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Fixed Point's Theorems for $\omega - \varphi -$ Contractions

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Using the notion of $\omega-$distance on the metric space, (M, d), we get some generalizations of results of Browder [3], Boyd-wong [2], Mukherjea [18] and Matkowski [14].

Introduction

In 1996, O. Kada- T. Suzuki- W. Takahashi [13] introduced the concept of $\omega-$distance on a metric space and using this notion they improved the Caristi fixed point theorem [4], Ekeland's $\varepsilon-$Variational Principle [10] and proved a fixed point theorem in a complete metric space which generalize the fixed point theorems of Subramanyan [24], Kannan [12] and Ciric [5].

T. Suzuki- W. Takahashi [25] using the notion of $\omega-$distance on a metric space proved a fixed point theorem for set-valued mapping a complete metric space which are related which Nadler's fixed point theorem [19] and Edelstein's theorem [9].

T. Suzuki [26] using the $\omega-$distance gave another fixed point theorems which are generalizations of the Banach Contraction Principle and Kanan's fixed point theorem.

Y. J. Cho - N. J. Huang - L. Xiang [6] introduced new classes of generalized contractive type set-valued mappings and weakly dissipative mappings and they proved some coincidence theorems for these mappings by using the concept of $\omega-$distance.

M. Hiromichi [11] in his thesis used the notion of $\omega-$distance and the concept of fixed point to characterize the mathematical structure of space metric completeness and finite dimensionality of Banach spaces.

The author in [16] and [17] gave other results referent to fixed point theorems.

Recently S. Park [20], using the $\omega-$distance concept, improved the equivalent formulation of Ekeland's Principle in various aspects and moreover, as a simple application, he gave an extended form of a fixed point theorem of Downing-Kirk [8].

Finally in this article our end is to generalize some fixed point theorems for $\varphi-$contractions using the concept of $\omega-$distance on a metric space.

1 Preliminares

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real number and $R_+ = [0, +\infty)$.

Definition 1.1 *Let (M, d) be a metric space. Then a function $p : M \times M \longrightarrow [0, +\infty)$ is called a $\omega-$distance on M if the following conditions are satisfied:*

ω *1.- $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in M$.*

ω *2.- For any $x \in M$, $p(x, \cdot) : M \longrightarrow [0, +\infty)$ is a lower semicontinuous function.*

ω *3.- For any $\varepsilon > 0$ exists $\delta = \delta(\varepsilon) > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply that $d(x, y) \leq \varepsilon$.*

The metric d is a $\omega-$distance on M. Some other examples of $\omega-$distances are given in [13], [25] and [26].

Notation 1.1 *By $W(M)$, we denote the set of all $\omega-$distances p on M and it is clear that $W(M) \neq \emptyset$.*

In [13] we found an example which show that p is not symetric, $p(x, y) \neq p(y, x)$ for all $x, y \in M$, so we denote by $W_0(M)$, the set of all $\omega-$distances p on M that are symetric. It is clear that $W_0(M) \neq \emptyset$.

The following results are crucial in the proof of our theorems. The next result was proved in [13].

Lemma 1.1 *Let (M, d) be a metric space and let p be a $\omega-$distance on M. Let (α_n) and (β_n) be sequences in $[0, +\infty)$ converging to 0, and let $x, y, z \in M$. Then the following hold:*

a.- If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$ then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$ then $y = z$.

b.- If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converge to z.

c.- If $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence in (M, d).

d.- If $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$ then (x_n) is a Cauchy sequence in (M, d).

The following result can be found in [26].

Lemma 1.2 *Let (M, d) be a metric space, let p be a $\omega-$distance on M and let (x_n) be a sequence in M.*

Suppose that

$$\lim_{n\to\infty} \sup_{m>n} \{p(x_n, x_m), p(x_m, x_n)\} = 0.$$

Then (x_n) is a Cauchy sequence in M. In particular the following hold:

a.- If $\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0$ *then* (x_n) *is a Cauchy sequence in* M.

b.- If $\lim_{n\to\infty} \sup_{m>n} p(x_m, x_n) = 0$ *then* (x_n) *is a Cauchy sequence in* M.

