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INTRODUCTION TO VON NEUMANN ALGEBRAS

PART I

(CHAPTERS 1-4)

POR

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Dedicated to
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and
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CONTENTS

CHAPTER 1

PRELIMINARIES

In this chapter we give definitions and results that are essential to build up the theory of von Neumann algebras. We assume that the reader is well informed about measure theory, theory of commutative B*-algebras, operator theory in Hilbert space including the spectral theorem for normal operators and elementary theory of Banach spaces and locally convex spaces.The reader may refer to Halmos [H_1] or Munroe [Mu] for measure theoretic results, Bachman and Narici [BN], Halmos [H_2], Naimark [Na],Riesz and Nagy [RN] ans Stone [St] for the theory of operators in Hilbert spaces, Schaefer [Sc] for locally convex spaces,Simmons [S] for Banach spaces, Rickart [Ri], Naimark [Na] and Simmons [S] for the theory of Banach algebras. One may also refer to Dunford and Schwartz [DS].

1.1. Hilbert space

Definition 1.1.1. A complex vector space H is called a Hilbert space if there is an inner product [.,.] on H such that H is a complete normed linear space under the norm $\|x\| = [x,x]^{\frac{1}{2}}$.

Throughout this chapter H will denote a Hilbert space with inner product $[.,.]$.

Theorem 1.1.2. (Cauchy-Schwarz) If $[.,.]$ is an inner product which is not necessarily strictly positive, then

$$|[x,y]| \leq \|x\|\|y\|.$$

If $[.,.]$ is strictly positive, then the equality holds if and only if x and y are linearly dependent.

Definition 1.1.3. A Hilbert space H is said to be *separable* if it has a countable dense subset.

Theorem 1.1.4. (Parallelogram law) For x, y in the Hilbert space H, $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Definition 1.1.5. Two vectors x and y in H are said to be *orthogonal* (in symbols $x \perp y$ or $y \perp x$) if $[x,y] = 0$.

Theorem 1.1.6. (Pythagorean theorem) If $x \perp y$ in H, then $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2$.

Definition 1.1.7. A non-void family F of vectors in H is said to be an *orthogonal family* if $x,y \in F$, $x \neq y$, then $x \perp y$. An orthogonal family F is said to be orthonormal if $x \in F$ implies $\|x\|=1$.

Theorem 1.1.8. (Polarization identity) For x,y in H, $4[x,y] = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$.

Definition 1.1.9. If S is a subset of H, the orthogonal complement of S, denoted by $S^{\perp}$, is defined as the set of $\{y : y \in H, [x,y] = 0$ for all $x \in S\}$. A *subspace* of H is a closed linear manifold in H.

Theorem 1.1.10.

(i) For any subset S of H, $S^{\perp}$ is a subspace of H.

(ii) $S \subset S^{\perp\perp}$.

(iii) $S^{\perp\perp} = S$ if and only if S is a subspace of H.

(iv) $S^{\perp} = S^{\perp\perp\perp}$.

Theorem 1.1.11. (Projection theorem) If S is a subspace of H, then $H = S \oplus S^{\perp}$.

1.2. Operators and sesqui-linear functionals

By an *operator* on H we mean a continuous linear mapping whose domain is all of H and range is contained in H. If T is an operator on H, then $\sup\limits_{\|x\|=1}\|Tx\| = \sup\limits_{\|x\|\leq 1}\|Tx\| = \sup\limits_{x\neq 0}\frac{\|Tx\|}{\|x\|} < \infty$ and $\sup\limits_{\|x\|=1}\|Tx\| = \|T\|$ (in symbols) is then called the norm of T. As a consequence, we have $\|Tx\| \leq \|T\|\ \|x\|$, for each $x \in H$. *We denote by* B(H) *the set of all operators on* H.

A continuous linear mapping $f : H \to \mathbb{C}$ is called a *continuous linear form* or *functional*. Norm of f, denoted by $\|f\|$, is given by $\sup\limits_{\|x\|=1}|f(x)|$ and is also equal to $\sup\limits_{\|x\|\leq 1}|f(x)|$ and $\sup\limits_{x\neq 0}\frac{|f(x)|}{\|x\|}$. The colletion of all such f is denoted by H^* and is called the dual of H.

Theorem 1.2.1. (The Riesz representation theorem) To every continuous linear form f on H, there corresponds a unique vector y_f in H such that

$$f(x) = [x, y_f] \qquad (1.2.1.1)$$

for each $x \in H$ and $\|f\| = \|y_f\|$. Conversely, the formula (1.2.1.1) defines a continuous linear form f on H for a fixed vector y_f and $\|f\| = \|y_f\|$.

Definition 1.2.2. A *sesqui-linear functional* $\psi(.,.)$ on a Hilbert space H is a map $\psi : H \times H \to \mathbb{C}$ such that ψ is linear in the first argument and conjugate linear in the second argument.

If ψ is a sesqui-linear functional on H, then the *quadratic form* induced by ψ is the functional $\hat{\psi}$ on H given by $\hat{\psi}(x) = \psi(x,x)$, for x in H.

The sesqui-linear functional ψ is said to be *positive* if $\hat{\psi}(x) \geq 0$, for each x in H and, is said to be *strictly positive* if $\hat{\psi}$ is positive and $\hat{\psi}(x) = 0$ implies $x = 0$.

Theorem 1.2.3. If ψ is a sesqui-linear functional on H, then $4\psi(x,y) = \hat{\psi}(x+y) - \hat{\psi}(x-y) + i\hat{\psi}(x+iy) - i\hat{\psi}(x-iy)$.

Consequently, two sesqui-linear functionals ψ and Ψ are equal if and only if $\hat{\psi} = \hat{\Psi}$.

Definition 1.2.4. A sesqui-linear functional ψ is said to be *symmetric* if $\psi(x,y) = \overline{\psi(y,x)}$, for all $x, y \in H$.

Theorem 1.2.5. The sesqui-linear functional ψ is symmetric if and only if $\hat{\psi}$ is real.

Definition 1.2.6. A sesqui-linear functional ψ is said to be *bounded* if there is a constant K such that $|\psi(x,y)| \leq K \|x\| \|y\|$, for all x, y in H. When ψ is bounded, the infimum of all such K s is called the *norm* of ψ and is denoted by $\|\psi\|$.

Theorem 1.2.7. The sesqui-linear functional ψ is bounded if and only if $\sup_{\|x\| = \|y\| = 1} |\psi(x,y)| < \infty$. If ψ is bounded, then $\|\psi\| = \sup_{\|x\| = \|y\| = 1} |\psi(x,y)|$.

Further, $|\psi(x,y)| \leq \|\psi\| \, \|x\| \, \|y\|$, for all x,y in H.

Definition 1.2.8. A quadratic form $\hat{\psi}$ is said to be *bounded* if there is a constant $k > 0$ such that $|\hat{\psi}(x)| \leq k \|x\|^2$, for all $x \in H$. If $\hat{\psi}$ is bounded, then the infimum of such k's is called the *norm* of $\hat{\psi}$ and is denoted by $\|\hat{\psi}\|$.

Theorem 1.2.9.

(i) $\hat{\psi}$ is bounded if and only if $\sup_{\|x\|=1} |\hat{\psi}(x)| < \infty$ and, when $\hat{\psi}$ is bounded, $\|\hat{\psi}\| = \sup_{\|x\|=1} |\hat{\psi}(x)|$ and $|\hat{\psi}(x)| \leq \|\hat{\psi}\| \, \|x\|$, for all x in H.

(ii) A sesqui-linear functional ψ is bounded if and only if $\hat{\psi}$ is bounded and, when ψ is symmetric, $\|\psi\| = \|\hat{\psi}\|$.

Theorem 1.2.10. If $A \in B(H)$, then the functional ψ, defined by $\psi(x,y) = [Ax,y]$, is a bounded sesqui-linear functional with $\|\psi\| = \|A\|$. Conversely, if ψ is a bounded sesqui-linear functional, then there exists a unique operator A on H such that $\psi(x,y) = [Ax,y]$ and $\|A\| = \|\psi\|$.

Theorem 1.2.11. If $A \in B(H)$, then there is a unique operator A^* in $B(H)$, called the adjoint of A, such that $[Ax,y] = [x,A^*y]$, for all x, y in H.

For A,B in $B(H)$ and α,β in $\mathbb{C}$, the following hold:
(a) $A^{**} = A$; (b) $(\alpha A)^* = \bar{\alpha} A^*$; (c) $(A + B)^* = A^* + B^*$; (d) $(A^{-1})^* = (A^*)^{-1}$; (e) $\|A^*\| = \|A\|$; (f) $\|A^*A\| = \|AA^*\| = \|A\|^2$; (g) $(AB)^* = B^*A^*$; (h) $\|AB\| \leq \|A\| \, \|B\|$.

Definition 1.2.12. If $A \in B(H)$, A is called *hermitian* or *self-adjoint* if $A^* = A$; *normal* if $AA^* = A^*A$; and *unitary* if $AA^* = A^*A = I$.

Theorem 1.2.13. Let A,B,C be in $B(H)$. Then:

(i) A is hermitian if and only if $\psi(x,y) = [Ax,y]$ is symmetric or, equivalently, $\hat{\psi}$ is real.

(ii) If A,B are hermitian and α,β real, then $\alpha A + \beta B$ is hermitian.

(iii) If A,B are hermitian, then AB is hermitian if and only if A and B commute.

(iv) Every $A \in B(H)$ can be put uniquely in the form $A = B + iC$, where B and C are hermitian.

(v) If A is hermitian, then $\|A\| = \sup_{\|x\|=1} |[Ax,x]|$.

(vi) A is normal if and only if $\|Ax\| = \|A^*x\|$, for each x in H.

(vii) A is unitary if and only if A is an isomorphism of H onto itself or, equivalently, is an onto isometric linear mapping.

1.3. Projections

Definition 1.3.1. Let S be a subspace of H. Then $S \oplus S^{\perp} = H$. Hence each x in H has a unique representation of the form $x = x_1 + x_2$, $x_1 \in S$, $x_2 \in S^{\perp}$. Define $Px = x_1$. Then P is called the *projection* of H on S.

Theorem 1.3.2. The projection P on the subspace S is an idempotent hermitian operator. Unless $S = \{0\}$, $\|P\| = 1$ and, when $S = \{0\}$, $P = 0$.

Theorem 1.3.3. Let P be the projection on S. Then:

(i) $\{z : Pz = z\}$ = The range of P = S.

(ii) $z \in S$ if and only if $\|Pz\| = \|z\|$.

(iii) For $x \in H$, $[Px,x] = \|Px\|^2$.

(iv) I-P is the projection on $S^{\perp}$.

(v) An operator P is a projection if and only if P is hermitian and idempotent.

Theorem 1.3.4. If S_1, S_2 are subspaces of H with P_1 and P_2 the respective projections on them, then the following are equivalent:

(i) $S_1 \perp S_2$;

(ii) $P_1P_2 = 0$;

(iii) $P_2P_1 = 0$;

(iv) $P_2(S_1) = 0$;

(v) $P_1(S_2) = 0$.

Definition 1.3.5. If P_1 and P_2 are two projections of H with $P_1P_2 = 0$, then P_1 and P_2 are said to be *orthogonal* to each other. In symbols, we write $P_1 \perp P_2$.

Theorem 1.3.6. If P_1, P_2 are projections on S_1 and S_2, respectively, then $P_1 + P_2$ is a projection if and only if $P_1 \perp P_2$. In that case, $P = P_1 + P_2$ is the projection on $S_1 \oplus S_2$.

This theorem extends to any orthogonal family of projections. To state the extension precisely, we need the following concept.

Definition 1.3.7. Let $\{x_\alpha\}_{\alpha \varepsilon A}$ be a family of vectors in H. We say that $\{x_\alpha\}_{\alpha \varepsilon A}$ is *summable with sum* x in H, if, for each $\varepsilon > 0$, there is a finite set $J_0 \subset A$ such that $\|\sum_{\alpha \varepsilon J_1} x_\alpha - x\| < \varepsilon$ for each finite subset J_1 of A containing J_0. In symbols, we write $\sum_{\alpha \varepsilon A} x_\alpha = x$.

Theorem 1.3.8. (Cauchy criterion) $\{x_\alpha\}_{\alpha \varepsilon A}$ is summable in H if and only if, for each $\varepsilon > 0$, there is a finite subset J_0 of A such that

$\|\sum_{\alpha \in J} x_\alpha\| < \varepsilon$, for all finite subsets J of A disjoint with J_0. Consequently, if $\{x_\alpha\}_{\alpha \in A}$ is summable in H, then all but a countable number of the x_α vanish.

Theorem 1.3.9.

(i) If $\{x_\alpha\}_\alpha$ is an orthogonal family of vectors in H, then $\{x_\alpha\}_\alpha$ is summable if and only if $\Sigma \|x_\alpha\|^2 < \infty$; and, if x is the sum, then $\|x\|^2 = \Sigma \|x_\alpha\|^2$.

(ii) (Bessel's inequality) If $\{x_\alpha\}_\alpha$ is an orthonormal family of vectors in H, then, for each x in H, $\Sigma |[x,x_\alpha]|^2 \leq \|x\|^2$.

(iii) If $\Sigma x_\alpha = x$, then $[x,y] = \Sigma [x_\alpha,y]$; $[y,x] = \Sigma [y,x_\alpha]$.

(iv) If $\{x_\alpha\}_\alpha$ is a maximal orthonormal family in H, then it is called an *orthonormal basis*. For such $\{x_\alpha\}_\alpha$ and for $x \in H$, we have $x = \Sigma [x,x_\alpha] x_\alpha$. The converse is also true. Further, $\|x\|^2 = \Sigma |[x,x_\alpha]|^2$, for each $x \in H$ (Parseval's identity). Orthonormal bases are also called *complete orthonormal systems*.

(v) Each Hilbert space has an orthonormal basis and all the orthonormal bases of a Hilbert space have the same cardinality. This cardinality is called the *dimensión* of the Hilbert space. We write dim H for dimension of H.

(vi) If $\{x_\alpha\}_\alpha$ is a complete orthonormal system, then, for x,y in H, $[x,y] = \Sigma_\alpha [x,x_\alpha][x_\alpha,y]$.

(vii) If $\{x_\alpha\}_\alpha$ is a complete orthonormal system, and if $x \perp x_\alpha$ for each α, then $x = 0$ and, conversely.

Definition 1.3.10. Let $\{S_\alpha\}_{\alpha \in A}$ be a family of linear manifolds of H. We define $\sum_{\alpha \in A} S_\alpha = \{x = \sum_{\alpha \in A} x_\alpha, x_\alpha \in S_\alpha$. i.e., $\{x_\alpha\}_{\alpha \in A}$ is summable with sum $x\}$.

Clearly, $\sum_{\alpha \in A} S_\alpha$ is a linear manifold.

Theorem 1.3.11.

(i) If $\{S_\alpha\}_{\alpha \in A}$ is a family of linear manifolds of H and $\Sigma S_\alpha = S$, then

$\bar{S} = \overline{\sum_{\alpha \in A} S_\alpha} = \overline{(\bigcup_{\alpha \in A} S_\alpha)}$, where $(\bigcup_{\alpha \in A} S_\alpha)$ denotes the linear manifold spanned by $\bigcup_{\alpha \in A} S_\alpha$.

(ii) If $\{S_\alpha\}_{\alpha \in A}$ is an orthogonal family of subspaces of H, then $\sum_{\alpha \in A} S_\alpha$ is a subspace and consequently, $\overline{(\bigcup_{\alpha \in A} S_\alpha)} = \sum S_\alpha$.

Thus, in case (ii) of Theorem 1.3.11, we denote $\sum_{\alpha \in A} S_\alpha$ by $\sum_{\alpha \in A} \oplus S_\alpha$ and call it the *direct sum* or *Hilbert sum* of $\{S_\alpha\}_{\alpha \in A}$. Clearly, $\sum_{\alpha \in A} \oplus S_\alpha = \{x = \sum_{\alpha \in A} x_\alpha, x_\alpha \in S_\alpha$ (so that $\sum_{\alpha \in A} \| x_\alpha \|^2 < \infty)\}$.

Definition 1.3.12. If $\{A_\alpha\}_{\alpha \in I}$ is a family of operators in B(H) and A is in B(H), we say that $\{A_\alpha\}_{\alpha \in I}$ is *summable* to A if, for each $x \in H$, $\{A_\alpha x\}_{\alpha \in I}$ is summable to Ax. Then we write $\sum_{\alpha \in I} A_\alpha = A$.

Theorem 1.3.13. The family $\{P_\alpha\}_{\alpha \in I}$ of projections is summable to a projection P if and only if $P_\alpha \perp P_\beta$, for $\alpha \neq \beta$ in I. In that case, P is the projection of H on $\sum_{\alpha \in I} \oplus P_\alpha(H)$.

Theorem 1.3.14.

(i) If P_1 and P_2 are two commuting projections, then $P = P_1 + P_2 - P_1P_2$ is the projection on $\overline{(S_1 \cup S_2)}$, where $P_iH_i = S_i$, i= 1,2.

(ii) If P_1 and P_2 are two commuting projections, then P_1P_2 is the projection on $S_1 \cap S_2$, with S_1 and S_2 as in (i).

Definition 1.3.15. If A and B are two operators on H, we say $A \leq B$ if $[Ax,x] \leq [Bx,x]$, for each x in H.

Theorem 1.3.16. The following statements are equivalent for projections P_i, i=1,2.

(i) $P_1 \leq P_2$.

(ii) $\|P_1x\| \leq \|P_2x\|$, for each $x \in H$.

(iii) $P_1(H) \subset P_2(H)$.

(iv) $P_2P_1 = P_1$.

(v) $P_1P_2 = P_1$.

Theorem 1.3.17. For two projections P_1 and P_2 of H, $P_2 - P_1$ is a projection if and only if $P_1 \leq P_2$. In that case $P_2 - P_1$ is the projection on $P_2(H) \cap (P_1(H))^{\perp}$.

Theorem 1.3.18. The class of all subspaces of H is a complete lattice under the p.o. (i.e., partial ordering) $S_1 \leq S_2$ if $S_1 \subset S_2$. Consequently, the class of all projections of H is a complete lattice L, with $P_1 \leq P_2$ if $P_1P_2 = P_1$. Then, for two commuting projections P_1 and P_2, $P_1 \vee P_2 = P_1 + P_2 - P_1P_2$ and $P_1 \wedge P_2 = P_1P_2$. If $\{P_\alpha\}_{\alpha \in A}$ is an orthogonal family of projections of H, then $\bigvee_{\alpha \in A} P_\alpha = \Sigma P_\alpha$ and $(\bigvee_{\alpha \in A} P_\alpha)(H) = \Sigma \oplus (P_\alpha(H))$.

Theorem 1.3.19. A collection B of commuting projections of H satisfies the following distributive law:

Suppose P is a projection of H and $\{P_j\}_j$ is a family of commuting projections of H. If $PP_j = P_jP$ for all j, then

$$P \wedge (\bigvee_j P_j) = \bigvee_j (P \wedge P_j).$$

1.4. Spectral measures and self-adjoint operators

Definition 1.4.1. A *spectral measure* on a measurable space (X,Σ) is a set function E(.) defined on Σ with values in projections of H and satisfies the following requirements:

(i) $E(\emptyset) = 0$, $E(X) = I$. (I denotes the identity operator.)

(ii) If $\{M_n\}$ is a disjoint sequence of sets in Σ, then

$$E(\bigcup_1^\infty M_n)x = \sum_1^\infty E(M_n)x, \text{ for each } x \in H.$$

The spectral measure $E(.)$ is called a *complex spectral measure* if $X = \mathbb{C}$ and Σ is the σ-algebra of Borel sets of $\mathbb{C}$.

Theorem 1.4.2. If $E(.)$ is a spectral measure on (X,Σ), then $E(M \cap N) = E(M)E(N)$ and the range of $E(.)$ is a σ-complete Boolean algebra of projections of H.

Theorem 1.4.3. A projection valued set function $E(.)$ on (X,Σ) is a spectral measure if and only if (i) $E(X)= I$ and (ii) for each pair x,y in H, $[E(.)x,y]$ is a countably additive set function on Σ or, equivalently, if and only if (i) $E(X)= I$ and (ii') for each $x \in H$, $[E(.)x,x]$ is a measure on Σ.

Let $E(.)$ be a spectral measure on (X,Σ). Let f be a simple function in the sense that $f = \sum_1^n \alpha_i \chi_{M_i}$, $M_i \in \Sigma$, $\alpha_i \in \mathbb{C}$. We define $\int_X f dE(\lambda) = \sum_1^n \alpha_i E(M_i)$. If f is a bounded Σ-measurable scalar function, then $f= (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$, where $f_1=$ Ref and $f_2=$ Imf. Then there exist sequences of simple functions $f_n^{(i)+}$, $f_n^{(i)-} \geq 0$ converging pointwise respectively to f_i^+ and f_i^-, $i = 1,2$. Then $(\lim_n \int_X f_n^{(1)+} dE - \lim_n \int_X f_n^{(1)-} dE) + i(\lim_n \int_X f_n^{(2)+} dE - \lim_n \int_X f_n^{(2)-} dE)$ exists in the norm topology of B(H) and is well-defined. This limit is denoted by $\int_X f dE$ or $\int_X f(\lambda)dE(\lambda)$. Note that $\int_X f\, dE \in B(H)$.

Theorem 1.4.4. Let f be a bounded Σ-measurable complex function on (X,Σ). Then

$$[\int_X f(\lambda)dE(\lambda)x,y] = \int_X f(\lambda)d[E(\lambda)x,y]$$

and

$$\| \int_X f(\lambda)dE(\lambda)\| \leq \sup_{\lambda \in X} |f(\lambda)|.$$

Theorem 1.4.5. If $E(.)$ is a spectral measure on (X,Σ) and if f,g are Σ-measurable bounded complex functions on (X,Σ), then, for $\alpha \in \mathbb{C}$, the following hold:

(i) $\int_X \alpha f dE = \alpha \int_X f dE$;

(ii) $\int_X (f + g)dE = \int_X f dE + \int_X g dE$;

(iii) $\int_X \bar{f} dE = (\int_X f dE)^*$;

(iv) $\int_X fgdE = (\int_X fdE)(\int_X gdE)$.

(v) If $B \in B(H)$ and if B commutes with $\int_X fdE$, then $E(M)B = BE(M)$ for each $M \in \Sigma$.

(vi) All such operators $\int_X fdE$ are normal and commute among themselves.

Definition 1.4.6. Let $A \in B(H)$. Then the *spectrum* $\sigma(A)$, the *point spectrum* $\sigma_p(A)$, the *continuous spectrum* $\sigma_c(A)$, the *residual spectrum* $\sigma_r(A)$ and the *resolvent set* $\rho(A)$ are defined as follows:

(i) $\sigma(A) = \{\lambda : (\lambda I - A)^{-1} \notin B(H)\}$.

(ii) $\sigma_p(A) = \{\lambda : (\lambda I - A)^{-1}$ does not exist $\}$.

(iii) $\sigma_c(A) = \{\lambda : (\lambda I - A)^{-1}$ exists with dense domain but not continuous$\}$.

(iv) $\sigma_r(A) = \{\lambda : (\lambda I - A)^{-1}$ exists with domain not dense in $H\}$.

(v) $\rho(A) = \mathbb{C} \setminus \sigma(A)$.

$\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$, $\rho(A)$ are pairwise disjoint and $\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$.

Theorem 1.4.7. Let $A \in B(H)$. Then:

(i) $\sigma(A)$ is non-void and compact.

(ii) If A is hermitian, $\sigma(A)$ is real, $\sigma(A) \subset [-\|A\|, \|A\|]$, and $\|A\| = \sup_{\lambda \in \sigma(A)} |\lambda|$; $\sigma(A) \subset [m,M]$, where $m = \inf_{\|x\|=1} [Ax,x]$ and $M = \sup_{\|x\|=1} [Ax,x]$.

(iii) If A is normal, then $\sigma_r(A) = \emptyset$.

(iv) If A is unitary, then $\sigma(A) \subset \{\lambda : |\lambda| = 1\}$.

Theorem 1.4.8. (The spectral theorem) Let A be normal. Then there is a unique complex spectral measure $E(.)$ on $\mathbb{C}$ with $E(\sigma(A)) = I$, called the resolution of the identity of A, such that $A = \int_{\mathbb{C}} \lambda dE = \int_{\sigma(A)} \lambda dE$. If A is unitary, $A = \int_{|\lambda|=1} \lambda dE$. If A is hermitian, then $E(.)$ is a real spectral measure with $E(\sigma(A)) = I$ and $A = \int_m^M \lambda dE$ =

$\int_{\sigma(A)} \lambda dE$, where m and M are as in Theorem 1.4.7.(ii). In the latter case, if $E_\lambda = E((-\infty,\lambda])$, then $\{E_\lambda : \lambda \in \mathbf{R}\}$ is called the *spectral family* of the hermitian operator A, and $\{E_\lambda\}_{\lambda \in \mathbf{R}}$ is continuous on the right in the sense that if $\lambda \leq \mu$, then $E_\mu x \to E_\lambda x$ as $\mu \to \lambda^+$, for each $x \in H$.

Theorem 1.4.9. If A and B are two commuting normal operators with resolutions of the identity $E_1(.)$ and $E_2(.)$, respectively, then $E_1(\sigma)E_2(\delta) = E_2(\delta)E_1(\sigma)$ for all Borel sets σ and δ of $\mathbb{C}$.

Definition 1.4.10. A linear transformation A with domain a linear manifold $\mathcal{D}(A)$ in H and range in H is said to be *closed* if, whenever $\{x_n\}_1^\infty \subset \mathcal{D}(A)$ converges to a vector x in H such that $Ax_n \to y$ for some $y \in H$, then $x \in \mathcal{D}(A)$ and $Ax = y$.

The graph of A, denoted by Γ_A, is defined as the set $\{(x,Ax) : x \in \mathcal{D}(A)\} \subset H \oplus H$. Obviously, A is closed if and only if Γ_A is closed in $H \oplus H$.

Definition 1.4.11. If A is a linear transformation with domain $\mathcal{D}(A)$ in H and range in H and if $\overline{\Gamma}_A$ (the closure of the graph of A in $H \oplus H$) is the graph of some linear transformation $\tilde{A}$, then $\tilde{A}$ is called the *closure* of A and in that case, A is said to admit *closure* or to be *preclosed*.

Clearly $\tilde{A}$ is the minimal closed extension of A, when A is preclosed.

Definition 1.4.12. Let A be a linear transformation with domain $\mathcal{D}(A)$ dense in H and range in H. For each $x \in \mathcal{D}(A)$ and for a given vector $y \in H$, if the representation $[Ax,y] = [x,z]$ holds, then we denote by $\mathcal{D}^*$ the set of all such vectors y in H and define the map $A^* : \mathcal{D}^* \to H$ by $A^*y = z$. Since $\mathcal{D}(A)$ is dense in H, A^* is well-defined and A^* is also linear as A is linear. A^* is called the *adjoint of A*. For the proof of the the following lemma the reader may refer to § 5, Naimark [Na].

Lemma 1.4.13. It is assummed that the linear transformations A,B on H have dense domains. If $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $Ax = Bx$ for $x \in \mathcal{D}(A)$, we write $A \subset B$. Then:

(i) If A^{-1} exists and $\mathcal{D}(A^{-1})$ is dense in H, then $(A^{-1})^* = (A^*)^{-1}$;

(ii) $(\lambda A)^* = \bar{\lambda} A^*$, $\lambda \in \mathbb{C}$;

(iii) If $A \subset B$, then $A^* \supset B^*$.

(iv) $(A + B)^* \supset A^* + B^*$.

(v) $(AB)^* \supset B^* A^*$.

(vi) $(A + \lambda I)^* = A^* + \bar{\lambda} I$.

Lemma 1.4.14. If the linear transformation A with dense domain in H and range in H has the closure $\tilde{A}$, then:

(i) $\tilde{A}^* = A^*$;

(ii) $\mathcal{D}(A^*)$ is dense in H;

(iii) $A^{**} = \tilde{A}$; in particular, if A is closed, then $A^{**} = A$;

(iv) A^* is a closed operator for any A with dense domain in H.

Proof. Let U be a a linear operator on $H \oplus H$, given by

$$U(x,y) = (iy, -ix).$$

Then U is clearly an isomorphism of $H \oplus H$ onto itself.

Let Γ_A be the graph of A. Let $U\Gamma_A = \Gamma'_A$.

Then

$$\Gamma_{A^*} = (H \oplus H) \ominus \Gamma'_A. \qquad (1.4.14.1)$$

To prove (1.4.14.1), the orthogonal complement of Γ'_A consists of those and only those pairs (y,z) satisfying the equation

$$[(y,z), (iAx, -ix)] = 0$$

for all $x \in \mathcal{D}(A)$; i.e., satisfying

$$[y, iAx] - [z, ix] = 0;$$

i.e., satisfying

$$[Ax,y] = [x,z].$$

This is equivalent to saying that $y \in \mathcal{D}(A^*)$, $A^*y = z$ and $(y,z) \in \Gamma_{A^*}$. Hence (1.4.14.1) holds.

Consequently, from (1.4.14.1) we also obtain that Γ_{A^*} is closed and hence A^* is always a closed operator. Thus (iv) holds.

Since $\Gamma_{\tilde{A}} = \overline{\Gamma}_A$, we have $\Gamma'_{\tilde{A}} = U\,\overline{\Gamma}_A = \overline{U\Gamma_A} = \overline{\Gamma}'_A$

and therefore from (1.4.14.1) we have

$$\Gamma_{\tilde{A}^*} = (H \oplus H) \ominus \Gamma'_{\tilde{A}} = (H \oplus H) \ominus \Gamma'_A = \Gamma_{A^*}.$$

Thus $\tilde{A}^* = A^*$. This proves (i) of the lemma.

