of a free union of spaces Y_i where $i \in S(\alpha)$ and each Y_i is of the form $X_i - \{x_i\}$ where X_i is a chandelier space with suspension point x_i for each $i \in S(\alpha)$. Again $\phi(\alpha)$ is the point at ∞ of X and hence we get the result. Thus we have the result if $cf(\alpha)$ is not countable.

Now let us assume that $cf(\alpha)$ is countable. Then we consider the case:

Case (d): If (α) is uncountable; $\phi^{-1}(\{\phi(\alpha)\}) - \{\alpha\}$ is cofinal with $S(\alpha)$ and contains a ray of the form $[\beta,\alpha)$ for some $\beta < \alpha$. In this case, ϕ maps $[1,\beta]$ onto X and $\phi(\beta) = \phi(\alpha)$ and hence by induction hypothesis we have that X is a chandelier space with $\phi(\alpha)$ as the suspension point.

Case (e): $\phi : S(\alpha + 1) \to X$ is a quotient map and $cf(\alpha)$ is countable and $\phi^{-1}(\{\phi(\alpha)\}) = \{\alpha\}$.

In this case we can find an increasing sequence $S_1 \subset S_2 \subset \ldots \subset S_n$ of compact open sets of $[1,\alpha)$ so that $\phi^{-1}(\phi(S_n)) = S_n$ for all $n = 1,2,3,\ldots$ and $\bigcup_{n=1}^{\infty} S_n = [1,\alpha)$. Putting $X_1 = \phi(S_1)$ and $X_n = \phi(S_n - S_{n-1})$ for all $n = 2,3,\ldots$ we get that each X_n is a chandelier space and X is the one point compactification of a free union of chandelier spaces with $\phi(\alpha)$ as the point at ∞.

So X is a chandelier space with $\phi(\alpha)$ as its suspension point. Now we come to the last case:-

Case (f): $cf(\alpha)$ is countable and $\phi^{-1}(\{\phi(\alpha)\}) - \{\phi(\alpha)\}$ is infinite and cofinal with $S(\alpha)$.

Now if there is an $\alpha_o < \alpha$ so that $\phi(\gamma) = \phi(\alpha)$ for all $\gamma > \alpha_o$ and $\gamma \leq \alpha$ then ϕ maps $[1,\alpha_o]$ onto X and $\phi(\alpha_o) = \phi(\alpha)$ and hence by induction assumption we have that X is a chandelier space with $\phi(\alpha)$ as suspension point. So we assume that for every $\alpha_o < \alpha$ there is a $\beta_o < \alpha$ so that $\alpha_o < \beta_o$ and $\phi(\beta_o) \neq \phi(\alpha)$. Now there is a strictly increasing sequence $\beta_1 < \beta_2 \dots < \beta_n < \dots$ of ordinals $< \alpha$ so that $\lim_{n\to\infty} \beta_n = \alpha$ and $\phi(\beta_n) = \phi(\alpha)$ for all $n = 1, 2, 3, \dots$

Put $X_1 = \phi([1,\beta_1])$; $X_2 = \phi([\beta_1 + 1, \beta_2]), \dots,$

$X_n = \phi([\beta_n + 1, \beta_{n+1}]) \dots$. Then $X_1, X_2, \dots, X_n, \dots$ are chandelier spaces with suspension point $\phi(\alpha)$ and with order $< \alpha$. Put $Y_n = X_n - \{\phi(\alpha)\}$ for $n = 1, 2, 3, \dots$. Then it is clear that X is the one point compactification of an interweave of the spaces (Y_n) with $\phi(\alpha)$ as its point at ∞. Thus we get again that X is a chandelier space with $\phi(\alpha)$ as its suspension point. So we get the theorem in all cases.

Remark 2.7. Theorem 2.6 has many interesting consequences and leads to furthur interesting problems. We do not inted to go into all of them here but leave it for a furture investigation. We give below some easy consequences of theorem 2.6.

COROLLARY 2.7. *Let α be an ordinal whose cofinality $cf(\alpha)$ is not countable. Then every Hausdorff perfect image X of $S(\alpha)$ is of the form $Y - \{Y\}$ where Y is a chandelier space and Y is the suspension point of Y.*

Proof: Since the perfect, continuous image of a locally compact space is locally compact (see [8]), it follows that X is locally compact. Therefore, X is completely regular. So, the map $\phi : S(\alpha) \to X$ extends to a map $f : \beta(S(\alpha)) \to \beta X$. Now $\beta(S(\alpha)) = S(\alpha + 1)$, and since ϕ is onto and f has compact range, f maps $\beta(S(\alpha))$ onto βX. Thus $\beta X = Y$ is a chandelier space, and $f(\alpha)$ is the point in $Y - X$, which is thus the suspension point of Y.

COROLLARY 2.8. *The class L of perfect images of ordinals is closed under the following operations:*

(a) *one point compactification of a free union of members of L.*

(b) *Well ordered sum of members of L.*

(c) *one point compactification of perfect images of countable discrete union of members of L.*

(d) *countable discrete union of compact members of L.*

We leave the proof to the reader.

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