The following definition is due to T. Suzuki - W. Takahashi [25].

DEFINITION 1.2 *Let* (M, d) *be a metric space and let* T *be a mapping from* M *into itself. We say that* T *is a* $\omega - B-$*contraction if there exists a* $\omega-$*distance* p *on* M *and* $k \in \mathbb{R}$, $0 \leq k \leq 1$ *such that*

$$p(Tx, Ty) \leq kp(x, y) \tag{1.1}$$

for all $x, y \in M$.

It is clear that if $p = d$ we get that T is a Banach contraction, (in short, $B-$contraction). In [25] we found the following result.

THEOREM 1.1

Let (M, d) *be a complete metric space. If a mapping* T *from* M *into itself is a* $\omega - B-$*contraction then* T *has a unique fixed point* $x_0 \in M$. *Moreover the* x_0 *satisfies* $p(x_0, x_0) = 0$.

It is clear that theorem 1.1 generalize the well known Banach contraction principle and for another similar results see [16].

In [17] the author introduced the following,

DEFINITION 1.3 *Let* (M, d) *be a metric space and let* T *be a mapping from* M *into itself. We say that* T *is a* $\omega - BR-$*contraction if there exists a* $\omega-$*distance* p *on* M *and a monotone decreasing function* $\alpha : \mathbb{R}_+ \longrightarrow [0, 1)$ *tal que*

$$p(Tx, Ty) \leq \alpha(p(x, y))p(x, y) \tag{1.2}$$

for all $x, y \in M$.

REMARK 1.1 *1.- If* $\alpha(t) = k$ *for all* $t \in \mathbb{R}$ *where* $0 \leq k \leq 1$ *we get (1.1).*

2.- If $p = d$ *then we get*

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \tag{1.3}$$

for all $x, y \in M$, *which is the Rakotch's condition, [21].*

The author in [17] proved the following, THEOREM 1.2

Let (M, d) be a complete metric space and let $T : M \longrightarrow M$ be a $\omega - BR-$contraction then there exists a unique $z \in M$ such that $z = Tz$ and $p(z, z) = 0$.

2 $\omega - \varphi-$contractions

Various concepts of comparison functions have been defined and intensevely studied in connection with the contraction mappings, see Rus, A. I. [22], Berinde, V. [1]. We are going to use the notions of $\varphi-$comparison function to define the concept of $\omega - \varphi-$contractions.

DEFINITION 2.1 (BOYD-WONG - [23 *)] A function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called φ_A-comparison function if:*

A1.- φ is upper semicontinuous function.

A2.- For each $t > 0$, $\varphi(t) < t$.

DEFINITION 2.2 (MUKHERJEA - [18 *)] A mapping $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called φ_B-comparison function if:*

B1.- φ is a right continuous function.

B2.- For each $t > 0$, $\varphi(t) < t$.

It is well known that if $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is monotone increasing function then φ right upper semicontinuous iff φ is right continuous.

Therefore if $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is monotone increasing function then the definition 2.1 is equivalent to definition 2.2.

DEFINITION 2.3 (BROWDER) *[3] A mapping $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a φ_C-comparison if:*

C1.- φ is right continuous.

C2.- For each $t > 0$, $\varphi(t) < t$.

C3.- φ is monotone increasing.

Thus definitions 2.1, 2.2 and 2.3 are equivalent if $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a monotone increasing function.

LEMMA 2.1 *If $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a φ_C-comparison function then $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $n \in \mathbb{N}$ and for each $t > 0$.*

Proof

See [22].

∎

Definition 2.4 (Matkowski - [15)] *A mapping* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is called a* φ_D-*comparison function if:*

D1.- φ *is monotone increasing.*

D2.- $\lim_{n\to\infty} \varphi^n(t) = 0$ *for each* $t > 0,\ n \in \mathbb{N}$.