Again from (1.4.14.1), we have

$$\overline{\Gamma}'_A = (H \oplus H) \ominus \Gamma_{A^*}.$$

Clearly, $U^{-1}\,\overline{\Gamma}'_A = \overline{\Gamma}_A$. Besides, $U^{-1}\,\Gamma_{A^*} = \Gamma'_{A^*}$, since $U^{-1}(x,A^*x) = [iA^*x, -ix] = U(x,A^*x)$.

Hence by applying U^{-1} on both sides of the last equation, we get

$$\overline{\Gamma}_A = (H \oplus H) \ominus \Gamma'_{A^*}. \qquad (1.4.14.2)$$

If follows from (1.4.14.2) that $\mathcal{D}(A^*)$ is dense in H. In fact, otherwise, there would exist a non-zero vector $z \in H$ which is orthogonal to $\mathcal{D}(A^*)$. Then

$U^{-1}(z,o) \perp \Gamma'_{A^*}$; i.e. $(0,-iz) \in \overline{\Gamma}_A$. But, as $\overline{\Gamma}_A = \Gamma_{\tilde{A}}$, $(o,-iz) \in \Gamma_{\tilde{A}}$. Since $\tilde{A}$ is linear, $z = o$, and hence a contraction. Thus (ii) of the lemma holds.

Thus (1.4.14.1) and (1.4.14.2) imply that $\overline{\Gamma}_A = \Gamma_{A^{**}}$; on the other hand, $\overline{\Gamma}_A = \Gamma_{\tilde{A}}$, as A admits closure. Thus $\tilde{A} = A^{**}$. This proves (iii) of the lemma.

Definition 1.4.15. A linear transformation with domain a linear manifold $\mathcal{D}(A)$ in H and with range in H is called an operator, and it is said to be *hermitian* if $[Ax,y] = [x,Ay]$, for all $x,y \in \mathcal{D}(A)$. A hermitian operator with domain dense in H is said to be *symmetric*. An operator A with dense domain in H is said to be *self-adjoint* if $A = A^*$. Thus an operator may be bounded or unbounded. The situation will be clear from the context.

Clearly, an operator A on H with dense domain is symmetric if and only if $A \subset A^*$.

Lemma 1.4.16.

(a) If A is a self-adjoint operator on H, then, for α,β real, the operator $\alpha A + \beta I$ is also self-adjoint.

(b) A symmetric operator A on H whose range R_A coincides with H is a self-adjoint operator.

Proof.

(a) Trivial.

(b) It suffices to show that $\mathcal{D}(A) = \mathcal{D}(A^*)$. As A is symmetric, $\mathcal{D}(A) \subset \mathcal{D}(A^*)$. Let $y \in \mathcal{D}(A^*)$ and let $z = A^*y$. Since $R_A = H$, there is a vector y' in $\mathcal{D}(A)$ such that $z = Ay'$. Now, for arbitrary $x \in \mathcal{D}(A)$, we have

$$[Ax,y] = [x,A^*y] = [x,z] = [x,Ay'] = [Ax,y']$$

and hence $y = y'$, as $R_A = H$. Thus $y \in \mathcal{D}(A)$. Hence $A = A^*$.

Definition 1.4.17. An operator A on H is said to be *positive definite* if $[Ax,x] \geq 0$, for all $x \in \mathcal{D}(A)$.

Lemma 1.4.18. If A is a closed operator on H with dense domain, then A^*A is a positive definite self-adjoint operator on H.

Proof. For $x \in \mathcal{D}(A^*A)$, $[A^*Ax,x] = [Ax,A^{**}x] = [Ax,Ax] \geq 0$ by Lemma 1.4.14(iii). Thus A^*A is positive definite.

We presently show that A^*A is self-adjoint . From equation(1.4.14.1) we have

$$\Gamma_A' \oplus \Gamma_{A^*} = H \oplus H. \qquad (1.4.18.1)$$

Hence the vector $(0,-ix)$ of $H \oplus H$ can be written in the form $(0,-ix) = (iAy,-iy) + (z, A^*z)$, $y \in \mathcal{D}(A)$, $z \in \mathcal{D}(A^*)$; i.e., $0 = iAy + z$, $-ix = -iy + A^*z$
$= -iy - iA^*Ay.$

Therefore it follows that $x = (I + A^*A)y$, so that the range of $I + A^*A$ is H. To show that A^*A is self-adjoint, in view of Lemma 1.4.16, it suffices to show that $I + A^*A$ is symmetric.

Obviously, $I + A^*A$ is hermitian. So, it suffices to show that $\mathcal{D}(I + A^*A)$ is dense in H. Let $x_0 \perp \mathcal{D}(I + A^*A)$. By what has been proved above, $x_0 = (I + A^*A)y_0$, for some $y_0 \in \mathcal{D}(A)$, so that $[(I + A^*A)y_0,y] = 0$, for all $y \in \mathcal{D}(I + A^*A)$. In particular, taking $y = y_0$, we obtain

$$0 = [(I + A^*A)y_0,y_0] = \|y_0\|^2 + \|Ay_0\|^2$$

which means $y_0 = 0$, and hence $x_0 = (I + A^*A)y_0 = 0$.

This completes the proof of the lemma.

Theorem 1.4.19. (The spectral theorem for arbitrary self-adjoint operators). For every self-adjoint operator A on the Hilbert space H with dense domain $\mathcal{D}(A)$, there exists a unique spectral measure $E(.)$ on the Borel sets of the real line with the following properties:

(i) If $E_\lambda = E((-\infty,\lambda])$, then $E_\lambda E_\mu = E_\lambda$ for $\lambda \leq \mu$; (E_λ is called the *spectral family* of A.)

(ii) $E(.)$, and hence E_λ, commutes with every operator $T \in B(H)$ which commutes with A.

(iii) $\lim_{\lambda\to-\infty} E_\lambda x = 0$, $\lim_{\lambda\to+\infty} E_\lambda x = x$, for each $x \in H$.

(iv) $E_\lambda x$ is a function which is continuous on the right for arbitrary $x \in H$.

(v) $x \in \mathcal{D}(A)$ if and only if $\int_{-\infty}^{\infty} \lambda^2 d\|E(\lambda)x\|^2 < \infty$ and, in that case, $Ax = \int_{-\infty}^{\infty} \lambda dE(\lambda)x$, $x \in \mathcal{D}(A)$.

Definition 1.4.20. An operator T on H is said to be *normal* if T is closed, densely defined and $TT^* = T^*T$.

Theorem 1.4.21. Let T be an unbounded closed densely defined operator on H. The following are equivalent:

(a) T is normal.

(b) $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $\|Tx\| = \|T^*x\|$ for each $x \in \mathcal{D}(T)$.

(c) There is a spectral measure E(.) on the Borel sets of $\mathbb{C}$ such that $\mathcal{D}(T) = \{x: \int_{\mathbb{C}} \lambda\, dE(\lambda)x \text{ exists}\}$ and $Tx = \int_{\mathbb{C}} \lambda dE(\lambda)x$, $x \in \mathcal{D}(T)$.

When T is normal, the spectral measure E(.) is unique, and E(.) is called the resolution of the identity of T. For such T, $\mathcal{D}(T)$ is also given by $\{x: \int_{\mathbb{C}} |\lambda|^2 \|dE(\lambda)x\|^2 < \infty\}$.

Lemma 1.4.22. If A is a positive definite self-adjoint operator on H, then there exists a unique positive definite self-adjoint operator P on H such that $P^2 = A$.

Proof. Suppose E(.) is the resolution of the identity of A. Since A is positive definite, $E_\lambda = 0$ for $\lambda < 0$. Let $\mathcal{D}(P) = \{x: \int_0^\infty \lambda\, d\|E(\lambda)x\|^2 < \infty\}$ and set

$$Px = \int_0^\infty \sqrt{\lambda}\, dE(\lambda)x$$

for $x \in \mathcal{D}(P)$. For $x \in \mathcal{D}(A)$,

$$Ax = \int_0^\infty \lambda\, dE(\lambda)x.$$

Hence

$$\infty > [Ax,x] = [\int_0^\infty \lambda dE(\lambda)x, x]$$
$$= \int_0^\infty \lambda d[E(\lambda)x,x]$$

$$= \int_0^\infty \lambda \, d\|E(\lambda)x\|^2$$

and hence $x \in \mathcal{D}(P)$. Thus $\mathcal{D}(P)$ is dense in H. For $x \in \mathcal{D}(P)$,

$$P^2x = \int_0^\infty \lambda \, dE(\lambda)x = Ax$$

and hence $\mathcal{D}(P) = \mathcal{D}(A)$ and $P^2 = A$. Clearly, P is a positive definite self-adjoint operator. The uniqueness of P follows from the uniqueness of the resolution of the identity E(.) of A.

We shall denote P by $A^{\frac{1}{2}}$.

1.5.Banach algebras

Theorem 1.5.1. B(H) is a Banach algebra with identity under the operator norm. It is a B*-algebra.

Definition 1.5.2. If $x \in A$, a Banach algebra with identity e, the resolvent set $\rho(x) = \{\lambda : (\lambda e - x)^{-1} \in A\}$. $\mathbb{C} \setminus \rho(x)$ is called the *spectrum* of x and is denoted by $\sigma(x)$.

Theorem 1.5.3. If A is a Banach algebra with identity e, then $\sigma(x) \neq \emptyset$, for each $x \in A$; and $\sigma(x)$ is compact. Further, $\max\{|\lambda| : \lambda \in \sigma(x)\} = r(x) = \lim_n \|x^n\|^{1/n}$, called the *spectral radius* of x. If p is a polynomial, then $\sigma(p(x)) = \{p(\lambda) : \lambda \in \sigma(x)\}$. (The last result is known as the *spectral mapping theorem*.)

Definition 1.5.4. An element x in a B*-algebra with identity e is called *hermitian* if $x^* = x$; *normal* if $x^*x = xx^*$; *unitary* if $x^*x = xx^* = e$.

It is known that, for an element x in a B*-algebra with identity, $\sigma(x)$ is real if x is hermitian; and $\|x\| = r(x)$ if x is normal.

Theorem 1.5.5. If A is a commutative division Banach algebra, then A is isometrically isomorphic to the Banach algebra of all complex numbers.

Theorem 1.5.6. (Gelfand-Naimark) If A is a commutative Banach algebra with identity, then its maximal ideal space $\mathfrak{M}$ is a compact Hausdorff space under the weak topology induced by the functionals $\{\hat{x}: x \in A\}$, where $\hat{x}(M)= x(M)$ given by the quotient map: $A \to A/M \cong \mathbb{C}: x \to x(M)= x+M \in \mathbb{C}$, for $M \in \mathfrak{M}$. If, further, A is a B*-algebra, then A is isometrically isomorphic to $C(\mathfrak{M})$ under the mapping: $x \to \hat{x}$ and $\|x\|=\|\hat{x}\| = \sup_{M \in \mathfrak{M}} |\hat{x}(M)|$. Also $\sigma(x)=\{\hat{x}(M): M \in \mathfrak{M}\}$.

Theorem 1.5.7. (Functional Calculus) Suppose x is a hermitian element of a B*-algebra A with identity and $C(\sigma(x))$ is the B*-algebra of all continuous complex functions on the spectrum $\sigma(x)$ of x. Then there is a unique mapping $f \to f(x): C(\sigma(x)) \to A$ such that the following hold:

(i) f(x) has its elementary meaning when f is a polynomial.

(ii) $\|f(x)\| = \|f\|$, for $f \in C(\sigma(x))$.

(iii) $(f + g)(x)= f(x) + g(x)$;

(iv) $(fg)(x)= f(x)g(x)$;

(v) $\bar{f}(x)= f(x)^*$.

(vi) f(x) is normal; and is hermitian if f is real.

(vii) $f(x)y= y f(x)$ for all $y \in A$ for which $yx= xy$ holds.

Theorem 1.5.8. If A is a B*-algebra with identity e, B is a closed *-subalgebra of A containing e and if $x \in B$, then $\sigma_A(x)= \sigma_B(x)$ where $\sigma_A(x)$ and $\sigma_B(x)$ denote the spectrum of x with respect to A and B, respectively.

Theorem 1.5.9. (Extended spectral mapping theorem) If x is hermitian in a B*-algebra A with identity, and if $f \in C(\sigma(x))$, then

$$\sigma(f(x)) = \{f(\lambda): \lambda \in \sigma(x)\} .$$

Theorem 1.5.10. Each element x in a B*-algebra A with identity is a finite linear combination of unitary elements of A.

Proof . It suffices to consider the case in which $x = x^*$ and $\|x\| \leq 1$. Then $\sigma(x) \subset [-1,1]$ and we can define f in $C(\sigma(x))$ by $f(\lambda)= \lambda + i(1-\lambda^2)^{\frac{1}{2}}$. Obviously $u= f(x) \in A$ and satisfies $x = \dfrac{u + u^*}{2}$; and u is unitary in A.

Definition 1.5.11. By a homomorphism from a B^*-algebra A with identity into a B^*-algebra B with identity we mean a linear, multiplicative and adjoint preserving mapping ψ from A into B, which carries the identity of A onto the identity of B. In particular, if ψ is further one-one, then we call ψ an isomorphism of A onto $\psi(A)$.

Theorem 1.5.12. Suppose that A and B are B^*-algebras with identity and ψ is a homomorphism from A into B. Then, for each $x \varepsilon A$, we have the following:

(i) $\sigma(\psi(x)) \subset \sigma(x)$;

(ii) $\|\psi(x)\| \leqq \|x\|$;

(iii) If $x = x^*$ and $f \varepsilon C(\sigma(x))$, then $\psi(f(x)) = f(\psi(x))$;

(iv) If ψ is an isomorphism, then $\|\psi(x)\| = \|x\|$.

CHAPTER 2

BASIC PROPERTIES OF VON NEUMANN ALGEBRAS

§2.1. Some topologies on B(H)

Throughout this chapter H denotes a Hilbert space and B(H) denotes the B^*-algebra of all bounded operators on H. In addition to the norm topology τ_n on B(H), we need some more topologies. These topologies are discussed in detail in this section.

Suppose E is a vector space over $\mathbb{K}$(= $\mathbb{R}$ or $\mathbb{C}$). A mapping $p:E \to \mathbb{R}^+$ is called a *semi-norm* on E if $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = |\alpha| p(x)$, for all x,y in E and $\alpha \in \mathbb{K}$. If Γ is a set of semi-norms on E, then there is a topology τ on E for which the sets of the form $V(x_0;p_1,\ldots,p_n;\varepsilon) = \{x : x \in E,\ p_i(x - x_0) < \varepsilon,\ i = 1,2,\ldots,n\}$ (with $p_1,\ldots,p_n \in \Gamma$, $\varepsilon > 0$) constitute a neighbourhood basis at $x_0 \in E$. The topology τ is Hausdorff if and only if $\{x \in E : p(x) = 0 \text{ for all } p \in \Gamma\} = \{0\}$. If τ is Hausdorff and if Γ consists of a single semi-norm, then τ is the usual norm topology. For the topology τ induced by the semi-norms of Γ, (E,τ) is a *locally convex space*.

Notation 1. For a subset K of H, let [K] be the subspace spanned by K. Then, for a finite subset K of H, [K] is the same as the linear manifold spanned by K. Moreover, $P_{[K]}$ denotes the projection on the subspace [K].

§2.1(A). The strong operator topology τ_s

For a fixed vector $x \in H$, let $p_x(T) = \|Tx\|$, for $T \in B(H)$. Then, clearly, p_x is a semi-norm on $B(H)$. The family $\{P_x : x \in H\}$ induces a locally convex Hausdorff topology τ_s on $B(H)$, called the *strong operator topology* or simply the *strong topology*. It is Hausdorff, because $p_x(T) = 0$ for each $x \in H$ implies $Tx = 0$ for each $x \in H$ and hence, $T = 0$.

Sets of the form $U = U(0; x_1, \ldots, x_n; \varepsilon) = \{T \in B(H) : (\sum_1^n \|Tx_i\|^2)^{\frac{1}{2}} < \varepsilon\}$, where

$x_1, x_2, \ldots, x_n \in H$ and $\varepsilon > 0$, form a τ_s-neighbourhood basis at 0. In fact, $V(0; p_{x_1}, \ldots, p_{x_n}; \delta) = \{T : \|Tx_i\| < \delta,\ i = 1, 2, \ldots, n\} \subset U$ if $\delta < \varepsilon/\sqrt{n}$ and $U \subset V(0; p_{x_1}, \ldots, p_{x_n}; \varepsilon)$. This establishes our claim.

Notation 2. $U(0; x_1, \ldots, x_n; \varepsilon) = \{T \in B(H) : \sum_1^n \|Tx_i\|^2 < \varepsilon^2\}$ and $V(0; x_1, \ldots, x_n; \varepsilon) = \{T \in B(H) : \|Tx_i\| < \varepsilon,\ i = 1, \ldots, n\}$.

In terms of convergence, $T_\alpha \to T$ (in τ_s) in $B(H)$ if and only if $\|T_\alpha x - Tx\| \to 0$, for each x in H.

Lemma 2.1.1. With a fixed S in $B(H)$, the mappings

(i) $T \to ST : B(H) \to B(H)$,

(ii) $T \to TS : B(H) \to B(H)$, are τ_s-continuous. If $S \in B(H)_1$ (= the unit ball of $B(H)$ = $\{T \in B(H) : \|T\| \leq 1\}$, then

(iii) $(S,T) \to ST : B(H)_1 \times B(H) \to B(H)$ is τ_s-continuous.

Proof. Let $T_\alpha \to T$ in τ_s-topology and let $x \in H$. Then:

(i) Clearly, $\|(ST_\alpha - ST)x\| \leq \|S\|\,\|(T_\alpha - T)x\| \to 0$. Hence (i) holds.

(ii) Proof is similar to that of (i).

(iii) If $S_\alpha \to S$ in τ_s in $B(H)_1$, then $\|(S_\alpha T_\alpha - ST)x\| \leq \|S_\alpha (T_\alpha - T)x\| + \|(S_\alpha - S)Tx\| \leq \|S_\alpha\|\,\|(T_\alpha - T)x\| + \|(S_\alpha - S)Tx\| \to 0$, since $\|S_\alpha\| \leq 1$.

Note 1. If H is infinite dimensional, (a) $T \to T^*: B(H)_1 \to B(H)_1$ is not continuous in τ_s-topology and (b) the mapping $(S,T) \to ST: B(H) \times B(H) \to B(H)$ is not continuous in τ_s-topology, as is shown in the following counterexamples.

(a) Let $\{e_n\}_1^\infty$ be an orthonormal basis in a separable infinite dimensional Hilbert space H. For $x \in H$, let $U_n(x) = [x, e_n] e_1$. Then $\|U_n x\| = |[x,e_n]| \to 0$ as $n \to \infty$, since $\|x\|^2 = \sum_1^\infty |[x,e_n]|^2$. Thus $U_n \to 0$ in τ_s. But, $[U_n^* e_1, x] = [e_1, U_n x] = [e_n, x]$, for each $x \in H$. Thus $U_n^* e_1 = e_n$, so that $\|U_n^* e_1\| = 1$, for each n, and hence $U_n^* \not\to 0$ in τ_s.

(b) With H as in (a), let $Ve_n = e_{n-1}$ $(n > 1)$ and $Ve_1 = 0$, and extend V linearly and continuously on H. Let $A_n = V^n (n = 1,2,\ldots)$. Then, $\|A_n x\|^2 = \|\sum_{i=1}^\infty [x,e_i] A_n e_i\|^2 = \sum_{i=n+1}^\infty |[x,e_i]|^2 < \varepsilon$, for $x \in H$, if n is sufficiently large. Thus $A_n \to 0$ in τ_s. Now,

$$[V^* e_n, x] = [e_n, Vx] = [e_n, \sum_{i=2}^\infty [x,e_i] e_{i-1}]$$

$$= \sum_{i=2}^\infty [e_n, e_{i-1}] \overline{[x,e_i]} = \overline{[x,e_{n+1}]} = [e_{n+1}, x]$$

so that $V^* e_n = e_{n+1}$. Hence $\|A_n^* x\|^2 = \|(V^*)^n x\|^2 = \|\sum_{i=1}^\infty [x,e_i] e_{i+n}\|^2 = \sum_{i=1}^\infty |[x, e_i]|^2 = \|x\|^2$, for $x \in H$. Thus $A_n^* \not\to 0$ in τ_s.

Let $x \in H$ with $\|x\| = 1$. Let $V(0; x, \varepsilon)$ be a τ_s-neighbourhood of 0, with $0 < \varepsilon < 1$. Further, let $A_{n,\delta} = \frac{1}{\delta} A_n$, $B_{n,\delta} = \delta A_n^*$ $(\delta > 0)$. Then

$$\|A_{n,\delta} B_{n,\delta} x\| = \|A_n A_n^* x\| = \|x\| = 1 > \varepsilon .$$

i.e. $A_{n,\delta} B_{n,\delta} \notin V(0; x; \varepsilon)$ for any n and δ. But, on the other hand, let $V(0; x_1, x_2, \ldots, x_k; \varepsilon_1)$ and $V(0; y_1, \ldots, y_m; \varepsilon_2)$ be arbitrary τ_s-neighbourhoods of 0. Let $0 < \delta < \varepsilon_2 / (\sum_1^m \|y_i\|^2)^{\frac{1}{2}}$. Then we have

$$\|B_{n,\delta}\, y_i\|^2 = \delta^2 \|A_n^* y_i\|^2 = \delta^2 \|y_i\|^2 < \varepsilon_2^2 ,$$

i.e., $\|B_{n,\delta}\, y_i\| < \varepsilon_2$, for any n; i.e., $B_{n,\delta} \varepsilon V(0;y_1,\ldots,y_m;\varepsilon_2)$. Now, as $A_{n,\delta} \to 0$ in τ_s (for the fixed δ chosen above), $\|A_{n,\delta} x_i\| < \varepsilon_1$, for i= 1,2,...,k, if $n \geq n_o(\varepsilon)$(say); i.e., $A_{n,\delta} \varepsilon V(0;x_1,\ldots,x_k;\varepsilon_1)$, for $n \geq n_o(\varepsilon)$. Thus, in any two strong neighbourhoods of 0, there exist operators $A_{n,\delta}$, $B_{n,\delta}$ such that their product $A_{n,\delta} B_{n,\delta}$ is not in the prescribed strong neighbourhood of 0. i.e., $(A,B) \to AB$: $B(H) \times B(H) \to B(H)$ is not continuous in τ_s at (0,0).

Note 2. $\tau_n = \tau_s$ on B(H) if and only if H is finite dimensional. If H is infinite dimensional, then τ_n is strictly finer than τ_s.

Since $U(0;x_1,\ldots,x_k;\varepsilon) = \{T: T \;\varepsilon\; B(H), (\sum_{i=1}^{k} \|Tx_i\|^2)^{\frac{1}{2}} < \varepsilon\} \supset \{T: \|T\| < \frac{\varepsilon}{(\sum_1^k \|x_i\|^2)^{\frac{1}{2}}}\}$,

$\tau_n \geq \tau_s$. If H is infinite dimensional, $\tau_n \neq \tau_s$ since the map $T \to T^*$ is continuous in τ_n, and not continuous in τ_s by Note 1.

If H is of dimension n, and if $\{e_i\}_1^n$ is an orthonormal basis in H, then $W = \{T: \|T\| < \varepsilon\} \supset U(0;e_1,\ldots,e_n;\ \varepsilon/2) = V$, since, for $T \;\varepsilon\; V$ and $x \varepsilon H$, $\|T^* x\|^2 = \sum_1^n |[T^*x, e_i]|^2 = \sum_1^n |[x, Te_i]|^2 < \frac{\varepsilon^2}{4} \|x\|^2$, so that $\|T\| = \|T^*\| \leq \varepsilon/2$. Thus $\tau_s \geq \tau_n$. Hence $\tau_s = \tau_n$ if H is finite dimensional.

The part of sufficiency in Note 2 is a particular case of the following:

If τ_1 and τ_2 are two Hausdorff topologies on a vector space E of finite dimension such that (E,τ_1) and (E, τ_2) are topological vector spaces, then $\tau_1 = \tau_2$. (See Theorem 1.21(a) of Rudin [R].)

Theorem 2.1.2. If H is separable, then $B(H)_1$, the unit ball of B(H), is metrizable in τ_s.

Proof. Let A be a countable dense subset of H. Let Γ be the family of semi-norms $p_x(T)= \|Tx\|$, for $x \in A$. Then Γ is countable. Since A is dense in H, Γ separates points. Moreover, the topology τ induced by Γ on B(H) is metrizable(see p.27 of [R]). Obviously, $\tau \leq \tau_s$ on B(H) and $B(H)_1$. Conversely, let $V(0; z_1,\ldots,z_n; \varepsilon)$ be a τ_s-neighbourhood of 0 in $B(H)_1$. Choose $x_i \in A$ such that $\|x_i - z_i\| < \varepsilon/2$, for $i = 1,2,\ldots,n$. Let $S \in B(H)_1 \cap V(0;x_1,\ldots,x_n; \frac{\varepsilon}{2})$, which is a τ-neighbourhood of 0 in $B(H)_1$. Then, $\|S z_i\| \leq \|S(z_i - x_i)\| + \|S x_i\| < \varepsilon$, so that $S \in V(0; z_1,\ldots,z_n;\varepsilon)$. Hence $\tau \geq \tau_s$ on $B(H)_1$.

Theorem 2.1.3. $B(H)_1$ is complete under τ_s.

Proof. Suppose $\{T_\alpha\}_\alpha$ is a Cauchy net in $B(H)_1$ for τ_s-topology. Then $\{T_\alpha x\}$ is Cauchy in H and hence $\lim T_\alpha x= Tx$ exists, for each x in H, as H is complete. Clearly, T is linear and $\|Tx\| = \lim_\alpha \|T_\alpha x\| \leq \lim_\alpha \sup \|T_\alpha\| \|x\| \leq \|x\|$, as $\|T_\alpha\| \leq 1$ for all α. Thus $T \in B(H)_1$ and hence $B(H_1)$ is complete under τ_s.

Note 3. B(H) is topologically complete under τ_s in the sense that every closed and totally bounded set $S \subset (B(H),\tau_s)$ is compact.

Note 4. When H is infinite dimensional, B(H) is not first countable for the τ_s-topology even if H is separable, as we see below.

Let H be separable with an orthornormal basis $\{e_i\}_1^\infty$. Let $A_{m,n}= P_{[e_m]} + mP_{[e_n]}$ and $S= \{A_{m,n}:m,n= 1,2,\ldots,m < n\}$. If $x_1, x_2,\ldots,x_k \in H$ are given, then $\sum_{p=1}^{\infty} |[x_i,e_p]|^2 = \|x_i\|^2$, $i= 1,2,\ldots,k$. Thus, $\sum_{p=1}^{\infty}\sum_{i=1}^{k} |[x_i,e_p]|^2 = \sum_{i=1}^{k} \|x_i\|^2$, so that $\lim_{p\to\infty} \sum_{i=1}^{k} |[x_i,e_p]|^2 = 0$. Let $\varepsilon > 0$. Choose m such that $\sum_{i=1}^{k}|[x_i,e_m]|^2 \leq \varepsilon/2$ and an $n > m$ with $\sum_{i=1}^{k} |[x_i,e_n]|^2 \leq \varepsilon/2m^2$. Then

$$\sum_{i=1}^{k} \|A_{m,n}x_i\|^2 = \sum_{i=1}^{k}\|(P_{[e_m]} + m P_{[e_n]})(\sum_{j=1}^{\infty}[x_i,e_j] e_j)\|^2 = \sum_{i=1}^{k} |[x_i,e_m]|^2 +$$

$m^2 \; |[x_i, e_n]|^2) < \varepsilon$. Thus $A_{m,n} \in U(0; x_1,x_2,\ldots,x_k;\varepsilon)$. Thus 0 is a strong accumulation point of the set S.

If $\{A_{m_r,n_r}\}_{r=1}^{\infty}$ tends to 0 in τ_s, then by the uniform boundedness principle $\sup_r \|A_{m_r,n_r}\| < \infty$. Then there will exist a subsequence $\{r_j\}_{j=1}^{\infty}$ and $\ell \in \mathbb{N}$ such that $m_{r_j} = \ell$, for $j \in \mathbb{N}$. Since $\|A_{m,n}e_m\| = 1$, $A_{m_{r_j},n_{r_j}} \notin U(0;e_\ell;1)$, for each $j \in \mathbb{N}$; i.e. $A_{m_r,n_r} \not\to 0$ in τ_s.

Theorem 2.1.4. If a bounded subset F of $B(H)^+$ is directed upward, then F has the least upper bound T in $B(H)^+$ and T is also the strong operator limit of the net $\{A, A \in (F, \geq)\}$. ($B(H)^+ = \{T \in B(H): T \geq 0\}$.)

Proof. A set F in $B(H)^+$ is said to be directed upward if,given $A,B \in F$, then there is $C \in F$ such that $A \leq C$, $B \leq C$. Since F is bounded, there is a real number M such that $\|A\| \leq M$, for $A \in F$. Given $x \in H$, $0 \leq [Ax,x] \leq [Mx,x]$, so that $A \leq MI$,for all $A \in F$. It follows then that the increasing net $\{[Ax,x]: A \in (F,\geq)\}$ is bounded above in $\mathbb{R}^+$ and so converges to its least upper bound. Let $p(x,x) = \sup\{[Ax,x], A \in (F,\geq)\} = \lim_{A \in (F,\geq)} [Ax,x]$. Since $4[Ax,y] =$

$[A(x+y), x+y] - [A(x-y), x-y)] - i[A(x-iy), x-iy] + i[A(x+iy), x+iy]$,

$\lim_{A \in (F,\geq)} [Ax,y]$ exists in $\mathbb{C}$,for $x,y \in H$. Call it $p(x,y)$. Since $[Ax,y]$ depends linearly on x and conjugate linearly on y,and since $|[Ax,y]| \leq M\|x\| \; \|y\|$, for $A \in F$, it follows that p is a bounded sesqui-linear functional on H, with $\|p\| \leq M$. Hence by Theorem 1.2.10 there is a unique operator T in B(H) such that $\|T\| \leq M$ and $p(x,y) = [Tx,y]$, for every x,y in H. Since $[Tx,x] = \sup\{[Ax,x]: A \in F\} \geq 0$, $T \in B(H)^+$ and T is an upper bounded of F. If $T' \geq A$,for each $A \in F$, and $T' \in B(H)$, then $[T'x,x] \geq \sup\{[Ax,x]: A \in F\} = [Tx,x]$, for each $x \in H$; i.e.,T is the least upper bound of F in $B(H)^+$.

Next, if $x \in H$ and $\varepsilon > 0$, there is an A_ε in F such that $[A_\varepsilon x,x] > [Tx,x] - \varepsilon^2$. If $A \in F$ and $A \geq A_\varepsilon$, then $(T-A) \in B(H)^+$ and $\|(T-A)x\|^2 \leq$

$\|(T - A)^{\frac{1}{2}}\|^2 \quad \|(T - A)^{\frac{1}{2}} x\|^2 = \|T - A\| \ [(T - A)x,x] \leq \| T - A\| [(T - A_{\varepsilon})x,x] <$ $\varepsilon^2\| T - A\|$. Thus $\| (T - A)x\| < \varepsilon\|T - A\| < \varepsilon \ (\| T\| + \| A\|) \leq 2M\varepsilon$. Hence $T = \lim \{A:A \ \varepsilon \ (F, \geqq)\}$ in the strong operator topology.

§2.1 (B). The strongest or ultra-strong operator topology $\tau_{\sigma s}$ on B(H)

Given a sequence $X = (x_i)_1^{\infty}$ of elements in H such that $\sum_1^{\infty}\| x_i \|^2 < \infty$, the function $p_X(T) = (\sum_1^{\infty}\|Tx_i\|^2)^{\frac{1}{2}}$ defines a semi-norm p_X on B(H). In fact, for T,S ε B(H),

$$p_X^2(T + S) \leqq \sum_1^{\infty}(\|Tx_i\| + \| Sx_i\|)^2$$

$$= \sum_1^{\infty}\|Tx_i\|^2 + 2 \sum_1^{\infty}\|Tx_i\| \ \|Sx_i\| + \sum_1^{\infty}\| Sx_i\|^2$$

$$\leqq p_X^2(T) + p_X^2(S) + 2p_X(T)p_X(S) = (p_X(T) + p_X(S))^2 < \infty,$$

since $\sum_1\| Tx_i\| \ \|Sx_i\| \leqq (\Sigma\| Tx_i\|^2)^{\frac{1}{2}} (\Sigma\| Sx_i\|^2)^{\frac{1}{2}}$. Thus $p_X(T + S) \leqq p_X(T)+p_X(S)$. Clearly, $p_X(\alpha T) = |\alpha| p_X(T), \alpha \ \varepsilon \ \mathbb{C}$. The locally convex topology induced on B(H) by all such possible semi-norms p_X is called the *strongest or ultra-strong operator topology* and is denoted by $\tau_{\sigma s}$. $\tau_{\sigma s}$ is Hausdorff, since $p_X(T) = 0$ for all such X implies, in particular, taking X= {x}, x ε H, Tx= 0 for each x ε H and hence T=0.

Proposition 2.1.5. B(H) is a locally convex algebra under $\tau_{\sigma s}$. Also $(S,T) \to ST$: $B(H)_1 \times B(H) \to B(H)$ is continuous in $\tau_{\sigma s}$.

Proof. Since $\tau_{\sigma s}$ is a locally convex topology, being induced by a family of semi-norms, the mappings $(S,T) \to S + T : B(H) \times B(H) \to B(H)$ and $(\alpha ,T) \to \alpha T : \mathbb{C} \times B(H) \to B(H)$ are continuous. Let $X = (x_i)_1^{\infty}$ of elements in H with $\sum_1^{\infty}\|x_i\|^2 < \infty$.

If $S_\alpha \in B(H)_1$ and $T_\alpha \in B(H)$, and if $S_\alpha \to S$ and $T_\alpha \to T$ in $\tau_{\sigma s}$, then $p_X^2(S_\alpha T_\alpha - ST) \leqq (p_X(S_\alpha(T_\alpha - T)) + p_X((S_\alpha - S)T))^2 \leqq (p_X(T_\alpha - T) + p_{TX}(S_\alpha - S))^2 \to 0$. If $T_\alpha \to T$ in $\tau_{\sigma s}$, then, similarly, $S_\alpha T \to ST$ and $T_\alpha S \to TS$ in $\tau_{\sigma s}$ for any $S \in B(H)$.

Proposition 2.1.6. τ_s and $\tau_{\sigma s}$ induce the same topology on $B(H)_1$. Consequently, $B(H)_1$ is complete under $\tau_{\sigma s}$. $B(H)_1$ is metrizable under $\tau_{\sigma s}$ if H is separable.

Proof. Clearly, $\tau_s \leqq \tau_{\sigma s}$ on B(H) and hence on $B(H)_1$. Let $S \in B(H)_1$ and let the $\tau_{\sigma s}$-neighbourhood V of S be given by $V=\{T:T \in B(H), \sum_1^\infty \|(T-S)x_i\|^2 < \varepsilon^2, \sum_1^\infty \|x_i\|^2 < \infty\} = V(S; \{x_i\}_1^\infty; \varepsilon)$. Choose N such that $\sum_{i=N+1}^{\infty} \|x_i\|^2 < \frac{\varepsilon^2}{8}$. Then $W=\{T:T \in B(H), \sum_1^N \|(T-S)x_i\|^2 < \frac{1}{2}\varepsilon^2\}$ is a τ_s-neighbourhood of S and, for $T \in W \cap B(H)_1$, $\sum_1^\infty \|(T-S)x_i\|^2 < \varepsilon^2$. Thus $W \cap B(H)_1$ is contained in $V \cap B(H)_1$. Hence $\tau_s = \tau_{\sigma s}$ on $B(H)_1$. Now the completeness of $B(H)_1$ under $\tau_{\sigma s}$ follows from Theorem 2.1.3.

The last part of the proposition is a consequence of the first part and Theorem 2.1.2.

Corollary 2.1.7. If $T_n \to T$ strongly, then $T_n \to T$ ultra-strongly.

Proof. Since $T_n x \to Tx$, for each x in H, $\sup_n \|T_n x\| < M_x < \infty$ and hence, by the uniform boundedness principle, $\sup_n \|T_n\| = M < \infty$. Now the corollary follows from the above proposition.

Note 5. B(H) is topologically complete under $\tau_{\sigma s}$ in the sense of Note 3. See von Neumann [4].

Note 6. If H is infinite dimensional, the mappings $T \to T^*: B(H) \to B(H)$ and $(S,T) \to ST: B(H) \times B(H) \to B(H)$ are not continuous in $\tau_{\sigma s}$, as is shown below. Consequently, $\tau_n \gneq \tau_{\sigma s}$ if H is infinite dimensional.

The transformations $A_n = V^n$ in (b) under Note 1 are bounded and $||A_n|| \leq 1$. Since $A_n \to 0$ in τ_s and $A_n^* \nrightarrow 0$ in τ_s, and $A_n, A_n^* \in B(H)_1$, by 2.1.6, $A_n \to 0$ in $\tau_{\sigma s}$ and $A_n^* \nrightarrow 0$ in $\tau_{\sigma s}$. Defining $A_{n,\delta}$, $B_{n,\delta}$ as in (b) under Note 1, for a given τ_s-neighbourhood $V(0; x;\varepsilon)$ of 0, which is also a $\tau_{\sigma s}$-neighbourhood of 0, $A_{n,\delta}B_{n,\delta} \notin V(0; x;\varepsilon)$, for any n and δ. If $V(0;\{x_i\}_1^\infty; \varepsilon_1)$ and $V(0;\{y_i\}_1^\infty;\varepsilon_2)$ are $\tau_{\sigma s}$-neighbourhoods of 0, taking $0 < \delta < \varepsilon_2 / (\sum_1^\infty ||y_i||^2)^{\frac{1}{2}}$, it can be shown as in (b) under Note 1 that $(A,B) \to AB$ is not continuous in $\tau_{\sigma s}$ at (0,0).

Note 7. For infinite dimensional H, B(H) is not first countable in $\tau_{\sigma s}$ even when H is separable.

The construction under Note 4 holds here. Let $A_{m,n} = P_{[e_m]} + mP_{[e_n]}$. If $\sum_{i=1}^\infty ||x_i||^2 < \infty$, then $\sum_{p=1}^\infty |[x_i,e_p]|^2 = ||x_i||^2$ and hence $\sum_{i=1}^\infty \sum_{p=1}^\infty |[x_i,e_p]|^2 = \sum_{i=1}^\infty ||x_i||^2 < \infty$, so that $\sum_{p=1}^\infty \sum_{i=1}^\infty |[x_i,e_p]|^2 < \infty$. Thus $\lim_{p\to\infty} \sum_{i=1}^\infty |[x_i,e_p]|^2 = 0$. Choose m such that $\sum_{i=1}^\infty |[x_i,e_p]|^2 < \varepsilon/2$, for $p \geq m$, and choose $n > m$ with $\sum_{i=1}^\infty |[x_i,e_n]|^2 < \frac{\varepsilon}{2m^2}$. Then

$$\sum_{i=1}^\infty ||A_{m,n}x_i||^2 = \sum_{i=1}^\infty ||(P_{[e_m]} + mP_{[e_n]})x_i||^2$$

$$= \sum_{i=1}^\infty ||(P_{[e_m]} + mP_{[e_n]})(\sum_{j=1}^\infty [x_i,e_j]e_j)||^2$$

$$= \sum_{i=1}^\infty (|[x_i,e_m]|^2 + m^2|[x_i,e_n]|^2)$$

$$= \sum_{i=1}^\infty |[x_i,e_m]|^2 + m^2 \sum_{i=1}^\infty |[x_i,e_n]|^2 < \varepsilon .$$

Thus 0 is a $\tau_{\sigma s}$-accumulation point of $\{A_{m,n}\}$. But 0 is not the limit of a sequence $\{A_{m_r,n_r}\}_{r=1}^\infty$ even in τ_s-topology and hence not in $\tau_{\sigma s}$-topology. Thus

B(H) is not first countable in $\tau_{\sigma s}$.

Note 8. $\tau_s = \tau_{\sigma s} = \tau_n$ if and only if H is finite dimensional. If H is infinite dimensional, then $\tau_n \underset{\neq}{>} \tau_{\sigma s} \underset{\neq}{>} \tau_s$.

Clearly, $\tau_s \leq \tau_{\sigma s} \leq \tau_n$. But, by Note 2, $\tau_s = \tau_n$ if and only if H is finite dimensional. Hence $\tau_{\sigma s} = \tau_n$ when H is finite dimensional. If $\tau_{\sigma s} = \tau_n$, then H is finite dimensional by Note 6. It suffices to show that $\tau_{\sigma s} \underset{\neq}{>} \tau_s$ when H is infinite dimensional. (The result that $\tau_s = \tau_{\sigma s} = \tau_n$ on B(H) when H is finite dimensional is also obvious from the theorem mentioned under Note 2.)

Let H be an infinite dimensional separable Hilbert space and let S = $\{256n^2(I - P_M): n = 1,2,\ldots,$ M is a subspace of H, dim M $\leq n\}$. If $x_1, x_2, \ldots, x_n$ are given in H, then $256n^2(I - P_{[x_1,\ldots,x_n]})x_i = 0$, for $i = 1,2,\ldots,n$, and hence $256n^2(I - P_{[x_1,\ldots,x_n]}) \varepsilon U(0;x_1,\ldots,x_n;\varepsilon)$. Thus 0 is a strong accumulation point of S. But 0 is not an ultrastrong accumulation point of S. In fact, let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for H. Let $x_n^o = \frac{1}{2n} e_n$, $n = 1,2,\ldots$. Then $\sum_1^{\infty}\|x_n^o\|^2 < 1$. Let $A = 256n^2(I-P_M) \varepsilon S$. Let $M = [f_1,\ldots,f_k], k \leq n$, $f_1, \ldots, f_k$ an orthonormal basis in M. Then, $\sum_{m=1}^{\infty}\|P_M e_m\|^2 = \sum_{m=1}^{\infty}(\sum_{j=1}^{k}\|P_{[f_j]}e_m\|^2) = \sum_{j=1}^{k}\sum_{m=1}^{\infty}|[e_m,f_j]|^2 = \sum_{j=1}^{k}\|f_j\|^2 = k \leq n$. So $\|P_M e_m\| > \frac{1}{2}$ cannot hold for 4n times or more. Hence there is an $m' \leq 4n$ such that $\|P_M e_{m'}\| \leq \frac{1}{2}$. Then $\|(I - P_M)x_{m'}^o\|^2 = \frac{1}{4m'^2}\|e_{m'} - P_M e_{m'}\|^2 \geq \frac{1}{4m'^2}(1 - \frac{1}{2})^2 = \frac{1}{16m'^2} \geq \frac{1}{256n^2}$, and so $\sum_1^{\infty}[Ax_m^o, x_m^o] = 256n^2(\sum_1^{\infty}[(I - P_M)x_m^o, x_m^o]) = 256n^2\sum_1^{\infty}\|(I - P_M)x_m^o\|^2 \geq 256n^2\|(I - P_M)x_{m'}^o\|^2 \geq 1(*)$. Hence $\sum_{m=1}^{\infty}\|Ax_m^o\|^2 \geq \sum_{m=1}^{\infty}\|Ax_m^o\|^2 \sum_{m=1}^{\infty}\|x_m^o\|^2 \geq (\sum_1^{\infty}\|Ax_m^o\|\,\|x_m^o\|)^2 \geq (\sum_1^{\infty}|[Ax_m^o, x_m^o]|)^2 \geq 1$ by (*), where we used the hypothesis that

$\sum_1^\infty \|x_m^o\|^2 < 1.$

Thus $A \notin U(0;\{x_m^o\}_1^\infty, 1)$. Since A is arbitrary in S, it follows that 0 is not a $\tau_{\sigma s}$-accumulation point of S. Thus $\tau_{\sigma s} \gneq \tau_s$.

§2.1(C). The weak operator topology τ_w

For x,y in H, let $p_{x,y}(T) = |[Tx,y]|$, $T \in \mathcal{B}(H)$. Then $p_{x,y}$ is a semi-norm on B(H). The family $\{p_{x,y} : x,y \in H\}$ induces a locally convex Hausdorff topology τ_w on B(H), called the *weak operator topology*. Sets of the form $V(0;x_1,x_2,\dots,x_n;y_1,\dots,y_n;\varepsilon) = \{T : T \in B(H) : |[Tx_i,y_i]| < \varepsilon,\ i = 1,2,\dots,n\}$, with $x_1,y_1;x_2,y_2;\dots;x_n,y_n$ in H and $\varepsilon > 0$, form a base of τ_w-neighbourhoods of 0. Then $T_\alpha \to T$ in τ_w if and only if $[T_\alpha x,y] \to [Tx,y]$, for all x,y in H.

Proposition 2.1.8. B(H) is a locally convex algebra in τ_w-topology. Further, $T \to T^*$ is continuous in τ_w.

Proof. Clearly, B(H) is a locally convex space under τ_w. For fixed $S \in B(H)$, $(S,T) \to ST$ and $(T,S) \to TS$ are continuous in τ_w. In fact, if $T_\alpha \to T$ in τ_w, $[S(T_\alpha - T)x,y] = [(T_\alpha - T)x,S^*y] \to 0$. Hence $ST_\alpha \to ST$ in τ_w. Similarly, $T_\alpha S \to TS$ in τ_w. $[T_\alpha^* x,y] = [x,T_\alpha y] = [\overline{T_\alpha y,x}] \to [\overline{Ty,x}] = [x,Ty] = [T^*x,y]$. Hence $T_\alpha^* \to T^*$ in τ_w.

Note 9. $(S,T) \to ST$ is not continuous even for $B(H)_1 \times B(H)_1 \to B(H)$ in τ_w, as is shown in the following counter-example.

Consider the operators A_n of (b) under Note 1. As $A_n \to 0$ strongly, $A_n \to 0$ weakly. Hence A_n^* tends to zero weakly. But $A_n A_n^* = I$ for each n and hence $A_n A_n^*$ does not tend to zero weakly. Note that $\|A_n\| = \|A_n^*\| = 1$, for each n.

Note 10. $\tau_w \lneq \tau_s \lneq \tau_{\sigma s} \lneq \tau_n$ if H is infinite dimensional.

Obviously, $\tau_w \leqq \tau_s$. If H is of infinite dimension, $T \to T^*$ is not continous in τ_s by Note 1, but $T \to T^*$ is always continuous in τ_w. Hence $\tau_w \lneqq \tau_s$.

Note 11. $\tau_w = \tau_s = \tau_{\sigma s} = \tau_n$ if and only if H is finite dimensional.

$\tau_w \leqq \tau_s$ always. If dim $\mathcal{H} = n < \infty$, let $\{e_i\}_1^n$ be an orthonormal basis. Let $U = U(0;x_1;\varepsilon)$ be a τ_s-neighbourhood of 0. Consider $V = (0;\underbrace{x_1,x_1,\ldots,x_1}_{(n \text{ times})};e_1,e_2,\ldots,e_n;\varepsilon')$ with $\varepsilon' < \frac{\varepsilon}{\sqrt{n}}$. For $T \in V$, $\|Tx_1\|^2 = \sum_{i=1}^{n} |[Tx_1,e_i]|^2 < n\,\frac{\varepsilon^2}{n} = \varepsilon^2$. Thus $V \subset U$. If U_0 is an arbitrary strong neighbourhood of 0, let $U_0 = \bigcap_{i=1}^{m} U_i(0;x_i;\varepsilon)$. Then the τ_w-neighbourhood V_0 of 0, given by $V_0 = \bigcap_{i=1}^{m} V_i(0;\underbrace{x_i,\ldots}_{(n \text{ times})};e_1,\ldots,e_n;\frac{\varepsilon}{\sqrt{n}})$, is contained in U_0 by the above argument. Hence $\tau_w = \tau_s$. (See the theorem mentioned under Note 2.)

If $\tau_w = \tau_s$, then, in the light of Note 10, H is finite dimensional.

Notation 3. $V(S_0;x_1,\ldots,x_n;y_1,\ldots,y_n;\varepsilon) = \{T \in B(H) : |[(T-S)x_i,y_i]| < \varepsilon, i = 1,2,\ldots,n\}$.

Theorem 2.1.9. Let H be separable. Then the weak operator topology τ_w restricted to $B(H)_1$ is metrizable.

Proof. Let A be a countable dense subset of H. The sets $V(S_0;x_1,\ldots,x_n;y_1,\ldots,y_n;\varepsilon)$, where $\varepsilon > 0$, the x_i and y_i are in A and ε is rational, form a countable base of neighbourhood of S_0 for a locally convex topology τ on B(H). This topology is induced by a countable family Γ of semi-norms of the form $p_{x,y}(S) = |[Sx,y]|$, $x,y \in A$. Since Γ is countable and separates points, τ is metrizable.

Clearly, $\tau \leqq \tau_w$. To prove the reverse inequality, let V be a weak neighbourhood of S_0 in $B(H)_1$ given by

$$V = V(S_0;\ x_1,\ldots,x_n;\ y_1,\ldots,y_n;\varepsilon) \cap B(H)_1.$$

Choose $K > \max_{1 \leq i \leq n} (\|x_i\|, \|y_i\|)$ and then choose x_i', y_i' in A such that

$$\|x_i - x_i'\| < \frac{\varepsilon}{8K}, \quad \|y_i - y_i'\| < \frac{\varepsilon}{8K}, \quad \|x_i'\| < K, \ (i \leq i \leq n).$$

Suppose $S \in V(S_0;\ x_1', \ldots, x_n'; y_1', \ldots, y_n'; \tfrac{1}{2}\varepsilon) \cap B(H)_1$. Then

$$|[(S - S_0)x_i, y_i]| \leq |[(S - S_0)x_i', y_i']| + |[(S - S_0)(x_i - x_i'), y_i]|$$

$$+ |[(S - S_0)x_i', y_i - y_i']| < \varepsilon \quad .$$

Thus $S \in V$. Hence $\tau_w = \tau$ and thus τ_w is metrizable.

Lemma 2.1.10. $B(H)_1$ is complete under τ_w.

Proof. Let $\{T_\alpha\}$ be a Cauchy net in $B(H)_1$ for τ_w. Then, for x,y in H, $\{[T_\alpha x,y]\}_\alpha$ is a Cauchy net in $\mathbb{C}$ and hence $\lim_\alpha [T_\alpha x,y]$ exists. Now call $\lim_\alpha [T_\alpha x,y] = p(x,y)$. Clearly, $p(x,y)$ is a sesqui-linear functional. Also

$$|p(x,y)| = |\lim_\alpha [T_\alpha x,y]| = \lim_\alpha |[T_\alpha x,y]| \leq \|x\| \, \|y\| \, .$$

Thus p is a bounded sesqui-linear functional, with $\|p\| \leq 1$. Hence there exists a unique operator T in $B(H)_1$ such that $p(x,y) = [Tx,y]$, so that $T_\alpha \to T$ in τ_w. Therefore $B(H)_1$ is complete under τ_w.

Note 12. B(H) is topologically complete under τ_w. (See von Neumann [4].)

Theorem 2.1.11. $B(H)_1$ is compact under τ_w.

Proof. Given x,y in H, let $D_{x,y}$ be the compact disc $\{\lambda : \lambda \in \mathbb{C}, |\lambda| \leq \|x\| \, \|y\|\}$. Let $Q = \prod_{(x,y) \in H \times H} D_{x,y}$. With the product topology on Q, by Tychonoff's theorem, Q is compact. If $T \in B(H)_1$, then $|[Tx,y]| \leq \|x\| \, \|y\|$, so that $[Tx,y] \in D_{x,y}$ $(x,y \in H)$. Consider the mapping $\psi : B(H)_1 \to Q$ defined by $\psi(T) = \{[Tx,y]\}_{(x,y) \in H \times H}$. With τ_w on

$B(H)_1$, ψ is a homeomorphism of $B(H)_1$ onto a subset Q_0 of Q, since, $T_\alpha \to T$ in τ_w in $B(H)_1 \iff [T_\alpha x,y] \to [Tx,y]$ for all $(x,y) \in H \times H \Leftrightarrow \psi(T_\alpha) \to \psi(T)$ in Q. Hence, to show that $B(H)_1$ is compact under τ_w, it suffices to show that Q_0 is closed in Q.

A general element q of Q has the form $\{q(x,y)\}_{(x,y)\in H\times H}$ where $q(x,y) \in D_{x,y}$ and, for fixed x and y, the projection $q \to q(x,y)$ is continuous. If $q \in Q_0$, then $q = \psi(T)$ for some T in $B(H)_1$ and so $q(x,y) = [Tx,y]$. Hence such a q is a bounded sesqui-linear functional on H, with $\|q\| \leqq 1$. Conversely, if $q \in Q$ and q is a bounded sesqui-linear functional with $\|q\| \leqq 1$, then $q(x,y)$ is expressible uniquely as $[Tx,y]$ for some $T \in B(H)_1$ and hence $q = \psi(T)$, i.e., $q \in Q_0$. Thus Q_0 is the collection of all sesqui-linear functionals q on H, with $\|q\| \leqq 1$. If $\{q_\alpha\}_\alpha \subset Q_0$ and $q_\alpha \to q$ in Q, then $q_\alpha(x,y) \to q(x,y)$, for all $(x,y) \in H \times H$, and hence q is sesqui-linear and $|q(x,y)| = \lim_\alpha |q_\alpha(x,y)| \leqq \|x\| \; \|y\|$. Thus $q \in Q_0$ so that Q_0 is closed in Q.

This completes the proof of the theorem.

Corollary 2.1.12. If H is separable, then $B(H)_1$ is a separable complete metrizable space in τ_w.

Proof. H separable $\Rightarrow B(H)_1$ is metrizable under τ_w by Theorem 2.1.9 . $B(H)_1$ is always compact under τ_w by Theorem 2.1.11. Since compactness and metrizability imply separability (Theorem 1.8.15 of Dunford and Schwartz Par I [ ]), and since $B(H)_1$ is complete under τ_w by Lemma 2.1.10, the corollary follows.

§ 2.1(D). The ultra-weak operator topology $\tau_{\sigma w}$.

If $X = (x_i)_1^\infty$ and $Y = (y_i)_1^\infty$ are sequences of elements of H such that $\sum_1^\infty \|x_i\|^2$ and $\sum_1^\infty \|y_i\|^2$ are convergent, then the function $p_{X,Y}(T) = |\sum_1^\infty [Tx_i, y_i]|$ defines a semi-norm $p_{X,Y}$ on $B(H)$. The family Γ of all such semi-norms induces a locally convex Hausdorff topology on $B(H)$, called the *ultra-weak operator topology* $\tau_{\sigma w}$.

A neighbourhood basis at 0 for $\tau_{\sigma w}$ can be given by

$$V(0;\{x_i\}_1^\infty;\ \{y_i\}_1^\infty;\varepsilon) = \{T : T \in B(H), |\sum_1^\infty [Tx_i, y_i]| < \varepsilon\}$$

where $\sum_1^\infty \|x_i\|^2 < \infty$ and $\sum_1^\infty \|y_i\|^2 < \infty$.

$T_\alpha \to T$ in $\tau_{\sigma w}$ if and only if, for every $X = (x_i)_1^\infty$, $Y = (y_i)_1^\infty$ in H with $\sum_1^\infty \|x_i\|^2 < \infty$ and $\sum_1^\infty \|y_i\|^2 < \infty$, $\sum_{i=1}^\infty [(T_\alpha - T)x_i, y_i] \to 0$.

Proposition 2.1.13.

(i) B(H) is a locally convex algebra under $\tau_{\sigma w}$.

(ii) $T \to T^*: B(H) \to B(H)$ is continuous under $\tau_{\sigma w}$.

Proof. Similar to that of Proposition 2.1.8.

Proposition 2.1.14. τ_w and $\tau_{\sigma w}$ induce the same topology on $B(H)_1$. Consequently, $B(H)_1$ is compact and complete under $\tau_{\sigma w}$. If H is separable, $B(H)_1$ is metrizable and separable in $\tau_{\sigma w}$.

Proof. Clearly, $\tau_w \leq \tau_{\sigma w}$ on B(H) and hence on $B(H)_1$. To prove the reverse inequality, let $S \in B(H)_1$ and V be a $\tau_{\sigma w}$ neighbourhood of S in $B(H)_1$ given by $V = \{T : T \in B(H)_1, |\sum_{i=1}^\infty [(T-S)x_i, y_i]| < \varepsilon\}$, with $\sum_1^\infty \|x_i\|^2 + \sum_1^\infty \|y_i\|^2 < \infty$. Choose N sufficiently large so that $\sum_{i=N+1}^\infty (\|x_i\|^2 + \|y_i\|^2) < \varepsilon/2$. For each $T \in B(H)_1$, then

$$|\sum_{i=N+1}^\infty [(T-S)x_i, y_i]| \leq 2\sum_{N+1}^\infty \|x_i\|\,\|y_i\| \leq \sum_{N+1}^\infty (\|x_i\|^2 + \|y_i\|^2) < \frac{\varepsilon}{2} .$$

If W is a neighbourhood of S in B(H) for τ_w, given by

$$W = \{T : T \in B(H), |[(T-S)x_i, y_i]| < \varepsilon/2N,\ i = 1,2,\ldots,N\}$$

then, for $T \in W \cap B(H)_1$, we have

$$|\sum_1^\infty [(T-S)x_i, y_i]| \leq |\sum_{i=1}^N [(T-S)x_i, y_i]| + |\sum_{i=N+1}^\infty [(T-S)x_i y_i]| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence $\tau_{\sigma w} \leqq \tau_w$ on $B(H)_1$. Therefore $\tau_{\sigma w} = \tau_w$ on $B(H)_1$.

Corollary 2.1.15. If $\{A_n\}_1^\infty$ is weakly convergent to $A \in B(H)$, then $\{A_n\}_1^\infty$ is ultra-weakly convergent.

By uniform boundedness principle applied twice, $\sup_n \|A_n\| < \infty$. Now the corollary follows from 2.1.14.

Note 13. $\tau_s = \tau_w = \tau_{\sigma w} = \tau_{\sigma s} = \tau_n$ if and only if H is finite dimensional.

If $\dim H < \infty$, then $\tau_w = \tau_n$ by Note 11 and hence $\tau_{\sigma w} = \tau_n$, as $\tau_w \leqq \tau_{\sigma w} \leqq \tau_n$. If $\tau_{\sigma w} = \tau_n = \tau_{\sigma s}$, then $\dim H < \infty$ by Note 11.

Note 14. $\tau_w \lneqq \tau_{\sigma w} \lneqq \tau_{\sigma s} \lneqq \tau_n$ if H is infinite dimensional.

Clearly, $\tau_{\sigma w} \leqq \tau_{\sigma s}$. Since * operation is not continuous for $\tau_{\sigma s}$ when H is infinite dimensional, $\tau_{\sigma w} \lneqq \tau_{\sigma s}$. Obviously, $\tau_w \leqq \tau_{\sigma w}$.

In the counterexample under Note 8, 0 is a strong and hence weak accumulation point of the subset S of B(H). The argument there leads to the conclusion that $A \notin V = U(0;\{x_m^0\}_1^\infty; \{x_m^0\}_1^\infty; 1)$, for any $A \in S$, and V is an ultra-weak neighbourhood of 0. Hence 0 is not an accumulation point of S for $\tau_{\sigma w}$. Thus $\tau_w \lneqq \tau_{\sigma w}$.

Note 15. B(H) is not first countable under τ_w if H is infinite dimensional.

The construction under Note 4, with the argument modified suitably at the end, proves this statement. The details are left to the reader.

Note 16. The topologies τ_s and $\tau_{\sigma w}$ are not comparable if H is infinite dimensional.