From lemma 2.1 it is clear that if $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a φ_C-comparison function then φ is φ_D-comparison function.

Lemma 2.2 *Let* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a* φ_D-*comparison function then*

a.- $\varphi(t) < t$ *for all* $t > 0$.

b.- $\varphi(0) = 0$.

Proof

See [22].

∎

Example 2.1 *1.- Let* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a mapping defined by* $\varphi(t) = at,\ 0 \leq a \leq 1,\ t \in \mathbb{R}_+$. *It is clear that* φ *is a* $\varphi_A - (\varphi_B, \varphi_C, \varphi_D)-$*comparison function.*

2.- Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a function defined by* $\varphi(t) = \dfrac{t}{1+t}$, $t \in \mathbb{R}_+$. *Also* φ *is* $\varphi_A - (\varphi_B, \varphi_C, \varphi_D)-$*comparison function.*

Now we introduce the notions of $\omega-\varphi-$contractions which generalize the well known $\varphi-$contractions

Definition 2.5 *Let* (M, d) *be a metric space. A mapping* $T : M \longrightarrow M$ *is called a* $\omega - \varphi_A-$*contraction, (respectively* $\omega - \varphi_B-$*contraction,* $\omega - \varphi_C-$*contraction,* $\omega - \varphi_D-$*contraction), if there exists a* φ_A-*comparison, (respectively* φ_B-*comparison,* φ_C-*comparison,* φ_D-*comparison) function such that*

$$p(Tx, Ty) \leq \varphi(p(x, y)) \tag{2.1}$$

for all $x, y \in M$.

The author in [16] introduced the following,

Definition 2.6 *Let* (M, d) *be a metric space with a* $\omega-$*distance* p *on* M *and let* $T : M \longrightarrow M$ *be a mapping. Then*

a.- An element $x \in M$ *is* $\omega-$*asymptotic regular for* T *if*

$$\lim_{n\to\infty} p(T^n x, T^{n+1} x) = 0. \tag{2.2}$$

b.- T is $\omega-$asymptotic regular if all element $x \in M$ are $\omega-$asymptotic regular for T.

c.- Two elements x and y of M are $\omega-$asymptotic equivalent under T if

$$\lim_{n\to\infty} p(T^n x, T^n y) = 0 \tag{2.3}$$

Now we have the following result,

PROPOSITION 2.1 *Let (M,d) be a metric space and let $T : M \longrightarrow M$ be a $\omega - \varphi_D-$contraction. Then*

a.- T is $\omega-$asymptotic regular.

b.- Each two elements $x, y \in M$ are $\omega-$asymptotic equivalent under T.

Proof

Since T is a $\omega - \varphi_D-$contraction there exists a $\omega-$distance p on M and φ_D-comparison function such that

$$p(Tx, Ty) \leq \varphi(p(x,y))$$

for all $x, y \in M$.

a.- Let $x \in M$ be an element of M. Let $x_n = T^n x$, $n \in \mathbb{N}$. Then we have $p(x_n, x_{n+1}) \leq \varphi^n[p(x, Tx)]$. If follows that

$$\lim_{n\to\infty} p(T^n x, T^{n+1} x) = 0$$

for all $x \in M$. Therefore T is a $\omega-$asymptotic regular.

b.- Let $x, y \in M$ be. We have that

$$p(T^n x, T^n y) \leq \varphi^n[p(x,y)]$$

for $n \in \mathbb{N}$, so

$$\lim_{n\to\infty} p(T^n x, T^n y) = 0, \quad n \in \mathbb{N}$$

Therefore x and y are $\omega-$asymptotic equivalent under T.

∎

3 Main Results

In this section using the $\omega-$distance p on (M,d) we give some generalizations of some well known fixed point theorems.

The following result generalize the Boyd-Wong's Theorem, [2].

THEOREM 3.1

Let (M,d) be a complete metric space and let $T : M \longrightarrow M$ be a $\omega - \varphi_A-$contraction. Then T has a unique fixed point.

Proof

Since T is a $\omega - \varphi_A-$contraction there exists a $\omega-$distance $p \in W_0(M)$ and φ_A-comparison function such that

$$p(Tx, Ty) \leq \varphi(p(x,y)) \tag{3.1}$$

for all $x, y \in M$.

For an $x \in M$ we put $x_n = T^n x,\ n \in \mathbb{N}$ and $a_n = p(x_n, x_{n+1})$. Then for $n > 1$,

$$a_n = p\left(Tx_{n-1}, Tx_n\right) \leq \varphi(p(x_{n-1}, x_n)) = \varphi(a_{n-1}) < a_{n-1} \tag{3.2}$$

So that the sequence (a_n) is decreasing. Let $a = \lim\limits_{n\to\infty} a_n$. Then $a = 0$, since that (3.2) implies that $a \leq \varphi(a)$ which is a contradiction and consequently

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Thus, for $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $\forall\ n > n_0$

$$p(x_n, x_{n+1}) \leq \varepsilon - \varphi(\varepsilon).$$

Put $K(x_n, \varepsilon) = \{x \in M|\ p(x, x_n) \leq \varepsilon\}$. It is clear that $K(x_n, \varepsilon) \subset X$ is a closed set and for any $z \in K(x_n, \varepsilon)$ we have

$$\begin{aligned} p(Tz, x_n) &\leq p(Tz, Tx_n) + P(Tx_n, x_n) \leq \varphi(p(z, x_n)) + p(x_{n+1}, x_n) \\ &\leq \varphi(\varepsilon) + (\varepsilon - \varphi(\varepsilon)) = \varepsilon, \end{aligned}$$

so $K(x_n, \varepsilon)$ is invariant under T, which implies that for $m > n > n_0,\ p(x_n, x_m) \leq 2\varepsilon$.

Consequently by lemma 1.2, (x_n) is a Cauchy sequence in (M,d), hence there exists $z \in M$ such that $x_n \to z$.

Since $p(x_n, \cdot)$ is a lower semicontinuous function

$$p(x_n, z) \leq \liminf_{m\to\infty} p(x_n, x_m)$$

and it follows,

$$\lim_{n\to\infty} p(x_n, z) = 0.$$

On the other hand,

$$p(x_n, Tz) = p(tx_{n-1}, Tz) \leq \varphi(p(x_{n-1}, z)) < p(x_{n-1}, z)$$

hence

$$\lim_{n\to\infty} p(x_n, Tz) = 0,$$

so by lemma 1.1, $Tz = z$.

Now $p(z,z) = p(Tz,Tz) \leq \varphi(p(z,z)) < p(z,z)$ and $p(z,z) = 0$.

Finally, if $y = Ty$ then

$$p(z,y) = p(Tz,Ty) \leq \varphi(p(z,y)) < p(z,y)$$

and $p(z,y) = 0$ so $z = y$, from lemma 1.1

∎

In similar way we can show the following generalization of a Mukherjen's theorem [18].

THEOREM 3.2

Let (M,d) be a complete metric space and let $T : M \longrightarrow M$ be a $\omega - \varphi_B-$contraction mapping. Then T has a unique fixed point.

Now we give a generalization of a Matkowski's result [14].

THEOREM 3.3

Let (M,d) be a complete metric space and let $T : M \longrightarrow M$ be a $\omega - \varphi_D-$contraction. Then T has a unique fixed point.

Proof

Since T is $\omega - \varphi_D-$contraction there exists $p \in W(M)$ and a φ_D-comparison function such that

$$p(Tx,Ty) \leq \varphi(p(x,y)) \tag{3.3}$$

for all $x, y \in M$.

and define $x_n = T^n x,\ n \in \mathbb{N}$ then by (3.3) we have

$$p(x_n, x_{n+1}) \leq \varphi^n(p(x,Tx))$$

and hence $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.

For $m > n$

$$\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0$$

so by lemma 1.2 (x_n) is a Cauchy sequence in (M,d).