Since * operation is continuous in $\tau_{\sigma w}$ and not continuous in τ_s when $\dim H = \infty$, $\tau_{\sigma w} \not\subset \tau_s$. Moreover, $\tau_s \not\subset \tau_{\sigma w}$ by Lemma 2.1.1(iii) and the following Note.

Note 17. $(S,T) \to ST : B(H)_1 \times B(H)_1$ is not continuous in $\tau_{\sigma w}$.

This follows from Proposition 2.1.14 and Note 9.

Note 18. B(H) is not first countable under $\tau_{\sigma w}$ if H is infinite dimensional.

Use the construction under Note 4 modifying the argument suitably to establish the present note.

Note 19. τ_s and $\tau_{\sigma s}$ (resp. τ_w and $\tau_{\sigma w}$) can coincide on some subalgebra of B(H), even if H is infinite dimensional.

In fact, let $H_0 = \sum_{i=1}^{\infty} \oplus H$, the Hilbert space of all sequences $X=(x_i)$ of elements of H such that $\|X\|^2 = \sum_1^{\infty} \|x_i\|^2 < \infty$. For each $T \in B(H)$, define $\Phi(T)$ in $B(H_0)$ by

$\Phi(T)X = (Tx_i)_1^{\infty}$. Let $R = \{\Phi(T) : T \in B(H)\}$. Then, clearly, R

is a *-subalgebra of $B(H_0)$. Then the topologies $\tau_{\sigma s}$ and $\tau_{\sigma w}$ are the inverse images under Φ of the topologies τ_s and τ_w on R, respectively. Furthermore, it is easily verified that $\tau_{\sigma s} = \tau_s$ on R and $\tau_{\sigma w} = \tau_w$ on R.

Note 20. If H is infinite dimensional

$$\begin{array}{ccccc} \tau_w & < & \tau_{\sigma w} & & \\ \wedge & & \wedge & & \\ \tau_s & < & \tau_{\sigma s} & < & \tau_n \end{array}$$

All the topologies coincide if and only if H is finite dimensional.

This follows from Notes 14, 16, 8 and 13. (< means strictly less than.)

2.2. Linear functionals on B(H)

We shall denote by L_α the set of all τ_α-continuous linear functionals on B(H). Hence L_n is B(H)*, the Banach dual of B(H), and L_s, L_w, $L_{\sigma s}$ and $L_{\sigma w}$ are linear subspaces of L_n. When x, y $\in$ H, we denote [Tx, y] by $w_{x,y}(T)$ for each $T \in B(H)$.

Proposition 2.2.1. $w_{x,y} \in L_n$ and $||w_{x,y}|| = ||x|| \, ||y||$.

Proof. Clearly, $w_{x,y}$ is a linear functional on B(H).

Also $|w_{x,y}(T)| = |[Tx,y]| \leq ||T|| \, ||x|| \, ||y||$, so that $||w_{x,y}|| \leq ||x|| \, ||y||$ and hence $w_{x,y} \in B(H)^* = L_n$.

Define the operator T_0 on H by $T_0 z = [z,x]y$. T_0 is linear and $||T_0 z|| = |[z,x]| \, ||y|| \leq ||z|| \, ||x|| \, ||y||$, so that $||T_0|| \leq ||x|| \, ||y||$. Thus $T_0 \in B(H)$. Further, $||T_0 x|| = ||x||^2 ||y|| = (||x|| \, ||y||) \, ||x||$, so that $||T_0|| = ||x|| \, ||y||$.

Now, $|w_{x,y}(T_0)| = |[T_0 x,y]| = ||x||^2 ||y||^2 = (||x|| \, ||y||) \, ||x|| \, ||y|| = ||T_0|| \, ||x|| \, ||y||$. Hence $||w_{x,y}|| = ||x|| \, ||y||$.

Theorem 2.2.2. Let f be a linear functional on B(H).

(i) The following three conditions are equivalent:

(ia) f has the form $\sum_{i=1}^{N} w_{x_i, y_i}$, with $x_1, \ldots, x_n; y_1, \ldots, y_n \in H$.

(ib) $f \in L_w$.

(ic) $f \in L_s$.

(ii) The following three conditions are equivalent:

(iia) f has the form $\sum_{i=1}^{\infty} w_{x_i, y_i}$, with $x_i, y_i \in H(i=1,2\ldots)$, and $\sum_1^{\infty} ||x_i||^2 + \sum ||y_i||^2 < \infty$.

(iib) $f \in L_{\sigma w}$.

(iic) $f \in L_{\sigma s}$.

Proof. (ia)$\Longrightarrow$(ib) Let $f = \sum_{i=1}^{n} w_{x_i, y_i}$. If $T_\alpha \to T$ wearkly, $[T_\alpha x_i, y_i] \to [T x_i, y_i], i = 1,2,\ldots,n$. Hence $\sum_1^n w_{x_i,y_i}(T_\alpha) \to \sum_1^n w_{x_i,y_i}(T)$; i.e., f is τ_w-continuous. Hence $f \in L_w$.

(ib)$\Longrightarrow$(ic) Since $\tau_w \leq \tau_s$, $L_w \subset L_s$ and hence $f \in L_w$ implies $f \in L_s$.

(ic)$\Longrightarrow$(ia) Let $f \in L_s$. Then there is a τ_s-neighbourhood V of 0 such that $V = U(0; x_1,\ldots,x_n; \varepsilon)$ and such that, for $T \in V, |f(T)| < 1$ (2.2.21). By homogeneity we have

$$|f(T)| < \frac{1}{\varepsilon}\left(\sum_1^n \|Tx_i\|^2\right)^{\frac{1}{2}}, \text{for } T \in B(H). \qquad (2.2.2.2)$$

For, $\dfrac{\varepsilon T}{\left(\sum_{i=1}^n \|Tx_i\|^2\right)^{\frac{1}{2}}} \in V$ and hence $\left|f\left(\dfrac{\varepsilon T}{\left(\sum_{i=1}^n \|Tx_i\|^2\right)^{\frac{1}{2}}}\right)\right| < \varepsilon$ for $T \neq 0$ in B(H);

i.e., $|f(T)| < \frac{1}{\varepsilon}\left(\sum_1^n \|Tx_i\|^2\right)^{\frac{1}{2}}$ if $T \neq 0$. If $T = 0$, $f(T) = 0$.

Hence (2.2.2.2) holds.

Now, let $H_0 = \sum_{i=1}^n \oplus H$. Given $T \in B(H)$, let $X_T = (Tx_1,\ldots,Tx_n) \in H_0$.

Then $M = \{X_T : T \in B(H)\}$ is a linear manifold in H_0 and (2.2.2.2) asserts

that $$|f(T)| \leq \frac{1}{\varepsilon}\left(\sum_1^n \|Tx_i\|^2\right)^{\frac{1}{2}} = \frac{1}{\varepsilon}\|X_T\|, \ T \in B(H). \qquad (2.2.2.3)$$

We define a linear functional F_0 on M by setting

$$F_o(X_T) = f(T).$$

Then F_o is well-defined, since, if $X_T = X_S$, then $X_{T-S} = 0$ so that $|f(T-S)| \leq \frac{1}{\varepsilon}\| X_{T-S}\| = 0$ by (2.2.2.3). Thus $F_o(X_T) = f(T) = f(S) = F_o(X_S)$.

Again by (2.2.2.3), $|F_o(X_T)| \leq \frac{1}{\varepsilon}\|X_T\|$, for each $X_T \varepsilon M$. Clearly, F_o is linear. Hence by the Hahn-Banach theorem F_o can be extended to a continuous linear functional F on H_o such that

$$|F(X)| \leq \frac{1}{\varepsilon}\| X\|, \text{for each } X \varepsilon H_o.$$

Therefore, by the Riesz representation theorem there is a unique vector Y in H_o such that

$$F(X) = [X,Y], \text{ for each } X \varepsilon H_o.$$

Let $Y = (y_1, y_2, \ldots, y_n)$. In particular, for $T \varepsilon B(H)$,

$$f(T) = F_o(X_T) = F(X_T) = [X_T, Y] = \sum_1^n [Tx_i, y_i] = \sum_{i=1}^n w_{x_i,y_i} (T).$$

Hence $f = \sum_{i=1}^n w_{x_i, y_i}$.

(ii)

(iia) $\Longrightarrow$ (iib) Let $f = \sum_{i=1}^{\infty} w_{x_i,y_i}$, $\sum_1^{\infty}\| x_i\|^2 < \infty$ and $\sum_1^{\infty}\| y_i\|^2 < \infty$.

If $f_n = \sum_{i=1}^n w_{x_i,y_i}$, then $\|f_{n+p} - f_n\| = \|\sum_{n+1}^{n+p} w_{x_i,y_i}\|$

$$\leq \sum_{n+1}^{n+p} \|w_{x_i,y_i}\| = \sum_{n+1}^{n+p} \|x_i\| \, \|y_i\| \leq (\sum_{n+1}^{n+p} \|x_i\|^2)^{\frac{1}{2}} (\sum_{n+1}^{n+p} \|y_i\|)^{\frac{1}{2}} < \varepsilon$$

if n is sufficiently large. Thus $\{f_n\}$ is Cauchy in L_n so that $\lim_n f_n = f \in L_n$. Hence, for $T \in B(H)$,

$$f(T) = \lim_n f_n(T) = \lim_n \sum_{i=1}^{n} w_{x_i,y_i}(T) = \sum_{i=1}^{\infty}[Tx_i, y_i] \in \mathbb{C}.$$

This shows that f(T) is well-defined for $T \in B(H)$.

Let $T_\alpha \to T$ ultraweakly. Then, by definition of $\tau_{\sigma W}$,

$$\sum_{i=1}^{\infty} [T_\alpha x_i, y_i] \to \sum_{i=1}^{\infty} [Tx_i, y_i]. \text{ Hence } f(T_\alpha) \to f(T).$$

Thus f is ultraweakly continuous.

(iib) $\Longrightarrow$ (iic) Since $\tau_{\sigma W} \leqq \tau_{\sigma S}$, $L_{\sigma W} \subset L_{\sigma S}$. Hence $f \in L_{\sigma W} \Longrightarrow f \in L_{\sigma S}$.

(iic) $\Longrightarrow$ (iia) The proof is exactly similar to that of (ic) $\Longrightarrow$ (ia) except that we must take $H_0 = \sum_{i=1}^{\infty} \oplus H$.

This completes the proof of the theorem.

Note 21. Theorem 2.2.2 will be again studied in Chapter 5, where B(H) is replaced by an arbitrary von Neumann algebra R.

Definition 2.2.3. Let X be a vector space and let Γ be a total subspace of X', the space of all linear functionals of X. Then the Γ-*topology* of X or the *weak topology* on X *induced* by Γ is the weakest locally convex topology on X in which every functional in Γ is continuous.

Corollary 2.2.4. τ_W is the weak topology on B(H) induced by the set of all τ_W-continuous linear functionals on B(H) and $\tau_{\sigma W}$ is the weak topology on B(H) induced by the set of all $\tau_{\sigma w}$-continuous linear functionals on B(H).

Proof. By Theorem 2.2.2(i), $L_w = \{f = \sum_{i=1}^{n} w_{x_i, y_i}, x_1, \ldots, x_n; y_1 \ldots, y_n \text{ in } H\}$. Thus, if τ is a locally convex topology on B(H) and if all members of L_w are continuous in τ, then $\tau \geq \tau_w$, since, $T_\alpha \to T$ in $\tau \Rightarrow f(T_\alpha) \to f(T)$ for $f \in L_w \Rightarrow [T_\alpha x, y] \to [Tx, y]$ for x,y in $H \Rightarrow T_\alpha \to T$ in τ_w. Thus τ_w is the weak topology induced by L_w. Similarly, the second statement follows by appealing to Theorem 2.2. (ii).

Corollary 2.2.5. Let C be a convex subset of B(H).

(i) The closures of C in τ_w and τ_s coincide.

(ii) The closures of C in $\tau_{\sigma w}$ and $\tau_{\sigma s}$ coincide.

(iii) If C is further norm bounded, then its closures in τ_w, τ_s, $\tau_{\sigma w}$ and $\tau_{\sigma s}$ coincide.

Proof. (i) and (ii) follow from Theorem 2.2.2 and Corollary 2.2.4, since the closure of C in a locally convex topology τ on B(H) is the same as its closure in the weak topology induced by the set of all τ-continuous linear functionals. (See Corollary V.2.14 of Dunford and Schwartz, Part I, [DS]).

If C is norm bounded, we may suppose that $C \subset S_M = \{T : T \in B(H), \|T\| \leq M\}$. Since S_M is τ_w-compact by 2.1.11 and $\tau_w = \tau_{\sigma w}$ on S_M by Proposition 2.1.14, it follows that S_M is closed for both τ_w and $\tau_{\sigma w}$. Hence the τ_w (respectively, $\tau_{\sigma w}$) closure of C in B(H) is the same as its relative closure in S_M. Again, since $\tau_w = \tau_{\sigma w}$ on S_M, the τ_w and $\tau_{\sigma w}$ closures of C coincide. This, together with (i) and (ii), proves (iii).

§2.3. The double commutant theorem for von Neumann algebras

Definition 2.3.1. Let H be a Hilbert space. A τ_w-closed *-subalgebra $\mathcal{R}$ of B(H) with $I \in \mathcal{R}$ is called a von Neumann algebra over H. If $\mathcal{F} \subset B(H)$, we define $\mathcal{F}^* = \{T^*: T \in \mathcal{F}\}$ and $\mathcal{F}' = \{S \in B(H): TS = ST \text{ for each } T \in \mathcal{F} \cup \mathcal{F}^*\}$ and call $\mathcal{F}'$ the commutant of $\mathcal{F}$.

$\mathcal{F}'$ is a von Neumann algebra. Also writing $\mathcal{F}''$ for $(\mathcal{F}')'$, etc, we have $\mathcal{F} \subset \mathcal{F}''$. Clearly, $\mathcal{F}_1 \subset \mathcal{F}_2 \Longrightarrow \mathcal{F}_2' \subset \mathcal{F}_1'$. Hence $(\mathcal{F}'')' \subset \mathcal{F}'$ and $(\mathcal{F}') \subset (\mathcal{F}')''$. Thus $\mathcal{F}' = \mathcal{F}'''$.

The von Neumann algebra $\mathcal{R}(\mathcal{F})$ generated by $\mathcal{F}$ is defined as the smallest von Neumann algebra containing $\mathcal{F}$ and is the τ_w-closure of the set of finite linear combinations of finite products of element of $\{I\} \cup \mathcal{F} \cup \mathcal{F}^*$.

Lemma 2.3.2. Let $\mathcal{R}$ be a von Neumann algebra, $\mathcal{P}$ the set of all projections in $\mathcal{R}$ and $\mathcal{U}$ the set of all unitary operators in $\mathcal{R}$. Then $\mathcal{R}$ is the linear subspace of B(H) generated algebraically by $\mathcal{U}$ and is the τ_n-closure of the linear manifold generated by $\mathcal{P}$. In particular, $\mathcal{R} = \mathcal{R}(\mathcal{P}) = \mathcal{R}(\mathcal{U})$.

Proof. Since $\mathcal{R}$ is a B*-algebra with identity, the statement regarding $\mathcal{U}$ follows from Theorem 1.5.10.

Let M be the τ_n-closed subspace of B(H) generated by $\mathcal{P}$. Then $M \subset \mathcal{R}(\mathcal{P}) \subset \mathcal{R}$. To show that $M = \mathcal{R}(\mathcal{P}) = \mathcal{R}$, it is sufficient to prove that $T \in M$ if $T = T^* \in \mathcal{R}$. But, by the spectral theorem, such T is of the form $T = \int_a^b \lambda dE(\lambda)$, with a, b real. Hence T can be approximated in norm by operators of the form $\sum_{i=1}^{n} \lambda_i E_i$ with E_i projections which are strong operator, and hence weak operator, limits of polynomials in T, so that each $E_i \in \mathcal{P}$. Hence $T \in M$.

Lemma 2.3.3. Suppose that $F \subset B(H)$ and E is the projection of H with range M. Then $E \varepsilon F'$ if and only if M is invariant under each operator in $F \cup F^*$.

Proof. If $E \varepsilon F'$ and $T \varepsilon F \cup F^*$, then ET = TE, so that T leaves M invariant. In fact, $x \varepsilon M \Longrightarrow Ex = x$, and hence $Tx = TEx = ETx \varepsilon M$. Conversely, suppose that each $T \varepsilon F \cup F^*$ leaves M invariant. Then TEx = ETEx, for each $x \varepsilon H$, so that TE = ETE and, since $T^* \varepsilon F \cup F^*$, $T^*E = ET^*E$. Hence $(T^*E)^* = (ET^*E)^*$; i.e., ET = ETE. Therefore, ET = TE for all $T \varepsilon F \cup F^*$.

Notation. When $X \subset H$, we denote by [X] both the closed subspace spanned by X and the projection of H with range [X]. When P is a projection, M = P(H), $x \varepsilon M$ and $T \varepsilon B(H)$, we shall write "$x \varepsilon P$" in place of "$x \varepsilon M$", "P is invariant under T" in place of "M is invariant under T", etc.

Lemma 2.3.4. Suppose that $\mathcal{R}$ is a *-subalgebra of B(H) and $E = [Ry: R \varepsilon \mathcal{R}, y \varepsilon H]$. Then (i) $E \varepsilon \mathcal{R}' \cap \mathcal{R}''$; (ii) R = RE = ER for each $R \varepsilon \mathcal{R}$. (iii) $Ex \varepsilon [\mathcal{R}x]$ for each $x \varepsilon H$.

Proof.

(i) Suppose that $R, S \varepsilon \mathcal{R}$, $S' \varepsilon \mathcal{R}'$ and $y \varepsilon H$. Then $S(Ry) = (SR)y \varepsilon E$, $S'(Ry) = R(S'y) \varepsilon E$ and hence E is invariant under $\mathcal{R}$ and $\mathcal{R}'$. Thus $E \varepsilon \mathcal{R}' \cap \mathcal{R}''$ by Lemma 2.3.3.

(ii) For each R in $\mathcal{R}$ and y in H, we have $Ry \varepsilon E$ and hence ERy = Ry. Thus ER = R. Since $E \varepsilon \mathcal{R}'$, ER = RE for $R \varepsilon \mathcal{R}$. Thus ER = RE = R.

(iii) Since $\mathcal{R}x$ is invariant under $\mathcal{R}$, $[\mathcal{R}x] = P$ is also invariant under $\mathcal{R}$ and hence $P \varepsilon \mathcal{R}'$. For each $R \varepsilon \mathcal{R}$, $PR^*x = R^*x$ and hence $R^*(I-P)x = (I-P)R^*x = 0$. Hence, for each $R \varepsilon \mathcal{R}$ and $y \varepsilon H$, $[(I-P)x, Ry] = [R^*(I-P)x, y] = 0$ so that

$E(I-P)x = 0$. Recalling that $P \in \mathcal{R}'$ and $E \in \mathcal{R}''$ we have $Ex = EPx = PEx \in P$. Hence $Ex \in [\mathcal{R}x]$.

Let $\mathcal{R}$ be a *-subalgebra of $B(H)$. Then:

Definition 2.3.5. The projection $E = [Ry\colon R \in \mathcal{R},\ y \in H]$ is called the principal identity of $\mathcal{R}$ (the name being justified by Lemma 2.3.4 (ii)). If $I \in \mathcal{R}$, then $E = I$. (Note that E can be I even if $I \notin \mathcal{R}$, if $[Ry\colon y \in H,\ R \in \mathcal{R}] = H$.)

Lemma 2.3.6. Let $\mathcal{R}$ be a *-subalgebra of $B(H)$ with the principal identity E. If $S \in \mathcal{R}''$, $S = SE$, $x \in H$ and $\varepsilon > 0$, then there exists an operator $R \in \mathcal{R}$ such that $\|Sx - Rx\| < \varepsilon$.

Proof. Let $P = [\mathcal{R}x] \in \mathcal{R}'$. Since $S \in \mathcal{R}''$, S leaves P invariant. By Lemma 2.3.4 (iii), $Ex \in P$. Hence $Sx = SEx \in P$, as $SE = S$. Since $\mathcal{R}x$ is a linear manifold in H and $Sx \in [\mathcal{R}x]$, there is $R \in \mathcal{R}$ such that $\|Sx - Rx\| < \varepsilon$.

The lemma just proved asserts that a certain τ_s-neighbourhood of S meets $\mathcal{R}$. Now we prove a much stronger version below.

Lemma 2.3.7. Suppose that $\mathcal{R}$ is a *-subalgebra of $B(H)$ with the principal identity E, $S \in \mathcal{R}''$ and $S = SE$. Then S lies in the $\tau_{\sigma s}$-closure of $\mathcal{R}$.

Proof. Let $\varepsilon > 0$. Suppose (x_i) is a sequence of elements of H such that $\sum_1^\infty \|x_i\|^2 < \infty$. Let $V = \{T \in B(H)\colon \sum_1^\infty \|(S - T)x_i\|^2 < \varepsilon^2\}$.
Since sets of this type form a base of $\tau_{\sigma s}$-neighbourhoods of S, it is sufficient to prove that V meets $\mathcal{R}$.

Let $H_o = \sum_{i=1}^{\infty} \oplus H_i$, $H_i = H$ for each i. Define the operators $V_j\colon H_o \to H_j = H$ by $V_j(x_1, x_2, \ldots) = x_j$ and $U_i\colon H \to H_o$ by $U_i(x) = (0, \ldots, x, 0, \ldots)$ (i^{th} place). Let A be in $B(H_o)$. Define $A_{ij} = V_i A U_j\colon H \to H$.

Let $AX = Y$, with $X=(x_i)_1^{\infty}$, $Y = (y_i)_1^{\infty}$, $\sum_1^{\infty}\|x_i\|^2 < \infty$ and $\sum_1^{\infty}\|y_i\|^2 < \infty$.

Now, $X = (x_i)_1^{\infty} = \sum_{j=i}^{\infty} U_jV_jX$.

$$y_i = V_iY = V_iAX = V_iA(\sum_{j=1}^{\infty} U_jV_jX) = \sum_{j=1}^{\infty} (V_iAU_j)V_jX$$

$= \sum_{j=1}^{\infty} A_{ij}V_jX$. Now, $(A_{ij})(x_i)_{i=1}^{\infty} = (\sum_{j=1}^{\infty} A_{ij}\, x_j)_{i=1}^{\infty}$ (by the usual matrix multiplication)

$= (\sum_{j=1}^{\infty} A_{ij}V_jX)_{i=1}^{\infty} = (y_i)_{i=1}^{\infty} = Y$. Thus $A = (A_{ij})_{i,j=1}^{\infty}$.

Consequently, each A in $B(H_0)$ can be identified with an infinite matrix (A_{ij}), where each $A_{ij} \in B(H)$. (For further details of matrix representation, see Chapter 4.) For $T \in B(H)$, let $\tilde{T}$ be the operator $(\delta_{ij}\, T)_{i,j=1}^{\infty}$ of $B(H_0)$. Then $\tilde{S}X - \tilde{T}X = (Sx_i - Tx_i)_{i=1}^{\infty}$, so that $\|\tilde{S}X - \tilde{T}X\|^2 = \sum_{i=1}^{\infty}\|(S - T)x_i\|^2$ and so it suffices to show that

(*) there exists an $R \in \mathcal{R}$ such that $\|(\tilde{S} - \tilde{R})X\| < \varepsilon$.

Let $\tilde{\mathcal{R}} = [\tilde{R}: R \in \mathcal{R}]$. Then, evidently, $\tilde{\mathcal{R}}$ is a *-subalgebra of $B(H_0)$ and routine matrix computations show that $(\tilde{\mathcal{R}})' = \{A=(A_{ij})_{i,j} \in B(H_0): A_{ij} \in \mathcal{R}'$ for all i and $j\}$, $\tilde{S} \in (\mathcal{R}'')^{\sim}$, $\tilde{S} = \tilde{S}\,\tilde{E}$. Finally, $\tilde{E}$ is the principal identity of $\tilde{\mathcal{R}}$. For this, we note first that if $Y=(y_i)_1^{\infty} \in H_0$ and $R \in \mathcal{R}$, then $\tilde{R}Y = (Ry_i)_1^{\infty} = (ERy_i)_1^{\infty} = \tilde{E}(\tilde{R}\,Y)$. Hence the principal identity F of $\tilde{\mathcal{R}}$ satisfies $\tilde{E}F = F$; i.e., $F \leq \tilde{E}$. Suppose conversely $Z = (z_i)_1^{\infty} \in \tilde{E}$. Then $(z_i)_1^{\infty} = \tilde{E}(z_i)_1^{\infty} = (Ez_i)_1^{\infty}$ so that $z_i \in E$, for each i. Given y in H and $R \in \mathcal{R}$, we may define Y in H_0 by $Y = (0, 0, \ldots, 0, \underset{(i^{th})}{y}, 0, \ldots)$. Then F contains the vector $\tilde{R}Y = (0, \ldots, 0, \underset{(i^{th})}{Ry}, 0, \ldots)$. By taking norm limits of linear combinations of such vectors, we find that F contains $(0, 0, \ldots, 0, z_i, 0, \ldots)$ and hence contains $(z_1, z_2, \ldots, z_k, 0, \ldots)$, for each k, and so contains $Z = \lim_k (z_1, z_2, \ldots, z_k, 0 \ldots)$. Thus $\tilde{E} \leq F$. Hence $\tilde{E} = F$.

The assertion (*) now follows from Lemma 2.3.6 with $\tilde{\mathcal{R}}$, $\tilde{R}$, $\tilde{S}$ and X in place of $\mathcal{R}$, R, S and x, respectively.

Theorem 2.3.8. (The double commutant theorem) If $\mathcal{R}$ is a *-subalgebra of B(H) with the principal identity E, then for each of the topologies τ_w, τ_s, $\tau_{\sigma s}$ and $\tau_{\sigma w}$ the corresponding closure of $\mathcal{R} = \{S \in \mathcal{R}'': S = SE\}$. In particular, when $\mathcal{R}$ is a von Neumann algebra, $\mathcal{R} = \mathcal{R}''$.

Proof. If τ is one of these topologies, then $\mathcal{R}_0 = \{S \in \mathcal{R}'': S = SE\}$ is τ-closed as the mapping $T \to T(I-E)$ is τ-continuous and the topology τ is Hausdorff. $\mathcal{R}_0$ contains $\mathcal{R}$ by Lemma 2.3.4 (ii). Hence $\mathcal{R}_0$ contains the τ-closure of $\mathcal{R}$.

Since
$$\begin{array}{ccc} \tau_w & \leq & \tau_{\sigma w} \\ \wedge & & \wedge \\ \tau_s & \leq & \tau_{\sigma s} \end{array}$$
it is now sufficient to show that $\tau_{\sigma s}$-closure of $\mathcal{R}$ contains $\mathcal{R}_0$. This follows from Lemma 2.3.7. If $\mathcal{R}$ is a von Neumann algebra, then $E = I \in \mathcal{R}$ and hence $\mathcal{R}_0 = \mathcal{R}''$. But, as $\mathcal{R}$ is weakly closed, it follows that $\mathcal{R}'' = \mathcal{R}$.

Corollary 2.3.9. Let $\mathcal{R}$ be a *-subalgebra of B(H). Then the following four conditions are equivalent:

(i) $\mathcal{R}$ is τ_w-closed.

(ii) $\mathcal{R}$ is τ_s-closed.

(iii) $\mathcal{R}$ is $\tau_{\sigma w}$-closed.

(iv) $\mathcal{R}$ is $\tau_{\sigma s}$-closed.

Proof. Each is equal to $\mathcal{R}_0 = \{S \in \mathcal{R}'': S = SE\}$, where E is the principal identity of $\mathcal{R}$.

Corollary 2.3.10. Suppose that $\mathcal{R}$ is a τ_W-closed *-subalgebra of B(H) with the principal identity E. Then $E \in \mathcal{R}$.

Proof. By Lemma 2.3.4 (i), $E \in \mathcal{R}' \cap \mathcal{R}''$. Hence $E \in \mathcal{R}''$ and E.E = E. Hence $E \in \mathcal{R}^{-W} = \mathcal{R}$.

Remarks.

(i) Since $E \in \mathcal{R}$ when $\mathcal{R}$ is a τ_W-closed *-subalgebra of B(H) and since ER = RE = R for each $R \in \mathcal{R}$, E is the identity of $\mathcal{R}$. Hence every τ_W-closed *subalgebra of B(H) has an identity (not necessarily I).

(ii) For each $R \in \mathcal{R}$, where $\mathcal{R}$ is a τ_W-closed *-subalgebra of B(H), let R_E be the restriction of R to E(H). Then $\mathcal{R}_E = \{ R_E : R \in \mathcal{R} \}$ is a von Neumann algebra over E(H). Since R(I - E) = 0 for each $R \in \mathcal{R}$, this reduces the study of τ_W-closed *-subalgebra of B(H) to that of von Neumann algebras.

Corollary 2.3.11. Let $\mathcal{R}$ be a *-subalgebra of B(H). If $\mathcal{R}$ has the principal identity I, then the closure of $\mathcal{R}$ in any of the topologies τ_W, τ_S, $\tau_{\sigma W}$ and $\tau_{\sigma S}$ is $\mathcal{R}''$.

Corollary 2.3.12. Let $\mathcal{R}$ be a *-subalgebra of B(H). Then the following conditions are equivalent:

(i) $\mathcal{R} = \mathcal{R}''$.

(ii) $\mathcal{R} = \mathcal{F}'$ for some $\mathcal{F} \subset B(H)$.

(iii) $\mathcal{R}$ is a von Neumann algebra.

Proof. (i) $\Longrightarrow$ (ii) Obvious. (ii) $\Rightarrow$ (iii) Obvious. (see 2.3.1.) (iii) $\Rightarrow$ (i) By the last part of Theorem 2.3.8.

Dixmier takes (i) above for the definition of von Neumann algebras in his treatise [1].

Corollary 2.3.13. If $\mathcal{R}$ is a von Neumann algebra, then $\mathcal{R}$ and $\mathcal{R}'$ have the same centre, namely $\mathcal{R} \cap \mathcal{R}'$.

Corollary 2.3.14. If $\Gamma \subset B(H)$, then $\Gamma'' = \mathcal{R}(\Gamma)$.

Proof. By 2.3.12, Γ'' is a von Neumann algebra containing Γ. Hence $\mathcal{R}(\Gamma) \subset \Gamma''$. If $\mathcal{R}$ is any von Neumann algebra containing Γ, then $\mathcal{R} \supset \Gamma \Longrightarrow \mathcal{R} = \mathcal{R}'' \supset \Gamma''$. Hence $\mathcal{R}(\Gamma) = \Gamma''$.

§2.4. The Kaplansky density theorem

In this section we study an important result due to Kaplansky, which finds application in several situations in the following chapters.

We recall that $B(H)_1$ denotes the unit ball of $B(H)$.

Lemma 2.4.1.

(i) Let $T = T^* \in B(H)$. Then $(I + T^2)^{-1}$ and $2T(I + T^2)^{-1}$ are in $B(H)_1$.

(ii) Let $S = S^* \in B(H)_1$. Then there exists a T in $B(H)_1$ such that $T = T^*$, T is in the C*-algebra generated by S and further, $S = 2T(1 + T^2)^{-1}$.

Proof.

(i) Since $\sigma(T)$ is real, we can define $f \in C(\sigma(T))$ by $f(t) = (1 + t^2)^{-1}$. Then $(1 + t^2)\, f(t) = f(t)\,(1 + t^2) = 1$, $|f(t)| \leq 1$ and $|2t\, f(t)| \leq 1$ for each t in $\sigma(T)$ and hence, by Theorem 1.5.7, we have $(I + T^2)\, f(T) = f(T)(I + T^2) = I$,

$||f(T)|| \leqq 1$ and $||2T\, f(T)|| \leqq 1$. Thus $I + T^2$ has the inverse $f(T)$ and (i) is proved.

(ii) Since $S = S^* \in B(H)_1$, we have $\sigma(S) \subset [-1, 1]$. With $g(t) = 2t(1+t^2)^{-1}$, g is a strictly increasing continous mapping from [-1, 1] onto [-1,1] and g(0)=0.Hence it has a continuos inverse f on [-1,1] with f(0)= 0.Since f(0) = 0, f can be approximated uniformly on [-1, 1] by real polinomials p, with p(0) = 0. Thus T = f(S) belongs to the C*-algebra generated by S (without identity). Hence T* = T and $||T|| \leq \sup_{t\in[-1,1]} |f(t)| = 1$.

Since $s = g(f(s)) = 2\,\dfrac{f(s)}{1+(f(s))^2}$,

$s(1 + (f(s))^2) = 2 f(s)$. Hence we have from the functional calculus theorem (Theorem 1.5.7) that $S = 2T(I + T^2)^{-1}$ and $S(I + T^2) = 2T$.

Theorem 2.4.2. (The Kaplansky density theorem) Suppose that A and $\mathcal{B}$ are *-subalgebras of B(H), such that $A \subset \mathcal{B}$ and A is τ_s-dense in $\mathcal{B}$. Let M and N be the sets of self-adjoint elements in A and $\mathcal{B}$, respectively. Then A_1 (respectively, M_1) is τ_s-dense in $\mathcal{B}_1$ (respectively, N_1) where X_1 denotes the unit ball in X.

Proof. First we can assume that A and B are norm closed *-subalgebras. For, if $\mathcal{C} = \bar{A}^{n}$ and $\mathcal{D} = \bar{\mathcal{B}}^{n}$, then $\mathcal{C}$ and $\mathcal{D}$ are norm closed *-subalgebras and $\mathcal{C} \subset \mathcal{D}$, $\bar{\mathcal{C}}^{s} \supset \bar{A}^{s} \supset \mathcal{B}$; $\bar{\mathcal{C}}^{s} \supset \bar{\mathcal{B}}^{s} \supset \bar{\mathcal{B}}^{n} = \mathcal{D}$. Also, $\mathcal{C}_1 = (\bar{A}^{n})_1$, $\mathcal{D}_1 = (\bar{\mathcal{B}}^{n})_1$ and if $\bar{\mathcal{C}}_1^{s} \supset \mathcal{D}_1$, then, as $\bar{A}_1^{s} \supset \bar{A}_1^{n} = (\bar{A}^{n})_1 = \mathcal{C}_1$, we have $\bar{A}_1^{s} \supset \bar{\mathcal{C}}_1^{s} \supset \mathcal{D}_1 = (\bar{\mathcal{B}}^{n})_1 \supset \mathcal{B}_1$.

Thus it suffices to prove that $\bar{\mathcal{C}}_1^{s} \supset \mathcal{D}_1$. Similarly, if $\bar{M}^{n} = E$, $\bar{N}^{n} = F$ and if $\bar{E}_1^{s} \supset F_1$, then $\bar{M}_1^{s} \supset N_1$.

Therefore, we assume that A and B are norm closed and show first that M_1 is τ_s-dense in N_1. Suppose $S_0 \in N_1$. Then $S_0 = S_0^* \in B$, $\|S_0\| \leq 1$, and hence by Lemma 2.4.1 (ii) there is a $T_0 = T_0^*$ in B such that $S_0 = 2T_0(I + T_0^2)^{-1}$. Since A is τ_s-dense in B, there is a net $\{T_\alpha\}$ in A such that $T_\alpha \to T_0$ in τ_s. Hence $T_\alpha \to T_0$ in τ_w and, since *-operation is continuous in τ_w, $\frac{1}{2}(T_\alpha + T_\alpha^*) \to T_0$ in τ_w. Since $\frac{1}{2}(T_\alpha + T_\alpha^*) \in M$, we have $T_0 \in \overline{M}^w$. Hence, by Corollary 2.2.5, $T_0 \in \overline{M}^s$. We may now assume that $T_\alpha \in M$ for each α and let $S_\alpha = 2T_\alpha(I + T_\alpha^2)^{-1}$. Then $S_\alpha \in M_1$ by 2.4.1 (i) and, for each $x \in H$,

$$\|(S_\alpha - S_0)x\| = \|\{2T_\alpha(I + T_\alpha^2)^{-1} - 2T_0(I + T_0^2)^{-1}\}x\|$$

$$= \|2(I + T_\alpha^2)^{-1}[T_\alpha(I + T_0^2) - (I + T_\alpha^2)T_0](I + T_0^2)^{-1}x\|$$

$$= \|2(I + T_\alpha^2)^{-1}(T_\alpha - T_0)(I+T_0^2)^{-1}x + 2(I+T_\alpha^2)^{-1}T_\alpha(T_0 - T_\alpha)T_0(I+T_0^2)^{-1}x\|.$$

By Lemma 2.4.1 (i) we have

$$\|(S_\alpha - S_0)x\| \leq 2\|(T_\alpha - T_0)(I+T_0^2)^{-1}x\| + \|(T_0 - T_\alpha)T_0(I+T_0^2)^{-1}x\|$$

and hence $S_\alpha \to S_0$ in τ_s. Thus $S_0 \in \overline{(M_1)}^s$. Thus M_1 is τ_s-dense in N_1.

Finally, suppose that $S \in B_1$. Let $\tilde{H} = H \oplus H$ and identify operators on $\tilde{H}$ by 2×2 matrices (S_{ij}) such that $S_{ij} \in B(H)$. Let

$\tilde{A} = \{(S_{ij}): S_{ij} \in A\}$; $\tilde{B} = \{(S_{ij}): S_{ij} \in B\}$; $\tilde{S} = \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \in \tilde{B}$. When $\overline{A}^s \supset B$, $\overline{\tilde{A}}^s \supset \tilde{B}$. For, $T \in \tilde{B} \Rightarrow T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ with $T_{ij} \in B$. Since $\overline{A}^s \supset B$, there is a net $A_{ij}^{(\alpha)} \xrightarrow[\tau_s]{} T_{ij}$ with $A_{ij}^{(\alpha)} \in A\,(i,j=1,2)$.

Now $A_\alpha = (A_{ij}^{(\alpha)}) \in \tilde{A}$ and, for $X = (x_1, x_2)$ in $\tilde{H}$,

$$\|A_\alpha X - TX\|^2 = \left\| \begin{pmatrix} (A_{11}^{(\alpha)} - T_{11})x_1 + (A_{12}^{(\alpha)} - T_{12})x_2 \\ (A_{21}^{(\alpha)} - T_{21})x_1 + (A_{22}^{(\alpha)} - T_{22})x_2 \end{pmatrix} \right\|^2$$

$$= \|(A_{11}^{(\alpha)} - T_{11})x_1 + (A_{12}^{(\alpha)} - T_{12})x_2\|^2$$
$$+ \|(A_{21}^{(\alpha)} - T_{21})x_1 + (A_{22}^{(\alpha)} - T_{22})x_2\|^2 \to 0.$$

Hence $T \in \tilde{\tilde{A}}^{s}$.

$S \in \mathcal{B}_1$ and $\tilde{S} \in \tilde{\mathcal{B}}$. Also $\tilde{S}^* = \tilde{S}$ and $\|\tilde{S}\| \leq 1$.

For, if $X = (x_1, x_2)$, $Y = (y_1, y_2)$ are in $\tilde{H}$, then

$$[\tilde{S}X, Y] = \left[\begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] = \left[\begin{pmatrix} S^*x_2 \\ S x_1 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right]$$
$$= [S^*x_2, y_1] + [Sx_1, y_2]$$
$$= [x_2, Sy_1] + [x_1, S^*y_2]$$
$$= \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} S^*y_2 \\ S y_1 \end{pmatrix}\right] = \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] = [X, \tilde{S}Y].$$

Thus $\tilde{S} = \tilde{S}^*$.

$\|\tilde{S}X\|^2 = \|S^*x_2\|^2 + \|Sx_1\|^2 \leq \|x_2\|^2 + \|x_1\|^2 = \|X\|^2$, so that $\|\tilde{S}\| \leq 1$.

Hence, by the result already proved there is a net $\{A^\alpha\}$ such that $A^\alpha = (A^\alpha)^*$, $A^\alpha = (A_{ij}^{(\alpha)}) \in \tilde{A}$, $\|A^\alpha\| \leq 1$, $A^\alpha \to \tilde{S}$ in τ_s. With S_α as the (2,1) element in the matrix representation of A^α, we have $S_\alpha \in A$.

Further $S_\alpha \to S$ in τ_s, since

$$A^\alpha = \begin{pmatrix} A_{11}^{(\alpha)} & A_{12}^{(\alpha)} \\ A_{21}^{(\alpha)} & A_{22}^{(\alpha)} \end{pmatrix} \underset{\tau_s}{\to} \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} = \tilde{S}$$

implies
$$\begin{pmatrix} A_{11}^{(\alpha)} & A_{12}^{(\alpha)} - S^* \\ A_{21}^{(\alpha)} - S & A_{22}^{(\alpha)} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} A_{11}^{(\alpha)} x \\ (A_{21}^{(\alpha)} - S)\, x \end{pmatrix} \longrightarrow 0$$

i.e., $A_{21}^{(\alpha)} \to S$ in τ_s; i.e., $S_\alpha = A_{21}^{(\alpha)} \to S$ in τ_s.

Again, for $X = (x,0), x \in H$,
$$\|A_{21}^{(\alpha)} x\|^2 \leq \left\| \begin{pmatrix} A_{11}^{(\alpha)} & A_{12}^{(\alpha)} \\ A_{21}^{(\alpha)} & A_{22}^{(\alpha)} \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^2 \leq \|A^\alpha\|^2 \|X\|^2 \leq \|X\|^2 = \|x\|^2$$

so that $\|A_{21}^{(\alpha)}\| = \|S_\alpha\| \leq 1$. Thus $S \in (A_1)^{-s}$.

This completes the proof of the theorem.

Corollary 2.4.3. $(A_1)^{-s} = (A^{-s})_1$ if A is a *-subalgebra of $B(H)$. Consequently, a *-subalgebra A of $B(H)$ containing the identity is a von Neumann algebra if and only if A_1 is closed in one of the τ_s, τ_w, $\tau_{\sigma w}$ and $\tau_{\sigma s}$ topologies.

Proof. As $A \subset \bar{A}^s$, $A_1 \subset (\bar{A}^s)_1$. We assert that $(\bar{A}^s)_1$ is strongly closed. For, if $T_\alpha \in (A^{-s})_1$ and $T_\alpha \xrightarrow[\tau_s]{} T$, then $\|Tx - T_\alpha x\| \to 0$, for each $x \in H$. Hence $\|Tx\| =$
$$= \lim_\alpha \|T_\alpha x\| \leq \lim_\alpha \sup \|T_\alpha\| \|x\| \leq \|x\|.$$

Thus $T \in (\bar{A}^s)_1$. Therefore, $(A_1)^{-s} \subseteq (\bar{A}^s)_1$ (2.4.3.1). But A is strongly dense in $\bar{A}^s$ and it can be shown that $\bar{A}^s$ is a *-subalgebra of $B(H)$, by using the facts that $\bar{A}^s = \bar{A}^w$ and that the *-operation is τ_w continuous, along with Lemma 2.1.1. Hence by the Kaplansky density theorem $(A_1)^{-s} \supset (\bar{A}^s)_1$. Now (2.4.3.1) implies $(A_1)^{-s} = (\bar{A}^s)_1$.

If A is a von Neumann algebra, $\overline{A}^s = A$ and hence $(A_1)^{-s} = (\overline{A}^s)_1 = A_1$. Thus A_1 is τ_s-closed. Since the unit ball is convex and bounded, by Corollary 2.2.5 (iii), A_1 is τ_w-closed, $\tau_{\sigma w}$-closed and $\tau_{\sigma s}$-closed.

Conversely, let A_1 be closed in anyone of the topologies τ_w, τ_s, $\tau_{\sigma w}$ and $\tau_{\sigma s}$. Then, by Corollary 2.2.5 (iii), A_1 is τ_s-closed and hence by the firs part of the present corollary, $A_1 = (A_1)^{-s} = (\overline{A}^s)_1$ and hence $\overline{A}^s = A$. In fact, if $\overline{A}^s \supsetneq A$, then there is a $T \in \overline{A}^s$, $T \notin A$. Then $\frac{T}{\|T\|} \in (\overline{A}^s)_1$ and $\frac{T}{\|T\|} \notin A_1$, a contradiction. Thus A is a τ_s-closed $*$-subalgebra of $B(H)$ containing the identity. Hence A is a von Neumann algebra by Corollaries 2.3.11 and 2.3.12.

Note 22. The above corollary is very effective in applications. The reader will find its use in later chapters. For instance, see the proof of Lemma 5.5.7.

CHAPTER 3

COMPARISON THEORY OF PROJECTIONS

The basic idea for this chapter is that two projections E,F in a von Neumann algebra R on a Hilbert space H should be considered to be of 'the same size', relative to R, if there is an operator V in R such that its restriction to the range of E is an isomorphism of E(H) onto F(H) and such that V(I - E)= 0. Such an operator V is called a partial isometry. Before discussing the comparison theory, we shall study in detail some of the properties of partial isometries in §3.1 below.

§3.1. Partial isometries and the polar decomposition of a closed operator

Though the polar decomposition of a bounded operator is sufficient for the immediate need in the succeeding sections,we study the decomposition in a more general set up to suit the needs of §5.9 also.

Definition 3.1.1. A bounded operator V on the Hilbert space H is called a *partial isometry* if there is a closed subspace M of H such that

$$\|Vx\| = \|x\| \text{ for all } x \in M$$

and

$$Vy = 0 \text{ for all } y \in H \ominus M.$$

M is called the *initial space* of V and the closed subspace $N = \{Vx: x \in H\} = \{Vx: x \in M\}$ is called the *final space* of V. If E and F are projections with ranges M and N, respectively, then E is called the *initial projection* and F is called the *final projection* of the partial isometry V.

Note 1. $V = FVE$ since $V = V(E + I - E) = VE = (F + I - F)VE = FVE$.

Note 2. The defining conditions of V can be replaced by the condition that $\|Vx\| = \|Ex\|$, for all $x \in H$.

Lemma 3.1.2. Suppose $V \in B(H)$, $E = V^*V$, $F = VV^*$. Then the following conditions are equivalent:

(a) E is a projection.

(b) F is a projection.

(c) V is a partial isometry.

(d) V^* is a partial isometry.

When one of these conditions holds, E and F are respectively the initial and final projections of the partial isometry V.

Proof. (a)=> (c) and (b) Let E be a projection. Then $(V(I-E))^*(V(I-E)) = (I-E)V^*V(I-E) = 0$, as $V^*V = E$. Thus $V(I-E) = 0$. Also $\|Vx\|^2 = [Vx,Vx] = [V^*Vx,x] = \|Ex\|^2$. Hence $\|Vx\| = \|Ex\| = \|x\|$ for all $x \in E$. Therefore, V is a partial isometry with initial projection E. Thus (a)=> (c). Further, as $V(I-E) = 0$, $F - F^2 = VV^* - VV^*VV^* = VV^* - VEV^* = V(I-E)V^* = 0$ and hence $F = F^2$. Clearly, $F = F^*$. Thus (a) implies (b) also.

Applying these results with V and V^* interchanged, we have (b)=> (a) and (d).

(c)=> (a) Let V be a partial isometry with initial projection $\tilde{E}$. Then $\|Vx\| = \|\tilde{E}x\| = \|x\|$, $x \in \tilde{E}(H)$ (by definition). By the polarization identity $[Vx,Vy] = [x,y]$, $x,y \in \tilde{E}(H)$. Thus, for $x,y \in H$, $[V^*Vx,y] = [Vx,Vy] = [V\tilde{E}x, V\tilde{E}y] =$

$(\tilde{E}x,\tilde{E}y] = [\tilde{E}x,y]$ and hence $E = V^*V = \tilde{E}$, a projection. Thus (a) holds.

Similarly, applying the above argument with V^* in place of V, we have (d)=>(b).

This completes the proof of the lemma.

Note 3. Obviously, the initial projection of the partial isometry V is the final projection of V^* and vice versa.

Lemma 3.1.3. Let U and V be partial isometries and E be a projection of H. Then:

(i) If $E \leqq U^*U$ (resp.(ii) If $E \leqq UU^*$), then UE(resp.EU) is a partial isometry with initial (resp.final) projection E and final (resp.initial) projection $\leqq UU^*$. (resp.$\leqq U^*U$.)

(iii) If $UU^* \leqq V^*V$, then VUV^* is a partial isometry with initial projection UU^* and final projection $\leqq VV^*$.

(iv) If $V^*V \leqq UU^*$, then U^*V^*V is a partial isometry with final projection VV^* and initial projection $\leqq U^*U$.

Proof.

(i) $(UE)^*UE = EU^*UE = E$, $(UE)(UE)^* = UEU^*$ and $(UE)(UE)^*UU^* = UEU^*$.

(ii) By (i), U^*E is a partial isometry with initial projection E and final projection $\leqq U^*U$. Hence $(U^*E)^* = EU$ is a partial isometry with desired properties.

(iii) This follows from (i).

(iv) This is immediate from (ii).

Lemma 3.1.4. Suppose that $\{V_\alpha\}_{\alpha \varepsilon J}$ is a family of partial isometries on H, $E_\alpha = V_\alpha^* V_\alpha$, $F_\alpha = V_\alpha V_\alpha^*$ and $\{E_\alpha\}_{\alpha \varepsilon J}$ and $\{F_\alpha\}_{\alpha \varepsilon J}$ are orthogonal families. Then $V = \sum_{\alpha \varepsilon J} V_\alpha$ exists in the strong operator topology and V is a partial isometry with $V^*V = \sum_{\alpha \varepsilon J} E_\alpha$ and $VV^* = \sum_{\alpha \varepsilon J} F_\alpha$.

Proof. For each $x \varepsilon H$, $V_\alpha x = F_\alpha V_\alpha E_\alpha x$ so that the terms of $\sum_{\alpha \varepsilon J} V_\alpha x$ are pairwise orthogonal and $\sum_{\alpha \in J} \|V_\alpha x\|^2 = \sum_{\alpha \in J} \|F_\alpha V_\alpha E_\alpha x\|^2 = \sum_{\alpha \in J} \|E_\alpha x\|^2 = \|Ex\|^2$, where $E = \sum_{\alpha \in J} E_\alpha$. Hence $\sum_{\alpha \in J} V_\alpha x$ is convergent and its sum Vx satisfies $\|Vx\| = \|Ex\|$, for $x \in H$. Thus V is a partial isometry with initial projection $E = \sum_{\alpha \in J} E_\alpha$ (see Note 2). The same argument with V_α^* in place of V_α shows that $\sum_{\alpha \in J} V_\alpha^*$ has initial projection $\sum_{\alpha \in J} F_\alpha$.

Definition 3.1.5. Given $T \in B(H)$, the *range projeccion* of T is defined to be $[T(H)]$= the closure of the range of T.

Note 4. $[T^*(H)] = [T^*T(H)]$.

For, clearly, $[T^*T(H)] \leq [T^*(H)]$. Let $[x, T^*Ty] = 0$, for each $y \in H$. Then

$$\|Tx\|^2 = [Tx, Tx] = [x, T^*Tx] = 0$$

and hence $Tx = 0$, whence

$$[x, T^*y] = [Tx, y] = 0, \text{for each } y \in H.$$

Thus $[T^*T(H)]^\perp \subset [T^*(H)]^\perp$, which means $[T^*(H)] \leq [T^*T(H)]$. Hence the note.

Theorem 3.1.6 (Polar decomposition). Every closed operator A on H with domain dense in H and range in H is uniquely expressible in the form $A = UP$, where P is a positive definite self-adjoint operator with $\mathcal{D}(P) = \mathcal{D}(A)$, kernel of P = kernel of A and U is a partial isometry with initial projection $\overline{R(A^*)}$ and final projection $\overline{R(A)}$. ($R(S)$ denotes the range of the operator S.)

Proof. By Lemma 1.4.18, A^*A is a positive definite self-adjoint operator on H. Consequently, in virtue of Lemma 1.4.22, $(A^*A)^{\frac{1}{2}} = P$ exists as a positive definite self-adjoint operator on H, with $\mathcal{D}(P) \supset \mathcal{D}(A^*A)$.

Let P_1 and A_1 be respectively the restrictions of P and A to $\mathcal{D}(P^2) = \mathcal{D}(A^*A)$. Thus, for $x \varepsilon \mathcal{D}(P^2)$,

$$\|P_1x\|^2 = [P_1x, P_1x] = [P_1^2x, x] = [P^2x, x] = [A^*Ax, x]$$

$$= [Ax, Ax] = \|Ax\|^2 = \|A_1x\|^2.$$

Consequently,

$$\|P_1x\| = \|A_1x\| \quad \text{for all } x \ \varepsilon\ \mathcal{D}(P_1) = \mathcal{D}(A_1). \quad (3.1.6.1)$$

Since $P_1 = P|\mathcal{D}(P^2)$ and $A_1 = A\,|\mathcal{D}(P^2)$, and since P, A are closed operators on H, P_1 and A_1 admit closures and hence from (3.1.6.1) we have

$$\|\tilde{P}_1x\| = \|\tilde{A}_1x\| \quad \text{for all} \quad x \,\varepsilon\, \mathcal{D}(\tilde{P}_1) = \mathcal{D}(\tilde{A}_1). \qquad (3.1.6.2)$$

We shall prove that $\tilde{P}_1 = P, \tilde{A}_1 = A$; i.e., $\Gamma_{\tilde{P}_1} = \Gamma_P$ and $\Gamma_{\tilde{A}_1} = \Gamma_A$, where Γ_A denotes the graph of A, etc. Clearly, $\Gamma_{\tilde{A}_1} \subset \Gamma_A$. If $\Gamma_{\tilde{A}_1} \neq \Gamma_A$, then there exists a non-zero vector $(x, Ax) \varepsilon\ \Gamma_A$, orthogonal to Γ_{A_1};

i.e., $[(x, Ax), (y, A_1y)] = 0$, for all $y \ \varepsilon\ \mathcal{D}(A_1)$.

This means that

$$0 = [x,y] + [Ax, A_1y]$$

$$= [x,y] + [Ax, Ay]$$

$$= [x,y] + [x, A^*Ay]$$

$$= [x, [I + A^*A)y].$$

Since, from the proof of Lemma 1.4.18, it is clear that $R(I + A^*A) = H$, it follows that $x = 0$. Hence $(x, Ax) = (0,0)$, a contradiction. Thus $\tilde{A}_1 = A$ and similarly, $\tilde{P}_1 = P$. Thus (3.1.6.2) assumes the form

$$\|Px\| = \|Ax\| \quad \text{for all } x \in \mathcal{D}(P) = \mathcal{D}(A). \qquad (3.1.6.3)$$

Let us define a linear operator U' as follows:

$$\left.\begin{array}{l} U'Px = Ax \text{ for all } x \in \mathcal{D}(P) \\ U'y = 0 \quad \text{for } y \perp R(P). \end{array}\right\} \qquad (3.1.6.4)$$

In view of (3.1.6.3), we can extend U' by continuity to $\overline{R(P)}$, and let us call this extension U. Then U is a partial isometry with initial projection $\overline{R(P)}$ and final projection $\overline{R(A)}$. It follows from (3.1.6.4) that A= UP.

To prove that $\overline{R(P)} = \overline{R(A^*)}$, it suffices to show that

$$H \ominus R(P) = H \ominus R(A^*).$$

Since P and A are closed operators, $N(P) = \{x \in \mathcal{D}(P) : Px = 0\}$ and $N(A) = \{x \in \mathcal{D}(A) : Ax = 0\}$ are closed in H; $P^{**} = P$ and $A^{**} = A$. Moreover, $P^* = P$, as P is self-adjoint. $H \ominus R(A^*) = \{y \in H : [y, A^*x] = 0 \text{ for all } x \in \mathcal{D}(A^*)\} = \{y \in H : [y, A^*x] = 0 = [A^{**}y, x] \text{ for all } x \in \mathcal{D}(A^*)\} = N(A^{**}) = N(A)$, since $\mathcal{D}(A^*)$ is dense in H by Lemma 1.4.14(ii). Similarly, $H \ominus R(P) = N(P)$. Since $N(P) = N(A)$ by (3.1.6.3), it follows that $H \ominus R(P) = H \ominus R(A^*)$.

It remains to prove the uniqueness of the operators U and P. If A= UP, then $A^* = PU^*$ and $A^*A = PU^*UP = PE_{\overline{R(P)}}P = P^2$, where $E_{\overline{R(P)}}$ is the projection onto the subspace $\overline{R(P)}$ in H. Consequently, P is uniquely fixed. Since U has initial space $\overline{R(P)}$, the equation A= UP fixes U also uniquely, as P is fixed and U is required to be continuous.

The above decomposition is called the *polar decomposition* of the closed operators A, which have domain dense in H.

The above result can be generalized to closed linear transformations with dense domain from one Hilbert space into another.

Definition 3.1.7. A closed linear operator A on H is said to be affliated to the von Neumann algebra R if A commutes with all $R' \varepsilon R'$, and we then write $A \eta R$ (A commutes with $R' \varepsilon R'$ means $R'A \subset AR'$). If $A \varepsilon B(H)$ and $A \eta R$, then $A \varepsilon R''$, since $R''=R$.

Theorem 3.1.8. Let A be a closed operator with domain dense in H, and let A= UP be the polar decomposition of A. If $A \eta R$, R a von Neumann algebra, then $P \eta R$ and U εR; consequently, $\overline{R(A)}$ and $\overline{R(A^*)}$ are in R. If $A \varepsilon R$, then both P and U are in R.

Proof. First we shall prove the following result (*):

(*) $A \eta R$ if and only if $A= U'AU'^{-1}$ for all unitary operators $U' \varepsilon R'$.

Proof of (*). If $A \eta R$ and U' is a unitary operator in R', then, by definition, $U'A \subset AU'$ (3.1.8.1).

As $U'^{-1} \varepsilon R'$, $U'^{-1}A \subset AU'^{-1}$. Hence $U'U'^{-1}A \subset U'AU'^{-1}$;

i.e., $A \subset U'AU'^{-1}$. Then $AU' \subset U'AU'^{-1}U' = U'A$.

Thus $U'A \supset AU'$ (3.1.8.2). Hence by (3.1.8.1) and (3.1.8.2) we have $U'A= AU'$ or, equivalently, $A= U'AU'^{-1}$.

Conversely, if $U'A= AU'$ for all unitary operators U' in R', then $U'A \subset AU'$ and hence, by Theorem 1.5.10, $R'A \subset AR'$ for all $R' \varepsilon R'$. Thus $A \eta R$.

Next we prove the following result:

If A= UP is the polar decomposition of A, then U and P are determined by the following properties (**):

$$(**) \begin{cases} P \text{ is positive definite and self-adjoint, } P^2= A^*A; \\ A= UP; \text{ and } Px= 0 \text{ implies } Ux= 0. \end{cases}$$

In fact, the first two properties in (**) determine P as $(A^*A)^{\frac{1}{2}}$. The last two properties in (**) determine U on the range of P and on $\{x \in \mathcal{D}(P): Px = 0\}$. As U is continuous, U is determined on $\overline{R(P)}$. As in the last part of the proof of Theorem 3.1.6 we observe that $N(P) = R(P)^{\perp}$ so that U is determined on $N(P) = R(P)^{\perp}$. This shows that U is determined on $H = \overline{R(P)} \oplus R(P)^{\perp}$.