In view of completeness of M there exists $z \in M$ such that $x_n \to z$. The rest of the proof follows since in the theorem 3.1.

∎

The following result generalize

COROLLARY 3.1 *Let* (M,d) *be a complete metric space. If a mapping* T *from* M *into itself is a* $\omega - B-$*contraction then* T *has a unique fixed point* $z \in M$*. Furthermore the point* z *satisfies* $p(z,z) = 0$.

Proof

Taking $\varphi(t) = kt,\ 0 \leq k \leq 1,\ t \in \mathbb{R}_+$ and since T is a $\omega - B-$contraction there exists $p \in W(M)$ such that

$$p(Tx,Ty) \leq kp(x,y) \tag{3.4}$$

for all $x, y \in M$.

The conclusion follows from theorem 3.3.

∎

THEOREM 3.4

Let (M,d) *be a complete metric space and* $T : M \longrightarrow M$ *is a mapping such that for some* $m \in \mathbb{N}$ T^m *is a* $\omega - \varphi_D-$*contraction. Then* T *has a unique fixed point in* M.

Proof

Since for some any $m \in \mathbb{N}$, T^m is a $\omega - \varphi_D-$contraction there exists $p \in W(M)$ and a φ_D-comparison mapping such that

$$p(T^m x, T^m y) \leq \varphi(p(x,y)) \tag{3.5}$$

for all $x, y \in M$.

Thus by theorem 3.1 there exists a unique $z \in M$ such that $z = T^m z$ and it follows that $z = Tz$.

∎

The following result generalize the theorem, of Chu-Diaz, [7].

COROLLARY 3.2 *Let* (M,d) *be a complete metric space and* $T : M \longrightarrow M$ *is a mapping such that for some* $m \in \mathbb{N}$, T^m *is a* $\omega - B-$*contraction. Then* T *has a unique fixed point in* M.

Proof

It is clear.

∎

The following result is a generalization of Browder's fixed point theorem [3].

THEOREM 3.5

Let (M,d) *be a complete metric space and let* $T : M \longrightarrow M$ *be a* $\omega - \varphi_C-$*contraction. Then* T *has a unique fixed point.*

Proof

Since T is a $\omega - \varphi_C-$contraction there exists $p \in W(M)$ and a φ_C-comparison function such that

$$p(Tx,Ty) \leq \varphi(p(x,y)) \tag{3.6}$$

for all $x, y \in M$.

By lemma 2.1 we have that

$$\lim_{n\to\infty} \varphi^n(t) = 0 \qquad \text{for } n \in \mathbb{N} \text{ and } t \in \mathbb{R}_+.$$

Now we apply theorem 3.3 to get the conclusion.

∎

Now we consider the following

EXAMPLE 3.1 *Let $M = [0,1] \subseteq \mathbb{R}$ be a complete metric space with the usual metric. We define a $\omega-$distance p on M by*

$$p(x,y) = \begin{cases} 0 & \text{if } \; x = 0 \\ y - x & \text{if } \; 0 < x \leq y \\ 3x - 3y & \text{if } \; x > y \end{cases}$$

for all $x, y \in M$.

Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a function defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } \; t = 0 \\ \\ \dfrac{1}{n+1} & \text{if } \; \dfrac{1}{n+1} < t \leq \dfrac{1}{n}, \; n = 1, \dots \end{cases}$$

It is clear that,

a.- φ is increasing function in $\mathbb{R}_+$.

b.- For all $t > 0$, $\varphi(t) < t$.

c.- $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$.

d.- φ is not upper semicontinuous from the right.

e.- φ is not continuous from the right.

Thus we have that, φ is a φ_D-comparison function but is not φ_A-comparison function neither φ_B-comparison function.

Suppose that $T : M \longrightarrow M$ is a mapping which satisfies (3.3) and we see that all assumptions of theorem 3.3 are full filled and this theorem generalize the theorem 3.1 since φ is not upper semicontinuous.

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