Now coming to the proof of the theorem, let A = UP be the polar decomposition of A. If U' is a unitary operator in $\mathcal{R}'$, then, as $A \eta \mathcal{R}$,

$$\begin{aligned} A &= U'AU'^{-1} \text{ (by (*))} \\ &= U'UPU'^{-1} \\ &= (U'UU'^{-1})(U'PU'^{-1}). \end{aligned}$$

But $(U'PU'^{-1})^* = U'PU'^{-1}$ and hence $U'PU'^{-1}$ is self-adjoint. Clearly, it is positive definite. Further, as (*) implies that $U'A^*U'^{-1} = A^*$, we have

$$(U'PU'^{-1})(U'PU'^{-1}) = U'P^2U'^{-1} = U'A^*AU'^{-1} = U'A^*U'^{-1}.U'AU'^{-1} = A^*A.$$

$U'PU'^{-1}x = 0 \Rightarrow PU'^{-1}x = 0 \Rightarrow UU'^{-1}x = 0$ (by the construction of the polar decomposition as in 3.1.6) $\Rightarrow U'UU'^{-1}x = 0$.

Thus $U'UU'^{-1}$ and $U'PU'^{-1}$ satisfy the conditions in (**) above and hence

$$U'PU'^{-1} = P$$

and

$$U'UU'^{-1} = U.$$

This holds for all unitary operators U' in $\mathcal{R}'$ and hence by (*) $P \eta \mathcal{R}$ and $U \eta \mathcal{R}$. Since U is bounded and, $\mathcal{R}'' = \mathcal{R}$, as $\mathcal{R}$ is a von Neumann algebra, $U \eta \mathcal{R}$ is the same as $U \in \mathcal{R}$. Since $U^*U = \overline{R(A)}$ and $UU^* = \overline{R(A)}$, both $\overline{R(A^*)}$ and $\overline{R(A)}$ are in $\mathcal{R}$. If A is a bounded operator, then P is bounded and hence $P \eta \mathcal{R}$ is the same as $P \in \mathcal{R}$.

Note 5. The special case of Theorem 3.1.8 when A is a bounded operator on H, will be needed in most part of these lecture notes. The simple direct proof for the polar decomposition of A when A is a bounded operator, is left to the reader.

Theorems 3.1.6 and 3.1.8 will be needed in §5.9.

§3.2. Comparison theory

Throughout this section R is a von Neumann algebra with centre Z, acting on a Hilbert space H.

Definition 3.2.1. Let E and F be projections in R. We say that

(i) E is *equivalent* to F (relative to R) and write $E \sim F$ if there is a partial isometry V in R such that $V^*V = E$ and $VV^* = F$.

(ii) $E \precsim F$ if there is a projection $E_1 \in R$ such that $E \sim E_1 \leqq F$.

(iii) $E \prec F$ if $E \leqq F$ and $E \not\sim F$.

Proposition 3.2.2. The relation $\sim$ in the above definition is an equivalence relation.

Proof.

(i) $E \sim E$, since $V = E$ gives the equivalence.

(ii) If $E \sim F$, $U^*U = E$, $UU^* = F$ and $U \in R$, then U^* gives that $F \sim E$.

(iii) Let $E \sim F$, $F \sim G$, with $E = U^*U$, $F = UU^*$; $F = W^*W$, $G = WW^*$, with $U, W \in R$. Then $(WU)^*(WU) = U^*FU = U^*UU^*U = E$ and $(WU)(WU)^* = WUU^*W^* = WFW^* = WW^*WW^* = G$. Hence $E \sim G$, since WU is a partial isometry by Lemma 3.1.2.

Example. When $R = B(H)$, for two projections $E, F \in R$,

$E \sim F \iff \dim E(H) = \dim F(H)$; $E \precsim F \iff \dim E(H) \leqq \dim F(H)$; $E \prec F \iff \dim E(H) \lneq \dim F(H)$, where 'dim' means cardinality of an orthonormal basis.

Lemma 3.2.3.

(i) Let E, F be projection in R and Q a projection in Z. If $E \sim F$ (respectively, $E \precsim F$)$QE \sim QF$ (respectively, $QE \precsim QF$).

(ii) Let $(E_\alpha)_{\alpha \varepsilon J}$, $(F_\alpha)_{\alpha \varepsilon J}$ be orthogonalfamilies of projections in R . If $E_\alpha \sim F_\alpha$ (respectively, $E_\alpha \precsim F_\alpha$) for each $\alpha \varepsilon J$ then $\sum_{\alpha \varepsilon J} E_\alpha \sim \sum_{\alpha \varepsilon J} F_\alpha$(respectively, $\sum_{\alpha \varepsilon J} E_\alpha \precsim \sum_{\alpha \varepsilon J} F_\alpha$).

Proof.

(i) Let V be in R such that $V^*V= E$, $VV^*= E_1$, where $E_1 \leq F$ and $E_1= F$ if $E \sim F$. Let $W= QV$. Then $W^*W= QE$ and $WW^*= QE_1 \leq QF$ and $QE_1= QF$ if $E_1= F$. Hence (i) holds.

(ii) Choose V_α in R such that $V_\alpha^*V_\alpha= E_\alpha$, $V_\alpha V_\alpha^*= G_\alpha$, where $G_\alpha \leq F_\alpha$ and $G_\alpha= F_\alpha$ if $E_\alpha \sim F_\alpha$, for each $\alpha \varepsilon J$. Then $V= \Sigma V_\alpha$ is a partial isometry in R by Lemma 3.1.4, since R is τ -closed. Also $V^*V= \Sigma E_\alpha$ and $VV^*= \Sigma G_\alpha$. Thus $\Sigma E_\alpha \sim \Sigma G_\alpha \leq \Sigma F_\alpha$. This proves (ii).

Theorem 3.2.4. Let E,F,G be projections in R . Then:

(i) $E \precsim E$.

(ii) $E \precsim F$, $F \precsim E$ imply $E \sim F$.

(iii) $E \precsim F$, $F \precsim G$ imply $E \precsim G$.

Hence $\precsim$ induces a partial ordering on the equivalence classes of projections in R .

Proof.

(i) Since $E \sim E$, $E \precsim E$. Before proving (ii), let us prove (iii).

(iii) Suppose $E \sim F_1 \leq F$, $F \sim G_1 \leq G$, F_1, G_1 in R. Then there exist partial isometries U and V in R such that $U^*U= E$, $UU^*= F_1$; $V^*V= F$, $VV^*= G_1$. Then $W= VU$ is a partial isometry in R with $W^*W= U^*V^*VU= U^*FU= U^*F_1U= U^*U= E$ and $WW^*= VUU^*V^*= VF_1V^* \leq VV^* = G_1 \leq G$. Hence $E \precsim G$.

(ii) Suppose that $E \sim F_1 \leq F$, $F \sim E_1 \leq E$. It suffices to show that $E_1 \sim E$, since, then $F \sim E_1$ and $E_1 \sim E$ and hence $F \sim E$. Choose U, V in R with $U^*U = E$, $UU^* = F_1$, $V^*V = F$, $VV^* = E_1$. Then $W = VU$ is a partial isometry in R, with $W^*W = U^*FU = U^*U = E$ and $WW^* = VUU^*V^* = VF_1V^* \leq VV^* = E_1$. Thus $W^*W = E$ and $WW^* = E_2$(say) $\leq E_1$.

Given any subprojection G of E, $G \in R$, WG is a partial isometry with initial and final projections G and WGW^*, respectively, as $(WG)^*(WG) = GEG = G$ and $(WG)(WG)^* = WGW^*$. Thus $G \sim WGW^*$ in R. We define E_n ($n = 1,2,\ldots$) inductively by

$$E_{n+2} = WE_nW^* (n=1,2,\ldots). \qquad (3.2.4.1)$$

(3.2.4.1) holds for $n = 0$ if we define $E_0 = E$, since $WEW^* = WW^* = E_2$.

We have $E = E_0 \geq E_1 \geq E_2$. If $E_n \geq E_{n+1}$, then $E_{n+2} = WE_nW^* \geq WE_{n+1}W^* = E_{n+3}$. Hence $E_{n+2} \geq E_{n+3}$. Thus by induction $E_n \geq E_{n+1}$ ($n=0,1,\ldots$)(3.2.4.2).

Define projections E_∞, G_0, $G_1, \ldots$ in R by

$$E_\infty = \lim_n E_n (\text{in } \tau_s \text{ topology}), \quad G_n = E_n - E_{n+1} (n=0,1,2,\ldots).$$

Then $G_n \sim WG_nW^* = WE_nW^* - WE_{n+1}W^* = E_{n+2} - E_{n+3} = G_{n+2}$ and so

$$G_0 \sim G_2 \sim G_4 \sim G_6 \sim \ldots$$

and

$$G_1 \sim G_3 \sim G_5 \sim \ldots.$$

Finally, since $\sum_0^\infty G_n = \lim_N \sum_0^N (E_n - E_{n+1})$ (in τ_s topology)

$$= \lim_N (E_0 - E_{N+1})$$

$$= E_0 - E_\infty,$$

$$E = E_\infty + \sum_0^\infty G_n = E_\infty + \sum_0^\infty G_{2n} + \sum_0^\infty G_{2n+1} \sim E_\infty + \sum_1^\infty G_{2n} + \sum_0^\infty G_{2n+1} = E_1.$$

This completes the proof.

Lemma 3.2.5. For each $T \in R$, $[T(H)] \sim [(T^*(H)]$.

Proof. If $T=UP$ is the polar decomposition of T in R, then $U \in R$, $U^*U = [T^*(H)]$ and $UU^* = [T(H)]$ by Theorems 3.1.6 and 3.1.8. Hence the lemma holds.

If $A \in R$, the set $\{RAx : R \in R, x \in H\}$ is invariant under each $S \in R$ and each $S' \in R'$. Hence the projection $C_A = [RAx : R \in R, x \in H]$ belongs to $R' \cap R'' = Z$ and clearly $C_A A = A$, since $I \in R$. If Q is any projection in Z such that $QA = A$, then $QRAx = RQAx = RAx$ (for $R \in R$, $x \in H$) and hence $Q \geq C_A$. Thus C_A is the smallest central projection Q of R such that $QA = A$.

Definition 3.2.6. The smallest projection C_A (as in the above) in Z among central projections Q with the property $QA = A$ is called the *central carrier* or *support* of A.

Lemma 3.2.7. Let E, F be projections in R. Then:

(i) $E \precsim F$ implies $C_E \leq C_F$.

(ii) $E \sim F$ implies $C_E = C_F$.

(iii) If $C_E C_F \neq 0$, then there exist projections E_1, F_1 in R such that $0 < E_1 \leq E$, $0 < F_1 \leq F$ and $E_1 \sim F_1$.

Proof.

(i) Let V be in R such that $V^*V = E$ and $VV^* = F_1 \leq F$. Since $V(H) = F_1(H) \subset F(H)$, we have $C_E = [REx : R \in R, x \in H] = [RV^*Vx : R \in R, x \in H] \leq [RVx : R \in R, x \in H] \leq [RFx : R \in R, x \in H] = C_F$. Hence $C_E \leq C_F$.

(ii) Follows from (i), since $E \sim F$ implies $E \precsim F$ and $F \precsim E$.

(iii) Since $C_E = [REx : R \in \mathcal{R}, x \in H]$, $C_F = [RFx : R \in \mathcal{R}, x \in H]$ and $C_E C_F \neq 0$, we can choose R, S in $\mathcal{R}$ and x, y in H such that $0 \neq [REx, SFy] = [FS^*REx, y]$. Let $T = FS^*RE$. Then $T \neq 0$, $T \in \mathcal{R}$ and $T^* = ER^*SF$. By Lemma 3.2.5, $E_1 = [T^*(H)] \sim [T(H)] = F_1$ (say). Clearly, $E_1 \leq E$ and $F_1 \leq F$. Hence (iii) holds.

Note 6. $C_E = C_F$ does not imply, in general, $E \sim F$. e.g. In B(H), let $0 < \dim E(H) < \dim F(H)$. Then $E \not\sim F$, though $C_E = C_F = I$.

Lemma 3.2.8. Suppose E and F are projections in $\mathcal{R}$ and $F \not\precsim E$. Then there exists a projection P in Z such that $0 < P \leq C_F$ and $PE \precnsim PF$.

Proof. Let $G = \{(E_\alpha, F_\alpha)\}_{\alpha \in J}$ be a family of pairs of projections in $\mathcal{R}$ which is maximal subject to the following conditions:

(a) $0 < E_\alpha \leq E$, $\{E_\alpha\}_{\alpha \in J}$ is an orthogonal family,

(b) $0 < F_\alpha \leq F$, $\{F_\alpha\}_{\alpha \in J}$ is an orthogonal family,

(c) $E_\alpha \sim F_\alpha$ for $\alpha \in J$.

Define $E_0 = \sum_{\alpha \in J} E_\alpha$, $F_0 = \sum_{\alpha \in J} F_\alpha$ if $G \neq \emptyset$. Otherwise, define $E_0 = 0 = F_0$. Let $E_1 = E - E_0$, $F_1 = F - F_0$. Since $F_0 \sim E_0 \leq E$, $F_0 \precsim E$ so that $F_0 \neq F$ by hypothesis. Consequently, $F_1 \neq 0$. Thus $0 < F_1 \leq F$ and hence $0 < C_{F_1} \leq C_F$. Furthermore, $C_{E_1} C_{F_1} = 0$, otherwise by 3.2.7(iii) there would exist projections E_2, F_2 in $\mathcal{R}$ with $0 < E_2 \leq E_1$, $0 < F_2 \leq F_1$ and $E_2 \sim F_2$. Thus (E_2, F_2) can be added to the family G, a contradiction to the maximility of G. Thus, with $P = C_{F_1}$, we have $0 < P \leq C_F$ and $PE - PE_0 = C_{F_1} E_1 = C_{F_1} C_{E_1} E_1 = 0$. Hence $PE = PE_0 \sim PF_0$ (by Lemma 3.2.3(i)) $\leqq PF$; i.e., $PE \precsim PF$.

Note 7. In a factor $\mathcal{R}$ (a von Neumann algebra with centre scalars) all projections are comparable. In fact, if E, F are projections in $\mathcal{R}$ and if $F \not\precsim E$, then by the above lemma there is a central projection P such that $0 < P \leq C_F = I$ such that $PE \precsim PF$; i.e., $E \precsim F$, since any non-zero central projection P in the factor $\mathcal{R}$ is I.

Lemma 3.2.9. For T in $\mathcal{R}$ and a central projection Q of $\mathcal{R}$, $C_{QT} = QC_T$.

Proof. Since QC_T is a central projection of R and since $QC_TQT = QT$, $QC_T \geq C_{QT}$. Hence $Q \geq C_{QT}$ so that $(I-Q)$ is orthogonal to C_{QT}. Next, observing that $C_{QT} + I - Q$ is a central projection and $T = QT + (I - Q)T = C_{QT}QT + (I - Q)T = C_{QT}T + (I-Q)T$ ($\because Q \geq C_{QT}$) $= (C_{QT} + I - Q)T$, we have $(C_{QT} + I-Q) \geq C_T$. Hence $QC_T \leq QC_{QT} = C_{QT}$. Thus $C_{QT} = QC_T$.

Theorem 3.2.10 (The comparison theorem). Suppose E and F are projections in R. Then there exist projections P, Q, R in Z such that $P + Q + R = I$, $PE \sim PF$, $Q_oE \prec Q_oF$ for each projection Q_o in Z such that $0 < Q_o \leq Q$ and $R_oF \prec R_oE$ for each projection R_o in Z such that $0 < R_o \leq R$, where P, Q and R are pairwise orthogonal.

Proof. Let $\{P_\alpha\}$ be a maximal orthogonal family of non-zero projections in Z such that $P_\alpha E \sim P_\alpha F$. If this collection is non-empty, take $P = \Sigma P_\alpha$, otherwise $P=0$. Then by Lemma 3.2.3(ii) $PE \sim PF$. If P_o is any central projection with $P_oE \sim P_oF$, then $P_o \leq P$ lest $P_o(I - P) \neq 0$ and hence $P_o(I - P)$ will be added to the family $\{P_\alpha\}$, contradicting the maximality.

Let $\{Q_\alpha\}$ be a maximal orthogonal family of non-zero projections in Z such that $Q_\alpha \leq I - P$ and $Q_\alpha E \precsim Q_\alpha F$ and let $Q = \Sigma Q_\alpha$ if this family is non-empty, otherwise $Q = 0$. Then $Q \leq I - P$ and $QE \precsim QF$ by Lemma 3.2.3(ii). If Q_o is any projection in Z with $0 < Q_o \leq I - P$ and $Q_oE \precsim Q_oF$, then $Q_o \leq Q$, since, otherwise, $(I - Q)Q_o$ will be a non-null projection in Z satisfying $(I - Q)Q_oE \precsim (I - Q)Q_oF$ so that $(I - Q)Q_o$ can be added to the maximal family $\{Q_\alpha\}$, a contradiction. If Q_o is a projection in Z with $0 < Q_o \leq Q$, then $0 < Q_o \leq I - P$ and hence $Q_o \nleq P$. Therefore, $Q_oE \nsim Q_oF$. However, $Q_oE = Q_oQE \precsim Q_oQF = Q_oF$. Hence $Q_oE \prec Q_oF$.

Let $R = I - P - Q$. If R_o is a projection in Z such that $0 < R_o \leq R$, we claim that $R_oF \prec R_oE$. To prove this, first note that $R_o \nleq P$ and hence $R_oE \nsim R_oF$. So it suffices to prove that $R_oF \precsim R_oE$. If $R_oF \not\precsim R_oE$, by Lemma 3.2.8 there is a projection R_1 in Z such that $0 < R_1 < C_{R_oF} = R_oC_F$ (by Lemma 3.2.9) and such that $R_1R_oE \precsim R_1R_oF$; i.e., $R_1E \precsim R_1F$. Hence $R_1 \leq (P + Q)$, a contradiction, since $R_1 \leq R_o \leq R = I-P-Q$.

Thus $R_0F \precsim R_0E$.

This completes the proof of the theorem.

Remark 1. With the notation used in Theorem 3.2.10, $E \precsim F$ if and only if $R=0$. When this is so, there are no projections R_0 to consider. If $E \prec F$, then $Q \neq 0$. (This is used later in the proofs of Theorem 3.2.14, Lemma 5.8.1, etc.)

Definition 3.2.11. If E and F are projections in R and $E \prec F$, we say $E \prec F$(purely) if $QE \prec QF$, whenever Q is a projection in Z such that $0 < Q \leqq C_F$.

Remark 2. With the notation in Theorem 3.2.10, $QE \prec QF$ (purely). For, $C_{QF} = QC_F \leqq Q$. If $C_{QF} < Q$, by taking $0 < Q_0 = Q - C_{QF} \leqq Q$, $Q_0E \prec Q_0F$; i.e., $QE - C_{QF}E \prec (Q - C_{QF})F = QF - QF = 0$, a contradiction. Hence $C_{QF} = Q$. Thus $QE \prec QF$ (purely) by Theorem 3.2.10.

Remark 3. If E and F are projections in a factor R , then exactly one of the relations

$$E \prec F, \quad E \sim F, \quad E \succ F$$

holds, since one of the projections P,Q,R of Theorem 3.2.10 is I and the other two are zero. Hence, in a factor, $\precsim$ induces a total ordering on the equivalence classes of projections.

Definition 3.2.12. A projection E in R is said to be

(a) *finite* if there is no subprojection E_1 in R such that $E \sim E_1 < E$;

(b) *infinite* if it is not finite;

(c) *properly infinite* if $E \neq 0$ and QE is infinite for each projection Q in Z such that $0 < Q \leqq C_E$.

Proposition 3.2.13. Let E and F be projections in R. Then:

(a) If $E \sim F$ and E is finite, infinite or properly infinite, then the same is true for F.

(b) If E is finite and $F \leqq E$, then F is finite. Consequently, if E is finite and $F \precsim E$, then F is finite.

Proof.

(a) If E is infinite, we have $E \sim E_1 < E$. Choose V in R such that $V^*V = E$, $VV^* = F$. With $W = VE_1$ and $F_1 = WW^*$, $W^*W = E_1$ and $F_1 = VE_1V^* < V(H) = F$; i.e., $F_1 < F$. Hence $F \sim E \sim E_1 \sim F_1 < F$. So F is infinite. If E is finite and if F is infinite, as $F \sim E$, E is infinite by the above argument and hence a contradiction. Hence F is also finite. Let E be properly infinite. As $E \sim F$, $C_E = C_F$ by Lemma 3.2.7 and for $0 < Q < C_E = C_F$ we have $QE \sim QF$. Since QE is infinite, QF is also infinite and it then follows that F is properly infinite.

(b) For, otherwise, $F \sim F_1 < F$. Then $E = (E - F) + F \sim (E - F) + F_1 < E$. Hence E is infinite, a contradiction. Thus (b) holds.

Theorem 3.2.14. Let E be a properly infinite projection in R. Then there is a projection F in R such that $F < E$ and $F \sim E - F \sim E$.

Proof. If Q is a projection in Z and $0 < Q \leqq C_E$, then $E_0 = QE$ is infinite by hypothesis. So there is a projection E_1 in R such that $E_0 \sim E_1 < E_0$. Choose V in R such that $V^*V = E_0$, $VV^* = E_1$. If G is a non-zero subprojection of E_0 in R, and if $W = VG$, then $W^*W = G$ and $WW^* = VGV^*$. Hence $G \sim VGV^*$. Define projections $E_n (n \geqq 2)$ inductively by $E_{n+1} = VE_nV^*$. This holds for $n = 0$ also, since $VE_0V^* = E_1$. Since $E_0 > E_1$ and $E_{n+1} - E_{n+2} = V(E_n - E_{n+1})V^*$, it follows from induction that $E_n > E_{n+1}$. Moreover, the projections $G_n = E_n - E_{n+1}$ satisfy $G_0 \sim G_1 \sim G_2 \sim \ldots$

Let $(G_\alpha)_{\alpha \in A}$ be a maximal orthogonal family of projections in R containing $G_0, G_1 \ldots$ (so that A is infinite) such that $G_\alpha \leqq E_0$, $G_\alpha \sim G_0$. Let $X = E_0 - \sum_{\alpha \in A} G_\alpha$. $G_0 \not\precsim X$, otherwise it will contradict the maximality of $\{G_\alpha\}_{\alpha \in A}$. Hence by Lemma 3.2.8 there is a projection P in Z such that $PX \precsim PG_0$, $0 < P < C_{G_0}$. Then $0 < P < C_{G_0} \leqq C_{QE}$ and $C_{QE} = QC_E = Q$.

We have

$$E_0 = \sum_{\alpha \in A} G_\alpha + X.$$

Hence $PE_0 = \sum_{\alpha \in A} PG_\alpha + PX$, $PX \precsim PG_0$, $0 < P \leq Q$. Since cardinality of A is infinite, we can partition the set A into disjoint sets B and C such that card.A= card.B= card.C and define

$$Y = \sum_{\alpha \in B} PG_\alpha + PX < PE_0$$

so that

$$PE_0 - Y = \sum_{\alpha \in C} PG_\alpha < PE_0.$$

Clearly, $Y \sim PE_0$ and $PE_0 - Y \precsim PE_0$. With r an element of C, card $\{\alpha: \alpha \in C, \alpha \neq r\}$= card.A, so that $PE_0 = \sum_{\alpha \in A} PG_\alpha + PX \precsim \sum_{\alpha \in C \setminus \{r\}} PG_\alpha + PG_r = PE_0 - Y$.

Hence by Theorem 3.2.4, $PE_0 \sim PE_0 - Y \sim Y$. Since $P \leq Q$ and $E_0 = QE$, this implies $PE \sim PE - Y \sim Y$. Thus, if Q is projection in Z and $0 < Q \leq C_E$, there exist projections P in Z and Y in R such that $0 < P \leq Q$, $Y \sim PE - Y \sim PE$.

Let $\{P_\alpha\}$ be a maximal orthogonal family of projections in Z such that $0 < P_\alpha \leq$ $\leq C_E$ and such that there is a projection Y_α in R with $Y_\alpha \leq P_\alpha E$, $Y_\alpha \sim P_\alpha E - Y_\alpha \sim P_\alpha E$. By what we have proved above and by Zorn's lemma, such a maximal family exists. Let $Q = C_E - \Sigma P_\alpha$. Clearly, $Q=0$ by the above argument and the maximality of $\{P_\alpha\}$. Then $C_E = \Sigma P_\alpha$ and $E = C_E E = \Sigma P_\alpha E$. Let $F = \Sigma Y_\alpha$ so that $E - F = \Sigma (P_\alpha E - Y_\alpha)$. Hence by Lemma 3.2.3(ii) $E \sim E - F \sim F$ and $F \leq \Sigma P_\alpha E = E$; i.e., $F \leq E$. Since $E - F \sim F$, $F < E$.

This completes the proof.

Note 8. For the study of properly infinite von Neumann algebras, a result stronger than the above theorem will be proved later in Chapter 6.

Lemma 3.2.15. Suppose $\{E_\alpha\}$ is an orthogonal family of projections in R, such that $\{C_{E_\alpha}\}$ is also an orthogonal family. If $E = \Sigma E_\alpha$, then $C_E = \Sigma C_{E_\alpha}$. Moreover, if each E_α is finite (respectively, properly infinite) then the same is true for E. (If at least one E_α is infinite, clearly E is infinite.)

Proof. With $Q = \Sigma C_{E_\alpha}$, Q is a projection in Z and $Q \geq C_{E_\alpha} \geq E_\alpha$ for each α and hence $Q \geq E$, so that $QE = E$. Thus $Q \geq C_E$. Since $C_E \geq E \geq E_\alpha$, we have $C_E \geq C_{E_\alpha}$ and hence $C_E Q = \Sigma C_E C_{E_\alpha} = \Sigma C_{E_\alpha} = Q$, so that $Q \leq C_E$. Thus $C_E = Q$.

Suppose each E_α is finite and $E \sim F \leq E$. Then $C_{E_\alpha} E \sim C_{E_\alpha} F \leq C_{E_\alpha} E$; i.e., $E_\alpha \sim C_{E_\alpha} F \leq E_\alpha$. Since each E_α is finite, $C_{E_\alpha} F = E_\alpha$. Thus $F.E_\alpha = FC_{E_\alpha} E_\alpha = E_\alpha$ so that $F \geq E_\alpha$. Hence $F \geq \Sigma E_\alpha = E$. Therefore, $F = E$ and hence E is finite.

Suppose next that each E_α is properly infinite. If P is a projection in Z and $0 < P \leq C_E$, then, for some α, $PC_{E_\alpha} \neq 0$. Since $0 < PC_{E_\alpha} \leq C_{E_\alpha}$ and E_α is properly infinite, it follows that $PC_{E_\alpha}.E_\alpha = PE_\alpha$ is infinite and hence PE is infinite. So E is properly infinite.

Theorem 3.2.16. If E is an infinite projection in R, then there is a unique projection Q in Z such that $0 < Q \leq C_E$, QE is properly infinite and $(I-Q)E$ is finite.

Proof. Suppose $E \sim E_1 < E$ and let $E_2 = E - E_1$, $P = C_{E_2}$. Then $0 < P \leq C_E$ and $C_{PE} = PC_E = P$. If Q is a projection in Z and $0 < Q \leq C_{PE}(=P)$, then $0 \neq Q = QP = QC_{E_2}$. Hence $QE_2 \neq 0$ by Lemma 3.2.9. Thus $0 \neq QE_2 = QE - QE_1$, so $QE_1 < QE$. Now $Q(PE) = QE \sim QE_1 < QE$. Thus QE is infinite so that PE is properly infinite. Thus we have proved the following statement(*):

(*) If E is an infinite projection in R, then there is a projection P in Z such that $0 < P \leq C_E$ and PE is properly infinite.

Now, by (*) and by Zorn's lemma, there exists a maximal orthogonal family $\{P_\alpha\}$ of projections in Z such that $0 < P_\alpha \leq C_E$ and $P_\alpha E$ is properly infinite. Since

$C_{P_\alpha E} = P_\alpha C_E = P_\alpha$, by Lemma 3.2.15, for $Q = \Sigma P_\alpha$, QE is properly infinite. If $F = (I-Q)E$ is infinite, then by(*) there is a projection P in Z such that $0 < P \leqq C_F = (I-Q)C_E$ and $PF(=P(I-Q)E=PE)$ is properly infinite. Hence P can be added to the maximal family $\{P_\alpha\}$, a contradiction. So (I-Q)E is finite.

If Q_1 is another projection in Z such that $0 < Q_1 \leqq C_E$, Q_1E is properly infinite and $(I-Q_1)E$ is finite, then $Q_1 \leqq Q$. Otherwise, $Q_1(I-Q) \neq 0$ and then $0 < Q_1(I-Q) \leqq Q_1 = Q_1C_E = C_{Q_1E}$ and so $Q_1(I-Q)E$ is an infinite subprojection of the finite projection (I-Q)E, a contradiction. Then $Q_1 \leqq Q$. Similarly, $Q \leqq Q_1$ and hence $Q = Q_1$. This proves the uniqueness of Q.

Definition 3.2.17. If $\{E_\alpha\}$ is a family of projections of H, then there is a smallest projection $\bigvee_\alpha E_\alpha = [\cup E_\alpha(H)]$ which contains each E_α and a largest projection $\bigwedge_\alpha E_\alpha = \bigcap_\alpha E_\alpha(H)$, which is contained in each E_α. When there are two projections E_1 and E_2 under consideration, we use the notation $E_1 \vee E_2$ and $E_1 \wedge E_2$.

Note 9.If each $E_\alpha \in R$, then each E_α is invariant under R' and hence $\bigvee_\alpha E_\alpha$ and $\bigwedge_\alpha E_\alpha$ are invariant under R' so that $\bigvee_\alpha E_\alpha$ and $\bigwedge_\alpha E_\alpha$ belong to $R''=R$.

Lemma 3.2.18. If E and F are projections in R, then $(E \vee F) - F \sim E - (E \wedge F)$.

Proof. Let $T = E(I-F)$ and $x \in H$. Then $Tx = E(I-F)x \in E$ and $(E \wedge F)Tx = (E \wedge F)E(I-F)x = (E \wedge F)(I-F)x = 0$, so that $Tx \in E - E \wedge F$. Thus $[T(H)] \leqq E - E \wedge F$ (3.2.18.1). Also, for $x \in [T(H)]^\perp$, $0 = [Ty,x]$ for each $y \in H$ so that $0 = [y, T^*x]$ for all $y \in H$. Hence $0 = T^*x = (I-F)Ex \Rightarrow Ex \in F \Rightarrow Ex \in E \wedge F \Rightarrow x = Ex + (I-E)x \in E \wedge F + (I-E)$. Thus $[T(H)]^\perp \leqq E \wedge F + (I-E)$. Consequently, $[T(H)] \geqq I - (E \wedge F) - (I-E) = E - E \wedge F$ (3.2.18.2). Thus $[T(H)] = E - E \wedge F$ by (3.2.18.1) and (3.2.18.2). Since $T^* = (I-F)E = (I-F)(I-(I-E))$, by the above result we have $[T^*(H)] = (I-F) - (I-F) \wedge (I-E) = (I-F) - \{(I-F)' \vee (I-E)'\}' = (I-F) - \{E \vee F\}' = (I-F) - \{I - E \vee F\} = E \vee F - F$, where $E' = I - E$, etc. Since by Lemma 3.2.5. $[T(H)] \sim [T^*(H)]$, the lemma holds.

Theorem 3.2.19. If E and F are finite projections in R, then $E \vee F$ and $E \wedge F$ are finite in R.

Proof. Since $E \wedge F \leqq E$ and E is finite, $E \wedge F$ is finite. If possible, let $E \vee F$ be infinite. As E is finite and as $EVF-F \sim E - E\wedge F$ by the above lemma, EVF-F is finite. Thus EVF is the sum of two orthogonal finite projections EVF-F and F.Hence we way assume henceforth that E and F are othogonal finite projections in R and that E + F is infinite in R.

By Theorem 3.2.16 there is a projection Q in Z such that Q(E + F) is properly infinite, of course, QE,QF are finite. Hence, without loss, we may assume that E and F are orthogonal finite projections and that E + F is properly infinite.

By Theorem 3.2.14 there exist orthogonal projections E_1 and F_1 in R such that

$$E_1 + F_1 = E + F,\ E_1 \sim F_1 \sim E + F \text{ so that } C_{E_1} = C_{F_1} = C_{E+F}.$$

By the comparison theorem applied to $E_1 \wedge F$ and $E \wedge F_1$, there exists a projection G in Z such that $0 < G \leqq C_{E+F}$ and either $G(E_1 \wedge F) \precsim G(F_1 \wedge E)$ or $G(F_1 \wedge E) \precsim G(E_1 \wedge F)$(take $G = C_{E+F}(P+Q)$,or $G=C_{E+F}(P+R)$, with P,Q,R as in the said theorem. $G \neq 0$ as $P + Q + R = I$.) We show then in the first case that $GE_1 \precsim GE$. Similarly, in the second case we would have $GF_1 \precsim GF$. In both cases we have a finite projection $\succsim$ an infinite projection (since E_1 is properly infinite and $0 < G \leqq C_{E+F} = C_{E_1}$, GE_1 is infinite and similarly,GF_1 is infinite), which is a contradiction.

Suppose $G(E_1 \wedge F) \precsim G(E \wedge F_1)$. (3.2.19.1)

Note that E_1 is orthogonal to F_1, F is orthogonal to E. Hence $E_1 \vee F$ is orthogonal to $F_1 \wedge E$. Thus $E_1 \vee F \leqq E + F - F_1 \wedge E$. Hence $E_1 \vee F - F \leqq E - F_1 \wedge E$ (3.2.19.2). Then by Lemma 3.2.18 and (3.2.19.2) we have $E_1 - E_1 \wedge F \sim E_1 \vee F - F \leqq E - F_1 \wedge E$. Hence $GE_1 - G(E_1 \wedge F) \precsim GE - G(F_1 \wedge E)$ (3.2.19.3).

By (3.2.19.1) and (3.2.19.2), $GE_1 \precsim GE$.

This completes the proof of the theorem.

Corollary 3.2.20. Suppose $E \sim F$, E, F finite in R. Then:

(i) If G is a projection in R with $G \geq E, G \geq F$, then $G - E \sim G - F$.

(ii) There is a unitary operator U in R such that $UEU^* = F$.

Proof.

(i) Since $G - E = G - E \vee F + E \vee F - E$ and $G - F = G - E \vee F + E \vee F - F$, it suffices to consider the case in which $G = E \vee F$ and therefore, in virtue of Theorem 3.2.9, we can assume that G is finite in R.

If $G - E \not\sim G - F$, then, in Theorem 3.2.10, $Q + R \neq 0$ and hence $Q(G - E) \prec Q(G - F)$, or $R(G - F) \prec R(G - E)$. We can assume the former to hold. Thus $Q(G - E) \sim X < Q(G - F)$. Since $E \sim F$, $QE \sim QF$. Thus $QG \sim X + QF < QG$; i.e., QG is infinite, a contradiction.

(ii) As $E \sim F$, by (i), $I - E \sim I - F$. Hence there are V, W in R with $V^*V = E$ $VV^* = F$, $W^*W = I - E$, $WW^* = I - F$. Then $U = V + W \in R$ and $U^*U = (V^* + W^*)(V + W) = V^*V + W^*V + V^*W + W^*W = E + I - E = I$; $UU^* = (V + W)(V^* + W^*) = VV^* + WV^* + VW^* + WW^* = F + I - F = I$. Hence U is unitary in R. Further,

$$\begin{aligned} UEU^* &= (V + W)E(V^* + W^*) \\ &= VEV^* + WEV^* + VEW^* + WEW^* \\ &= VEV^* = VV^*VV^* = F. \end{aligned}$$

Corollary 3.2.21. If R is finite (i.e., if I is finite), $E_i \leq F_i (i = 1,2)$ and $E_1 \precsim E_2$, $F_2 \precsim F_1$, then $F_2 - E_2 \precsim F_1 - E_1$.

Proof. $E_1 \leq F_1$, $E_1 \sim X \leq E_2$, $E_2 \leq F_2$, $F_2 \sim Y \leq F_1$. Hence, by Corollary 3.2.20, $I - F_2 \sim I - Y \geq I - F_1$. Then $I - F_1 \precsim I - F_2$. In fact, if $U: I - F_2 \to I - Y$, then $\{U^*(I - F_1)\}^* (U^*(I - F_1)) = (I - F_1)UU^*(I - F_1) = I - F_1$ and $U^*(I - F_1)U \leq I - F_2$. By hypothesis, $E_1 \perp I - F_1$, $E_2 \perp I - F_2$. Hence $I - F_1 + E_1 \precsim I - F_2 + E_2$; i.e., $I - (F_1 - E_1) \precsim I - (F_2 - E_2)$; i.e., $I - (F_1 - E_1) \sim P \leq I - (F_2 - E_2)$. Again by

Corollary 3.2.20, $F_1 - E_1 \sim I - P \geq F_2 - E_2$; i.e., $F_2 - E_2 \precsim F_1 - E_1$.

§3.3. Cyclic and countably decomposable projections

H will denote a Hilbert space throughout this section.

Definition 3.3.1. Suppose $\mathfrak{A}$ is a *-subalgebra of B(H), $I \in \mathfrak{A}$ and $X \subset H$. The set $\mathfrak{A}X = \{Ax: A \in \mathfrak{A}, x \in X\}$ is invariant under each $T \in \mathfrak{A}$; so the projection E' = $[\mathfrak{A}X]$ is in $\mathfrak{A}'$. We say that X is a *generating set* for $\mathfrak{A}$ if E' = I. We say that X is a *separating set* for $\mathfrak{A}$ if 0 is the only operator in $\mathfrak{A}$ which annhilates each x in X. If X consists of a single vector x, we use the terms *generating vector* or *totalisator* and *separating vector*, respectively.

Theorem 3.3.2. If A is a *sulbalgebra of B(H), $I \in A$ and $X \subset H$, then X is a generating set for A if and only if X is a separting set for A'. If $R (\subset B(H))$ is a von Neumann algebra, then X is a separating set for R if and only if X is a genarating set for R'.

Proof. The second statement follows from the first if we take $A = R'$, so that $A' = R'' = R$.

Suppose that X is a generating set for A, so that $[Ax: x \in X] = I$. If $T' \in A'$ and $T'x = 0$ for each $x \in X$, then $T'[AX] = T'I = T'$, but $T'Tx = TT'x = 0$ for each $x \in X$ and $T \in A$. Hence $T'[AX] = 0$. Thus $T' = 0$; i.e., X is a separating set for A'.

Conversely, suppose X is a separating set for A'. With $E' = [AX]$ we have $E' \in A'$ and $E'x = x$ for $x \in X$, as $I \in A$. Hence $(I - E')x = 0$ for each $x \in X$. Since X is a separating set for A' and $I - E' \in A'$, we conclude that $E' = I$.

Definition 3.3.3. If R is a von Neumann algebra acting on H and $X \subset H$, then $[R'X] \in R'' = R$. In particular, if $x \in H$, then $E = [R'x]$ is called a *cyclic projection* in R and is said to be *cyclic under* R'.

Theorem 3.3.4. Every projection E in a von Neumann algebra R is the sum of an orthogonal family of cyclic projections.

Proof. If $E=0$, the theorem holds trivially. Hence let $E \neq 0$. Then there is a non-zero vector x in $E(H)$. Let us consider $[\mathcal{R}'x]$. Then $[\mathcal{R}'x]$ is cyclic and $0 \neq [\mathcal{R}'x] \leq E$. Hence by Zorn's lemma, there exists a maximal orthogonal family $\{E_\alpha\}_{\alpha \varepsilon I}$ of cyclic projections in $\mathcal{R}$, which are majorised by E. If $\sum_{\alpha \varepsilon I} E_\alpha \neq E$, then there is a non-zero vector x in $(E - \sum_{\alpha \varepsilon I} E_\alpha)$ and the non-zero projection $[\mathcal{R}'x]$ is cyclic, orthogonal to $\sum_{\alpha \varepsilon I} E_\alpha$ and majorised by E, a contradiction. Hence $E= \sum_{\alpha \varepsilon I} E_\alpha$.

Proposition 3.3.5. Let A be a *-subalgebra of $B(H)$, containing I and let $M \subset H$. Then $[AM]$ is the smallest closed subspace N of H containing M such that $P_N \varepsilon A'$, where P_N is the projection on H with range N.

Proof. Clearly, $[AM] \supset M$, as $I \varepsilon A$ and, since each $A \varepsilon A$ leaves $[AM]$ invariant, $P= [AM] \varepsilon A'$. In fact, $PA^*P= A^*P$ for $A \varepsilon A$ and hence $PA= (A^*P)^*= PAP= AP$ for $A \varepsilon A$. Let N be a closed subspace containing M and let $P_N \varepsilon A'$. Then, as $P_N \varepsilon A'$, for $x \varepsilon M$, $A \varepsilon A$, we have $P_N Ax= AP_N x= Ax$, so that $[AM] \subset N$ and consequently, $[AM] \leq P_N$. Hence the proposition.

Proposition 3.3.6. Let $M \subset H$ and let $\mathcal{R}$ be a von Neumann algebra with centre Z. If $E'= [\mathcal{R}M]$, then $C_{E'}= [Z'M]$. If $E= [\mathcal{R}'M]$, then $C_E= C_{E'}= [Z'M]$. Consequently, if M is the singleton x in H, then $[\mathcal{R}'x]$ and $[\mathcal{R}x]$ have the same central carrier which is given by $[Z'x]$. Therefore, if E is a cyclic projection in $\mathcal{R}$, then C_E is cyclic in Z (under Z').

Proof. $[Z'M]$ is the smallest closed subspace N of H containing M such that $P_N \varepsilon Z'' = Z$, by Proposition 3.3.5. (It is easy to check that Z is a von Neumann algebra.) But, as $C_{E'}$ is a central projection of $\mathcal{R}'$ containing E' and in particular, the set M, it follows that $C_{E'}[Z'M]= [C_{E'}Z'M]= [Z'C_{E'}M]= [Z'M]$ and hence $C_{E'} \geq [Z'M]$. Moreover,

$$\begin{aligned} C_{E'} &= [R'E'y : y \varepsilon H, R' \varepsilon \mathcal{R}'] \\ &= [R'Rx : R \varepsilon \mathcal{R}, x \varepsilon M, R' \varepsilon \mathcal{R}'] \\ &\leq [Tx : T \varepsilon Z', x \varepsilon M] (\text{since } R'R \varepsilon Z' \text{ for } R \varepsilon \mathcal{R} \text{ and } R' \varepsilon \mathcal{R}') \\ &= [Z'M]. \end{aligned}$$

Hence $C_{E'} = [Z'M]$. Similarly, $C_E = [Z'M]$, since $\mathcal{R}'' = \mathcal{R}$ and $Z = \mathcal{R}' \cap \mathcal{R} = \mathcal{R}' \cap \mathcal{R}''$.

Lemma 3.3.7. Suppose $\mathcal{R}$ is a von Neumann algebra over H, $x_n \in H$, $E_n = [\mathcal{R}'x_n]$, $E'_n = [\mathcal{R}x_n]$ (n= 1,2,...). Then:

(i) If $\{E'_n\}$ is an orthogonal family, then $\bigvee_n E_n$ is a cyclic projection in $\mathcal{R}$.

(ii) If both $\{E_n\}$ and $\{E'_n\}$ are orthogonal families, then there is a vector x in H such that $\Sigma E_n = [\mathcal{R}'x]$, $\Sigma E'_n = [\mathcal{R}x]$.

(iii) If $\{C_{E_n}\}$ is an orthogonal family, then ΣE_n is cyclic in $\mathcal{R}$.

Proof.

(i) We may assume on multiplication by suitable scalars that $\|x_n\| = \frac{1}{n}$ and define $x = \sum_1^\infty x_n$. Since $\{E'_n\}$ is an orthogonal family, and since $E'_n x_n = x_n$, we have $E'_n x = x_n$. For each $R' \in \mathcal{R}'$, $R'x = \sum_1^\infty R'x_n \in \bigvee_1^\infty E_n$. Thus $[\mathcal{R}'x] \leq \bigvee_1^\infty E_n$. However, $[\mathcal{R}'x] \geq [\mathcal{R}'E'_n x] = [\mathcal{R}'x_n] = E_n$, so that $[\mathcal{R}'x] \geq \bigvee_1^\infty E_n$. Hence $[\mathcal{R}'x] = \bigvee_1^\infty E_n$ and thus $\bigvee_1^\infty E_n$ is cyclic.

(ii) If $\{E'_n\}$ is an orthogonal family, then $\bigvee_1^\infty E_n = [\mathcal{R}'x]$ from (i). But as $\{E_n\}$ is also orthogonal, $\bigvee_1^\infty E_n = \sum_1^\infty E_n$. Hence $\sum_1^\infty E_n = [\mathcal{R}'x]$. Interchanging the roles of E_n and E'_n, we get $\Sigma E'_n = [\mathcal{R}x]$.

(iii) If $\{C_{E_n}\}$ is orthogonal, then, as $C_{E_n} = C_{E'_n}$ by Proposition 3.3.6, the orthogonality of $\{C_{E_n}\}$ implies the orthogonality of $\{E_n\}_1^\infty$ and $\{E'_n\}_1^\infty$. Hence, by (ii), (iii) holds.

Definition 3.3.8. Let E be a projection in a von Neumann algebra $\mathcal{R}$. We say that E is *countably decomposable* (in $\mathcal{R}$) if every orthogonal family $\{E_\alpha\}$ of non-zero subprojections of E in $\mathcal{R}$ is at most countable. If I is a countably decomposable projection in $\mathcal{R}$, then $\mathcal{R}$ is said to be *countably decomposable*.

Note 10. Every von Neumann algebra acting on a separable Hilbert space H is countably decomposable. Every subprojection of a countably decomposable projection is itself countably decomposable.

Lemma 3.3.9. A projection E in a von Neumann algebra R is countably decomposable if and only if E= $[R'X]$ for some countable set X of vectors in H. In particular, every cyclic projection in R is countably decomposable. If E is cyclic, C_E is countably decomposable in Z and conversely, if P is countably decomposable in Z, then P is cyclic in Z;if P= [Z'x] and E= $[R'x]$, then P= C_E(i.e.,every countably decomposable projection P in Z is of the form C_E with E cyclic in R.)

Proof. Suppose that E is countably decomposable in R. From this and Theorem 3.3.4, $E=\sum_1^{\infty} E_n$, where $\{E_n\}$ is an orthogonal family of cyclic projections in R.Let $E_n=[R'x_n]$. Let $X=\{x_n:n=1,2,\ldots\}$. For each n, $x_n \in$ E. So $E_n= [R'x_n] \leqq [R'X]$ $\leqq$ E, since $X \subset E$ and $R'X \subset R'E=E$, as R' leaves E invariant. Hence $E=\sum E_n \leqq$ $[R'X] \leqq E$; i.e.,$[R'X]=E$.

Conversely, suppose $E=[R'X]$, X a countable subset of H. Let $\{E_\alpha\}_{\alpha \in A}$ be an orthogonal family of non-zero subprojections of E in R. Then,for each α, $E_\alpha E = E_\alpha \neq 0$. So there is a vector y in E such that $E_\alpha y \neq 0$. Hence there exist a vector $x \in X$ and an element $T' \in R'$such that $E_\alpha T'x \neq 0$. Thus $0 \neq E_\alpha T'x= T'E_\alpha x$, so that $E_\alpha x \neq 0$. Let $A_x=\{\alpha:\alpha \in A,\ E_\alpha x \neq 0\}$. Then $A=\bigcup\{A_x:x \in X\}$ (3.3.9.1).Since $\sum_{\alpha \in A} \|E_\alpha x\|^2 \leqq \|x\|^2$, the set A_x is countable. Since X itself is countable, it follows from (3.3.9.1) that A is countable. Thus E is countably decomposable.

The direct part of the last statement follows from Proposition 3.3.6 and from the first part of the lemma.

Conversely, let P be a central projection which is countably decomposable in Z. Then $P=\sum_1^{\infty} P_n$, $P_n=[Z'x_n]$, $P_nP_m=0$ if $n \neq m$, and $\|x_n\|=1$ by Theorem 3.3.4. Let $x=\sum_n \frac{x_n}{n}$. Then [Z'x]= P by Lemma 3.3.7 applied to Z, since Z is further abelian. Let $[R'x]=E$. We have $E \leqq P$, since $R' \subset Z'$. If Q is a central projection of R with QE= E, then Qx= x so that P= [Z'x]= [Z'Qx]= [QZ'x]= QP. Thus $P \leqq Q$. Hence P=

C_E. Note that E is cyclic in $\mathcal{R}$.

Note 11. Countably decomposable projections in an abelian von Neumann algebra $\mathcal{R}$ are necessarily cyclic, since $\mathcal{R}$= Z. (See also Theorem 3.3.12.)

Lemma 3.3.10. Let M be a subset of H and N= $[\mathcal{R}M]$, where $\mathcal{R}$ is a von Neumann algebra on H. Then $[\mathcal{R}'N]$= $[Z'M]$.

Proof. Since $Z' \supset \mathcal{R}$, $[Z'M] \supset N$. Again, as $Z' \supset \mathcal{R}'$, $[Z'M] \supset [\mathcal{R}'N]$ (3.3.10.1). Since, by Lemma 3.3.5, $[Z'M]$ is the smallest closed subspace containing M such that $[Z'M] \in Z''= Z$, by (3.3.10.1) it suffices to show that $[\mathcal{R}'N] \in Z$; i.e., to show that $[\mathcal{R}'N]$ is left invariant by $\mathcal{R} \cup \mathcal{R}'$. Clearly, $[\mathcal{R}'N]$ is invariant under $\mathcal{R}'$. For $R \in \mathcal{R}$, $R[\mathcal{R}'N]= [RR'y : y \in N, R' \in \mathcal{R}']= [R'Ry : R' \in \mathcal{R}', y \in N] \subset [\mathcal{R}'N]$, since N is invariant under $\mathcal{R}$. Thus $[\mathcal{R}'N] \in Z$. Hence the lemma.

Theorem 3.3.11. Let $\mathcal{R}$ be a countably decomposable von Neumann algebra. Then there is a central projection G such that $\mathcal{R}G$ has a generating vector and $\mathcal{R}(I - G)$ has a separating vector. If $\mathcal{R}$ is further abelian, then $\mathcal{R}$ has a separating vector.

Proof. Let $(x_i)_{i \in A}$ be a maximal family of non-zero vectors in H such that

(i) $[\mathcal{R}'x_i]= E_i$ are pairwise orthogonal, and

(ii) $[\mathcal{R}x_i]= E_i'$ are pairwise orthogonal.

$$\text{Let } E= \sum_{i \in A} E_i,\ E'= \sum_{i \in A} E_i',\ F= I - E \text{ and } F'= I - E'.$$

If $FF' \neq 0$, then there is a non-zero vector $y \in F(H) \cap F'(H)$. Then $[\mathcal{R}'y] \perp E$; $[\mathcal{R}y] \perp E'$ and hence y can be added to $(x_i)_{i \in A}$, a contradiction to the maximality of the family. Thus $FF'= 0$. Hence $C_F C_{F'}= 0$, for, $F'(RFx)= RF'Fx= RFF'x= 0$ for each $R \in \mathcal{R}$, $x \in H$. Hence $F'C_F= 0$, as F' is bounded . Thus $C_{F'}C_F= 0$ by Lemma 3.2.9. (C_F is the central carrier of F in $\mathcal{R}$ and is a member of Z.) Put $G= C_F$. Then $E= I - F \geq I - G$; $E'= I - F' \geq I - C_{F'} \geq I-(I - C_F)= C_F= G$. As $\mathcal{R}$ is countably decomposable, the index set A is at most countable and hence by proper scalar multiplication, we can assume $\Sigma x_i= x \in H$ with $\Sigma \|x_i\|^2 < \infty$. As $x_i= E_i x= E_i' x$, $[\mathcal{R}x] \geq [\mathcal{R}E_i x]= [\mathcal{R}x_i]= E_i'$; so $[\mathcal{R}x] \geq$

$E' \geq G$ and $[R'x] \geq [R'E_i'x] = [R'x_i] = E_i$, so $[R'x] \geq E \geq I - G$. Thus Gx is a generating vector for RG and $(I - G)x$ is a generating for $R'(I - G)$. Then x is a separating vector for $R(I - G)$. For, if $R(I - G)x = 0$ for some $R \in R$, then $0 = R'R(I - G)x = R(R'(I - G)x)$ for each $R' \in R'$. Hence $R(I - G) = 0$.

Suppose R is further abelian. Then with the above notation, $[R'x] \geq I-G$ and $[Rx] \geq G$. Since R is abelian, $R \subset R'$. Hence $[R'x] \geq G$. Then $[R'x] = I$; i.e., x is a generating vector for R' and hence x is a separating vector for R by Theorem 3.3.2.

Theorem 3.3.12. Let R be an abelian von Neumann algebra, acting on H. Then:

(i) Each countably decomposable projection in R is cyclic.

(ii) If R is countably decomposable, then R has a separating vector.

(iii) If R is countably decomposable and is also a maximal abelian *-subalgebra of $B(H)$, then R has a separating-generating vector x.

Proof.

(i) By Theorem 3.3.4 each countably decomposable projection E in R can be expressed as $E = \sum_1^\infty E_n$, where the E_n are pairwise orthogonal cyclic projections in R. Since R is abelian, $C_{E_n} = E_n$ and hence it follows from Lemma 3.3.7(iii) that $E = \sum_1^\infty E_n$ is cyclic.

(ii) If R is countably decomposable, then, by (i), I is cyclic and hence $I = [R'x]$ for some $x \in H$. Hence x is a separating vector for R by Theorem 3.3.2. ((ii) follows also from Theorem 3.3.11.)

(iii) If T' is any self-adjoint element of R', then, as R is abelian, the set $\{R, T'\}$ generates an abelian *-subalgebra of $B(H)$, containing R. Consequently, $T' \in R$, as R is maximal abelian. Varying T', we observe that $R' \subset R$. The reverse inclusion is clear, as R is abelian. Thus $R = R'$. With x chosen as in (ii), x is a separating vector for R and is consequently a generating vector for $R' = R$ by Theorem 3.3.2.

This proves the theorem.

§3.4. Comparison theory for cyclic projections

Throughout this section R wil denote a von Neumann algebra, with centre Z, acting on a Hilbert space H.

Lemma 3.4.1. If E and F are projections in R, with E $\precsim$ F and F cyclic (respectively, countably decomposable) then so is E.

Proof. Choose a partial isometry $V \varepsilon R$, with $V^*V = E$ and $VV^* = F_1 \leq F$. If F is cyclic, choose $x \varepsilon H$ such that $F = [R'x]$. Since $V^* = V^*F_1 = V^*F_1F$, we have $E = [V^*(H)] = [V^*F_1F(H)] = [V^*F_1 R'x] = [R'V^*F_1x]$. Thus E is cyclic with the generating vector V^*F_1x.

Suppose F is countably decomposable. Let $(E_\alpha)_{\alpha \varepsilon A}$ be an orthogonal family of non-zero subprojections of E in R. Let $G_\alpha = VE_\alpha V^*$. Then $G_\alpha G_\beta = 0$ if $\alpha \neq \beta$. Also $G_\alpha^2 = G_\alpha$ and $G_\alpha^* = G_\alpha$. Since $E_\alpha \neq 0$, $E_\alpha V^* \neq 0$ and hence $VE_\alpha V^* \neq 0$, as V is an isometry on E(H) and hence on $E_\alpha(H)$. Thus $\{G_\alpha\}_{\alpha \varepsilon A}$ is an orthogonal family of non-zero projections in R and, as $G_\alpha F_1 = VE_\alpha V^* = G_\alpha$, $G_\alpha \leq F$. Hence A is at most countable. Thus E is countably decomposable.

The following theorem gives a sufficient condition for $C_E \leq C_F$ to imply E $\precsim$ F. (See the note below Lemma 3.2.7.)

Theorem 3.4.2. If E and F are projections in R, with E countably decomposable, F properly infinite and $C_E \leq C_F$, then E $\precsim$ F.

Proof. Suppose E $\not\precsim$ F. By Theorem 3.2.10 there is a non-zero projection Q in Z such that PF $\prec$ PE whenever P is a projection in Z with $0 < P \leq Q$. In particular, QE $\succ$ QF. Then $QC_FF = QF \prec QE = QC_EE = QC_FC_EE = QC_FE$. Thus we can assume that $0 < Q \leq C_F$. Let $QF \sim G < QE$. Then, by Proposition 3.2.13, G is properly infinite, being equivalent to a properly infinite projection QF. (QF is properly infinite, for, if $0 < Q_0 \leq C_{QF} = QC_F = Q \leq C_F$, then $Q_0QF = Q_0F$ is infinite.) Hence by Theorem 3.2.14 there exist two orthogonal projections G_1 and H_1 in R such that $G_1 + H_1 = G \sim G_1 \sim H_1$.

Since H_1 is properly infinite, there exist orthogonal projections G_2 and H_2 in R such that

$$G_2 + H_2 \sim H_1 \sim G_2 \sim H_2.$$

Continuing in this way, we obtain an orthogonal sequence $\{G_n\}_1^\infty$ of projections in R such that $G \sim G_n \leq G < QE$. Let $\{G_\alpha\}_{\alpha \in A}$ be a maximal orthogonal family of projections in R which contains the sequence $\{G_n\}$ and which satisfies the condition $G \sim G_\alpha \leq QE$. Since E is countably decomposable, A is countably infinite. By maximality, $G \not\precsim QE - \sum_{\alpha \in A} G_\alpha$. So by Lemma 3.2.8 there is a projection P in Z, with $0 < P \leq C_G = C_{QF} \leq Q$ and $P(QE - \sum_{\alpha \in A} G_\alpha) \precsim PG$; i.e., $PE - \sum_{\alpha \in A} PG_\alpha \precsim PG$. Thus $PE = \sum_{\alpha \in A} PG_\alpha + (PE - \sum_{\alpha \in A} PG_\alpha) \precsim \sum_2^\infty PG_n + PG_1 = P(\sum_1^\infty G_n) \leq PG \sim PQF = PF$. Thus $PE \precsim PF$, contrary to our assumption that $PE \succ PF$ for $0 < P \leq Q$. Hence $E \precsim F$.

Corollary 3.4.3.

(i) If E is a properly inifinite projection in R and if C_E is countably decomposable in R, then $E \sim C_E$.

(ii) All infinite projections in a factor are properly infinite and hence are equivalent to each other when the factor acts on a separable Hilbert space.

Proof.

(i) Take E and F in Theorem 3.4.2 as C_E and E, respectively. Then $C_E \precsim E$. But $E \leq C_E$, so that $E \precsim C_E$. Hence $C_E \sim E$ by Theorem 3.2.4(ii).

(ii) Since I and 0 are the only central projections in a factor, infinite projections in a factor are properly infinite. If H is separable, then I is countably decomposable in R and hence by (i) every infinite projection $E \sim C_E = I$.

We proceed to relate the comparison theory of cyclic projections in R with the corresponding theory in R'. To this end we need the following two lemmas.

Lemma 3.4.4. If $x, y \in H$ and $y \in [Rx]$, then there exist S, T in R and $z \in H$ such that $Sz = y$, $Tz = x$ and $z \in [T^*(H)]$.

Proof. We split the argument into several stages.

(a) Since $y \in [\mathcal{R}x]$, there exist $A_0(=I), A_1, A_2, \ldots,$ in $\mathcal{R}$ such that $y = \lim_n A_n x$. Passing to a subsequence, if necessary, we may suppose that $\|y - A_n x\| < 4^{-n} (n \geq 1)$. With $R_0 = A_0$ and $R_n = A_n - A_{n-1} (n \geq 1)$, we have $\|R_n x\| \leq \|y - A_n x\| + \|y - A_{n-1}x\| < 4^{-n} + 4^{-(n-1)} = 5(4^{-n})$. Hence $\Sigma R_n x$ is convergent in H, with

$$y = \sum_0^\infty R_n x \quad \text{and} \quad \sum_0^\infty 2^{2n} \|Rx\|^2 < \infty. \qquad (3.4.4.1)$$

Let $K = \{u : u \in H, \sum_0^\infty 2^{2n} \|R_n u\|^2 < \infty\}$. Then K is a vector subspace of H and $x \in K$. Define an inner product $[.,.]_1$, and the associated norm $\|\cdot\|_1$ on K, by

$$[u,v]_1 = \sum_0^\infty 2^{2n}[R_n u, R_n v], \; (u, v \in K).$$

The series is absolutely convergent, since

$$\sum_0^\infty |2^{2n}[R_n u, R_n v]| \leq \tfrac{1}{2} \{\sum_0^\infty 2^{2n} \|R_n u\|^2 + \sum_0^\infty 2^{2n} \|R_n v\|^2\} < \infty,$$

as $u, v \in K$.

$$\|u\|_1^2 = \sum_0^\infty 2^{2n}\|R_n u\|^2 = \|u\|^2 + \sum_1^\infty 2^{2n}\|R_n u\|^2$$

so that

$$\|u\| \leq \|u\|_1, \text{ for every } u \in K. \qquad (3.4.4.2)$$

Next observe that K is a Hilbert space under $[.,.]_1$. In fact, let $(u_n)_1^\infty$ be a Cauchy sequence in $(K, [.,.]_1)$. By (3.4.4.2), $(u_n)_1^\infty$ is a Cauchy sequence in H and hence $(u_n)_1^\infty$ converges in $\|\cdot\|$ to some element u in H. Also, given $\varepsilon > 0$, there is a $P_\varepsilon \in \mathbb{N}$ such that, for $\ell, m \geq P_\varepsilon$,

$$\|u_\ell - u_m\|_1 < \varepsilon.$$

For each $q \geq 0$ and $\ell \geq P_\varepsilon$,

$$\sum_{n=0}^{q} 2^{2n}\|R_n(u_\ell - u)\|^2 = \lim_{m\to\infty} \sum_{n=0}^{q} 2^{2n}\|(R_n(u - u_m)\|^2$$

so that $\sum_0^\infty 2^{2n} \|R_n(u_\ell - u)\|^2 \leq \varepsilon^2$. Thus we have proved that $u_\ell - u \in K$ and $\|u_\ell - u\|_1 \leq \varepsilon$, whenever $\ell \geq P_\varepsilon$. Hence $u = u_\ell - (u_\ell - u) \in K$ and $u_\ell - u \to 0$ in $(K, \|.\|_1)$. Thus $(K, [,]_1)$ is a Hilbert space.

(b) If $R' \in \mathcal{R}'$, then R' leaves K invariant and $(R'_K)^* = (R'^*)_K$, where $A_K = A|K$. For, if $u \in K$, then

$\sum_0^\infty 2^{2n}\|R_n R'u\|^2 = \sum_0^\infty 2^{2n}\|R'R_n u\|^2 \leq \|R'\|^2 \|u\|_1^2 < \infty$ and hence $R'u \in K$. Moreover,

$\|R'u\|_1 \leq \|R'\| \|u\|_1$.

For $u,v \in K$,
$$[R'_K u,v]_1 = \sum_0^\infty 2^{2n}[R_n R'u, R_n v]$$
$$= \sum_0^\infty 2^{2n}[R_n u, R_n R'^* v]$$
$$= [u, (R'^*)_K v]_1$$

so that $(R'_K)^* = (R'^*)_K$.

(c) On the Hilbert space $(K, [.,.]_1)$, define $\psi(u,v) = [u,v]$. Then, clearly, ψ is a symmetric bilinear form on K. Also $\|\psi\|_1 \leq 1$ by (3.4.4.2). Hence ψ corresponds to a hermitian operator B on $(K, [,]_1)$ such that

$$[u,v] = \psi(u,v) = [Bu,v]_1$$

with $\|B\|_1 = \|\psi\| \leq 1$. Since $\psi(u,u) = [u,u] \geq 0$, $B \geq 0$ and hence $B^{\frac{1}{2}}$ exists, and $[u,v] = [B^{\frac{1}{2}}u, B^{\frac{1}{2}}v]_1$, $u,v \in K$ (3.4.4.3).

If $u \in K$ and $B^{\frac{1}{2}}u = 0$, then $[u,u] = [B^{\frac{1}{2}}u, B^{\frac{1}{2}}u]_1 = 0$ and hence $\|u\|^2 = 0$. Thus $u = 0$ and hence $B^{\frac{1}{2}}$ is an injective hermitian operator on K and the range of $B^{\frac{1}{2}}$ is dense in K, as $0 \in \sigma_c(B^{\frac{1}{2}})$ or $0 \in \rho(B^{\frac{1}{2}})$. (This is so, since for normal operators the residual spectrum is empty; and the point spectrum is empty when it is moreover injective.) Consider the linear map $B^{\frac{1}{2}}:(K,\|.\|) \to (B^{\frac{1}{2}}(K),\|.\|_1)$. By (3.4.4.3), $B^{\frac{1}{2}}$ is an onto isometry map from $(K,\|.\|)$ onto $(B^{\frac{1}{2}}(K),\|.\|_1)$ and so extends to an isometry W from $(M,\|.\|)$ onto $(K,\|.\|_1)$, where M is the closure of K in $(H,\|.\|)$. Choose z in M such that Wz = x (3.4.4.4).

(d) Since K is invariant under $\mathcal{R}'$, so is M and hence $E \in \mathcal{R}$, where E is the projection from H onto M. In view of (3.4.4.2), we can consider W as a norm decreasing map from $(M,\|.\|)$ onto $(K,\|.\|)$. Thus T= WE is a norm decreasing map from $(H,\|.\|)$ onto $(K,\|.\|)$ and hence can be considered as a norm decreasing operator T on H. We shall now show that $T \in \mathcal{R}$.

If $u, v \in K$ and $R' \in \mathcal{R}'$, then $R'u \in K$, and by (3.4.4.3) we have $[BR'_K u,v]_1 = [BR'u,v]_1 = [R'u,v] = [u,R'^*v] = [Bu, (R'^*)_K v]_1 = [R'_K Bu,v]_1$, since $(R'^*)_K = (R'_K)^*$. Hence B and $B^{\frac{1}{2}}$ commute with R'_K. Thus $WR'u = B^{\frac{1}{2}}R'_K u = R'_K B^{\frac{1}{2}} u = R'_K Wu$ (3.4.4.5) for each $u \in K$. Consequently, for each $u \in M$, $WR'u = R'Wu$.

For every $a \in H$, $Ea \in M$. Hence $R'Ta = R'WEa = WR'Ea = WER'a = TR'a$, whenever $R' \in \mathcal{R}'$. Thus $T \in \mathcal{R}'' = \mathcal{R}$.

(e) We have by (3.4.4.4) that

$$Tz = WEz = Wz = x.$$

If $P = [T^*(H)]$, then $(I-P)T^* = 0$. Thus $T(I-P) = 0$ and hence $TPz = Tz = x$. If we now replace z by Pz, we have $z \in [T^*(H)]$ and $Tz = x$.

For each $a \in H$, $Ta = WEa \in K$ so that $\sum_0^\infty 2^{2n}\|R_n Ta\|^2 < \infty$. Hence $\sup_n 2^n\|R_n Ta\| < \infty$. Now by the principle of uniform boundedness, $\sup_n 2^n\|R_n T\| < \infty$ and consequently, the series $\Sigma R_n T$ is convergent in norm to some operator S in $\mathcal{R}$. We have by (3.4.4.1)

$$Sz = \sum_{n=0}^{\infty} R_n Tz = \sum_{n=0}^{\infty} R_n x = y.$$

Lemma 3.4.5. Let $\mathcal{R}$ be a von Neumann algebra over H, T ε $\mathcal{R}$, x ε H. Then $[\mathcal{R}'Tx] \precsim [\mathcal{R}'x]$. Besides, $[\mathcal{R}'Tx] \sim [\mathcal{R}'x]$ if x ε [T*(H)].

Proof. Let $E = [\mathcal{R}'x]$. Then $[\mathcal{R}'Tx] = [T\mathcal{R}'x] = [TE(H)]$ and by Lemma 3.2.5 $[TE(H)] \sim [ET^*(H)]$, so that $[\mathcal{R}'Tx] \sim [ET^*(H)] \leqq E$. Hence $[\mathcal{R}'Tx] \precsim [\mathcal{R}'x]$.

If, further, x ε [T*(H)], then x ε [ET*(H)] and so R'x ε [ET*(H)] for each R' ε $\mathcal{R}'$. Thus $E = [\mathcal{R}'x] \leqq [ET^*(H)] \leqq E$ and hence in this case $[\mathcal{R}'Tx] \sim [ET^*(H)] = E = [\mathcal{R}'x]$.

Theorem 3.4.6. Let $\mathcal{R}$ be a von Neumann algebra over H and let x,y ε H. Then:

(i) $[\mathcal{R}'x] \precsim [\mathcal{R}'y]$ (in $\mathcal{R}$) if and only if $[\mathcal{R}x] \precsim [\mathcal{R}y]$ (in $\mathcal{R}'$).

(ii) $[\mathcal{R}'x] \sim [\mathcal{R}'y]$ (in $\mathcal{R}$) if and only if $[\mathcal{R}x] \sim [\mathcal{R}y]$ (in $\mathcal{R}'$).

(iii) $[\mathcal{R}'x] \prec [\mathcal{R}'y]$ (in $\mathcal{R}$) if and only if $[\mathcal{R}x] \prec [\mathcal{R}y]$ (in $\mathcal{R}'$).

Proof. It is clear that (ii) follows from (i) since $E \precsim F$, $F \precsim E$ imply $E \sim F$. Now (iii) follows from (i) and (ii). Hence we shall prove (i). To prove (i) it suffices to show that, if $[\mathcal{R}x] \precsim [\mathcal{R}y]$, then $[\mathcal{R}'x] \precsim [\mathcal{R}'y]$.

If $[\mathcal{R}x] \precsim [\mathcal{R}y]$, choose U' in $\mathcal{R}'$ such that $U'^*U' = [\mathcal{R}x]$ and $U'U'^* \leqq [\mathcal{R}y]$. Then $U'^*U'x = x$, so that $[\mathcal{R}'U'x] \geqq [\mathcal{R}'U'^*U'x] = [\mathcal{R}'x]$. But, $\mathcal{R}'U' \subset \mathcal{R}'$ and hence $[\mathcal{R}'U'x] \leqq [\mathcal{R}'x]$. Thus $[\mathcal{R}'U'x] = [\mathcal{R}'x]$ (3.4.6.1).

$U'x = U'U'^*U'x$ ε $[\mathcal{R}y]$. By Lemma 3.4.4 there exist operators S and T in $\mathcal{R}$ and a vector z ε H such that z ε [T*(H)], Sz= U'x and Tz= y. By Lemma 3.4.5, $[\mathcal{R}'Sz] \precsim [\mathcal{R}'z] \sim [\mathcal{R}'Tz]$. Hence, by (3.4.6.1), $[\mathcal{R}'x] = [\mathcal{R}'U'x] = [\mathcal{R}'Sz] \precsim [\mathcal{R}'Tz] = [\mathcal{R}'y]$. Thus $[\mathcal{R}'x] \precsim [\mathcal{R}'y]$.

This completes the proof of the theorem.

Lemma 3.4.7. Suppose x ε H, $E = [\mathcal{R}'x]$, $E' = [\mathcal{R}x]$ and [ABx,x]= [BAx,x] whenever A, B ε $E\mathcal{R}E$. Then [A'B'x,x]= [B'A'x,x] whenever A', B' ε $E'\mathcal{R}'E'$.

Proof. Obviously, it suffices to prove the lemma for self-adjoint elements $A', B' \in E'R'E'$. Since $A' = E'R'E'$ for some $R' \in R'$ whenever $A' \in E'R'E'$, $A'x \in E' = [Rx]$. Hence there is a sequence $A_n \in R$ such that $A_n x \to A'x$. Since $A'x$, x are in $[R'x] = E$, $A'x = EA'Ex = \lim_n EA_n Ex$, and so we may assume that $A_n \in ERE$. Then $A_n^* \in ERE$ and, as $A'^* = A'$, we have by hypothesis

$$\|A'x - A_n^* x\|^2 = [A'x - A_n^* x, A'x - A_n^* x]$$

$$= [A'^2 x, x] - [A_n^* x, A'x] - [A'x, A_n^* x] + [A_n A_n^* x, x]$$

$$= [A'^2 x, x] - [A_n^* A'x, x] - [A'A_n x, x] + [A_n^* A_n x, x]$$

$$= \|A'x - A_n x\|^2 \to 0.$$

Thus $A_n x \to A'x$ and $A_n^* x \to A'x$. Therefore, replacing A_n by $\frac{A_n + A_n^*}{2}$, we can assume that $A_n = A_n^* \in ERE$ and $A'x = \lim_n A_n x$. Similarly, we can choose hermitian operators B_n in ERE such that $B'x = \lim_n B_n x$. Thus

$$[A'B'x, x] = [B'x, A'x] = \lim_n \lim_k [B_k x, A_n x]$$

$$= \lim_n \lim_k [A_n B_k x, x]$$

$$= \lim_n \lim_k [B_k A_n x, x]$$

$$= \lim_n \lim_k [A_n x, B_k x]$$

$$= [A'x, B'x] = [B'A'x, x].$$

Hence the lemma.

Theorem 3.4.8. Suppose $x \in H$, $E = [R'x]$ and $E' = [Rx]$. Then E is finite (respectively, infinite or properly infinite) in R if and only if E' has the same property relative to R'.

Proof. In view of the symmetry between R and R' it suffices to prove that, if E is finite (or infinite or properly infinite) in R, the same is true of E' in R'. If this is not so, then one of the following situations occurs.

(a) E is finite and E' is infinite.

(b) E is infinite and E' is finite.

(c) E is properly infinite and E' is not properly infinite.

In case (a) by Theorem 3.2.16 there exists a central projection Q with QE' properly infinite while QE is finite; in case (b) QE is properly infinite while QE' is finite. In case (c), by the definition of properly infinite projections and by the fact that $C_E = C_{E'}$ (see 3.3.6), there is a central projection Q with QE properly infinite and QE' finite. Thus one of QE, QE' is finite and the other is properly infinite. Now [QE]= [QR'x]= [R'Qx] and QE'= [RQx]. Replacing x, E,E' by Qx, QE and QE', respectively, we may suppose that one of E,E' is finite, while the other is properly infinite. Finally, by the symmetry between R and R', we may assume that E is finite, while E' is properly infinite. We derive a contradiction by considering separately two cases.

Case (i). Every projection in ERE has the form QE, with Q a projection in Z.

In this case any two projection in ERE commute. Since $ERE = \{A: A \in R, A = EAE\}$, ERE is a τ_W-closed *-subalgebra of B(H). The restrictions $\{A|_{E(H)}: A \in ERE\}$ form a von Neumann algebra over E(H) which is abelian, since, as any two of its projections commute, by the spectral theorem all the hermitian elements commute. Hence [ABx,x]= [BAx,x] for A,B $\in ERE$. By Lemma 3.4.7, [A'B'x,x]=[B'A'x,x] (3.4.8.1) whenever A',B' $\in E'R'E'$.

If $E' \sim F' \leq E'$, then $F' = F'E' = F'[\mathcal{R}x] = [\mathcal{R}F'x]$. With V' in $\mathcal{R}'$ such that $V'^*V' = E'$, $V'V'^* = F'$, we have $E'V'E' = E'V'V'^*V' = E'F'V' = F'V' = V'V'^*V' = V'E' = V'$. Hence V' and $V'^* \in E'\mathcal{R}'E'$. Since $x \in E'$, by using (3.4.8.1)we have:

$$\|x\|^2 = \|V'x\|^2 = [V'^*V'x,x] = [V'V'^*x,x] = [F'x,x] = \|F'x\|^2.$$

Hence $F'x = x$, whence $F' = [\mathcal{R}F'x] = [\mathcal{R}x] = E'$. Thus E' is finite.

Case (ii) There is a projection F in $E\mathcal{R}E$ which is not of the form QE, where Q is a projection in Z.

In particular, $0 \neq F \neq C_F E$, since $C_F \in Z$. So $0 < F < C_F E \leqq C_F \leqq C_E$. With $F_1 = C_F E - F$ and $Q = C_{F_1}$, we have $0 < F_1 < C_F E \leq C_F$ and $0 < Q = C_{F_1} \leq C_F \leqq C_E$. Let $F_2 = QF$ and note that $C_{F_2} = QC_F = Q = C_{F_1} = C_{F_1} C_E = C_{QE}$. Since $F_1 + F_2 = C_{F_1}(F_1 + F) = C_{F_1} C_F E = C_{F_1} E = QE$, we have $F_1 + F_2 = QE = Q[\mathcal{R}'x] = [\mathcal{R}'Qx]$. Moreover, QE is finite while $QE' = [\mathcal{R}Qx]$ is properly infinite, since $0 < Q \leqq C_E = C_{E'}$ and E' is properly infinite by our assumption. Replacing x, E, E' by Qx, QE, QE', respectively, we may suppose that

$$E = F_1 + F_2 \text{ where } C_{F_1} = C_{F_2} = C_E.$$

Note that $F_j = F_j E = F_j[\mathcal{R}'x] = [\mathcal{R}'F_j x] = [\mathcal{R}'x_j]$, where $x_j = F_j x$, $j = 1,2$. Let $F_j' = [\mathcal{R}x_j]$, $j = 1,2$, so that $C_{F_j'} = C_{F_j} = C_E = C_{E'}$ $(j = 1,2)$. We claim that the F_j' are finite. For, on the contrary, there is a projection P in Z such that $0 < P \leqq C_{F_j'} = C_E$, and PF_j' is properly infinite for $j=1,2$. Since $C_{PF_j'} = PC_{F_j'} = P = PC_E = PC_{E'} = C_{PE'}$, by Theorem 3.4.2 $PF_j' \sim PE'$. Since the PF_j' are properly infinite, it follows that $PF_1 \neq 0 \neq PF_2$. Moreover, by Theorem 3.4.6, $[R' Px] \sim [R' Px_j]$, since $PF_j' = [R Px_j] \sim PE' = [R Px]$. Thus $PE \sim PF_j < PF_1 + PF_2 = PE$. This means that PE, hence also E, is infinite, a contradiction. Thus F_j' is finite for $j = 1,2$.

Since E' is properly infinite, by Theorem 3.2.14 there exist projections E_1', E_2' in R' such that $E_1'E_2' = 0$ and $E' = E_1' + E_2' \sim E_1' \sim E_2'$. Since $F_j' = [R x_j] = [R F_j x] \leq [Rx] = E' \sim E_j'$, there are projections F_j'' in R' such that $F_j' \sim F_j'' \leq E_j'$ and F_j'' finite ($j = 1,2$). Let $V_j': F_j' \to F_j''$ be a partial isometry in R'. Since $x_j \in F_j'$ ($j=1,2$), we have

$$F_j'' = [V_j' F_j'(H)] = [V_j' R x_j] = [R V_j' x_j]$$

and

$$F_j = [R' x_j] \geq [R' V_j' x_j] \geq [R' V_j'^* V_j' x_j] = [R' x_j] = F_j.$$

Thus, with $y_j = V_j' x_j$, we have $F_j = [R' y_j]$ and $F_j'' = [R y_j]$. Since (F_1, F_2) and (F_1'', F_2'') are both orthogonal pairs of projections, we have $F_1 + F_2 = [R' y]$, $F_1'' + F_2'' = [Ry]$ by Lemma 3.3.7(iii), where $y = y_1 + y_2$. (Or directly, $[R' y] \leq F_1 + F_2$, as $A'y = A'y_1 + A'y_2 \in [R' y_1] + [R' y_2] = F_1 + F_2 (A' \in R')$. $[R' y] \geq [R' F_j y] = [R' y_j] = F_j (j=1,2)$ whence $[R' y] \geq F_1 + F_2$.)

Since $E = F_1 + F_2$, we have $[R' y] = E = [R' x]$. By Theorem 3.4.6, $[Ry] \sim [Rx]$; i.e., $F_1'' + F_2'' \sim E'$. This is impossible, since $F_1'' + F_2''$ is finite by Theorem 3.2.19 and E' is properly infinite.

We shall conclude this chapter after proving an important theorem known as 'The Dixmier approximation theorem' which is of great use in the classification theory of von Neumann algebras. The next section deals with this theorem.

§3.5. The Dixmier approximation theorem

Let $\mathcal{R}$ be a Von Neumann algebra, Z its centre and $\mathcal{U}$ its unitary group. Given $A \in \mathcal{R}$, let $Co_{\mathcal{R}}(A)$ be the convex hull of $\{UAU^*: U \in \mathcal{U}\}$ and $\overline{Co}_{\mathcal{R}}(A)$ be the norm closure of $Co_{\mathcal{R}}(A)$. In this section we will mainly prove that $\overline{Co}_{\mathcal{R}}(A) \cap Z$ is non empty.

Definition 3.5.1. If G is a projection in Z and A is a hermitian operator in $\mathcal{R}$, we define

$$M_G(A) = \sup\{[Ax,x]: x = Gx,\ \|x\| = 1\}$$

$$m_G(A) = \inf\{[Ax,x]: x = Gx,\ \|x\| = 1\}.$$

When G= I we write M(A) and m(A), respectively. Further, we define

$$w_G(A) = M_G(A) - m_G(A)$$

$$w(A) = M(A) - m(A)$$

and

$$w_0(A) = 0.$$

Lemma 3.5.2. Let $\mathcal{R}$ be a von Neumann algebra with Z its centre and A a hermitian operator of $\mathcal{R}$. Then there exist projections P,Q in Z and an operator $U \in \mathcal{U}$ such that P and Q are orthogonal, P + Q= I,

$$w_P(\tfrac{1}{2}P(A + UAU^*)) \leq \tfrac{3}{4}w(A)$$

and

$$w_Q(\tfrac{1}{2}Q(A + UAU^*)) \leq \tfrac{3}{4}w(A).$$

Proof. Let $A = \int_{m(A)}^{M(A)} \lambda dE(\lambda)$ and $n(A) = \frac{1}{2}[m(A) + M(A)]$. With $E_t = E((-\infty, t])$, let $E = E_{n(A)-0}$, $F = I - E$. By the comparison theorem there exist orthogonal projections P, Q in Z with $P + Q = I$, $PE \precsim PF$ and $QF \prec QE$. Choose partial isometries V and W in R such that $V: PE \to F_1 \leq PF$, $W: QF \to E_1 < QE$. Let $U \doteq V + V^* + W + W^* + I - PE - QF - E_1 - F_1$.

PE	$\underset{\to}{V}$	F_1
F_1	$\underset{\to}{V^*}$	PE
QF	$\underset{\to}{W}$	E_1
E_1	$\underset{\to}{W^*}$	QF
$I-PE-E_1-QF-F_1$	$\underset{\to}{I}$	$I-PE-E_1-QF-F_1$

Then, clearly, U gives the equivalence of I with I so that U is a unitary operator in R; i.e., $U \in U$.

Further, $UPEU^* = F_1$, $UF_1U^* = PE$, $UQFU^* = E_1$, $UE_1U^* = QF$ and U acts as identity on $I - PE - E_1 - QF - F_1$. $\Big\} (*)$

We have $\qquad A \geq m(A)E + n(A)F.$

For,

$$A = \int_{m(A)}^{M(A)} \lambda dE(\lambda) = \int_{m(A)}^{n(A)} \lambda dE(\lambda) + \int_{n(A)}^{M(A)} \lambda dE(\lambda)$$

$$\geq m(A)\{E_{n(A)-0} - E_{m(A)-0}\}$$

$$+ n(A)\{E_{M(A)} - E_{n(A)-0}\}$$

$$= m(A)E + n(A)F,$$

since $E_{m(A)-0} = 0$ and $E_{M(A)} = I$.

Hence $\quad PA \geq m(A)PE + n(A)PF$

$$= m(A)PE + n(A)F_1 + n(A)(PF - F_1).$$

$$PUAU^* = UPAU^*$$

$$\geq m(A)UPEU^* + n(A)UF_1U^* + n(A)U(PF-F_1)U^*$$

$$= m(A)F_1 + n(A)PE + n(A)(PF - F_1)$$

by (*), since U acts as identity on $(F-(F_1+ QF))H= (PF-F_1)(H)$.

Therefore,

$$\tfrac{1}{2}P(A + UAU^*) \geq \tfrac{1}{2}(m(A) + n(A))PE + n(A)(PF - F_1) + \tfrac{1}{2}(m(A) + n(A))F_1$$

$$= \tfrac{1}{2}(m(A) + n(A))(PE + F_1) + n(A)(PF - F_1)$$

$$\geq \tfrac{1}{2}(m(A) + n(A))(PE + PF)$$

$$= \tfrac{1}{2}(m(A) + n(A))P.$$

But, $\tfrac{1}{2}(m(A) + n(A))= \tfrac{1}{2}(m(A) + \frac{m(A) + M(A)}{2})$

$$= \tfrac{1}{2}(\frac{3}{2}M(A) - \frac{3}{2}(M(A) - m(A)) + \tfrac{1}{2}M(A))$$

$$= \tfrac{1}{2}(2M(A) - \frac{3}{2}w(A))$$

$$= M(A) - \frac{3}{4}w(A).$$

Hence $\quad \tfrac{1}{2}P(A + UAU^*) \geq (M(A) - \frac{3}{4}w(A))P. \qquad (3.5.2.1)$

But $A \leq M(A)I$, so that $UAU^* \leq M(A)I$. Therefore,

$$\tfrac{1}{2}P(A + UAU^*) \leq M(A)P. \qquad (3.5.2.2)$$

$$(3.5.2.1) \Longrightarrow \quad m_p \{\tfrac{1}{2}P(A + UAU^*)\} \geq M(A) - \frac{3}{4} w(A).$$

$$(3.5.2.2) \Longrightarrow \quad \bar{M}_p \{\tfrac{1}{2}P(A + UAU^*)\} \leq M(A)$$

Hence $w_p \{\tfrac{1}{2} P(A + UAU^*)\} \leq \frac{3}{4} w(A)$.

Similarly, $w_p \{\tfrac{1}{2}Q(A + UAU^*)\} \leq \frac{3}{4} w(A)$.

This completes the proof of the lemma.

Lemma 3.5.3. Let $\mathcal{D}$ denote the set of all mappings $\alpha: \mathcal{R} \to \mathcal{R}$ of the form $\alpha(A) = \sum_{j=1}^{n} a_j U_j A U_j^*$, where $a_j > 0$, $\sum_1^n a_j = 1$ and $U_j \in \mathcal{U}$. Then:

(i) If $\alpha, \beta \in \mathcal{D}$ then $\alpha\beta (= \alpha\circ\beta) \in \mathcal{D}$.

(ii) α is linear and $\|\alpha\| = 1$ $(\alpha \in \mathcal{D})$.

(iii) $\alpha(z) = z$ $(\alpha \in \mathcal{D}, z \in Z)$.

(iv) $\alpha(zA) = z\alpha(A)$ $(\alpha \in \mathcal{D}, z \in Z, A \in \mathcal{R})$.

(v) $Co_{\mathcal{R}}(A) = \{\alpha(A) : \alpha \in \mathcal{D}\}$, $A \in \mathcal{R}$.

Proof.

(i) Let $\alpha(A) = \sum_{j=1}^{n} a_j U_j A U_j^*$, $a_j > 0$, $\sum_1^n a_j = 1$, $U_j \in \mathcal{U}$

and $\beta(A) = \sum_{k=1}^{\ell} b_k V_k A V_k^*$, $b_k > 0$, $\sum_1^{\ell} b_k = 1$, $V_k \in \mathcal{U}$.

$$(\alpha\beta)(A) = \sum_{k=1}^{\ell} b_k \sum_{j=1}^{n} a_j \, U_j V_k A V_k^* U_j^*$$

$$= \sum_{k=1}^{\ell} b_k \sum_{j=1}^{n} a_j (U_j V_k) A (U_j V_k)^*$$

$$= \sum_{\substack{j=1,\dots,n\\ k=1,\dots,\ell}} a_j b_k (U_j V_k) A (U_j V_k)^*, \; a_j b_k > 0, j=1,\dots,n; k=1,\dots,\ell$$

and $\sum_{j,k} a_j b_k = (\sum_1^n a_j)(\sum_1^\ell b_k) = 1$

so that $\alpha\beta \in \mathcal{D}$.

(ii) $\alpha(\lambda A + \mu B) = \sum_{j=1}^n a_j U_j (\lambda A + \mu B) U_j^*$

$$= \lambda\alpha(A) + \mu\alpha(B)$$

for $\lambda,\mu \in \mathbb{C}$, and hence α is linear.

$\|\alpha(A)\| = \|\sum_1^n a_j U_j A U_j^*\| \leq \sum_1^n a_j \|A\| = \|A\|$, so that $\|\alpha\| \leq 1$.

$\|\alpha(I)\| = \sum_1^n a_j U_j U_j^* = \sum_1^n a_j = 1$. Hence $\|\alpha\| = 1$.

(iii) For $z \in Z$, $\alpha(z) = \sum_1^n a_j U_j z U_j^* = z \sum_1^n a_j = z$.

(iv) $\alpha(zA) = \sum a_j U_j zA U_j^* = z\alpha(A)$, for $z \in Z$.

(v) $Co_{\mathcal{R}}(A)$ = convex hull of $\{UAU^* = U \in \mathcal{U}\}$

$$= \{\sum_{j=1}^n a_j U_j A U_j^* : n \text{ arbitrary}, \sum_1^n a_j = 1, a_j > 0, U_j \in \mathcal{U}\}$$

$$= \{\alpha(A) : \alpha \in \mathcal{D}\}.$$

Lemma 3.5.4. Suppose A is a hermitian operator in the von Neumann algebra $\mathcal{R}$ and let $n \in \mathbb{N} \cup \{0\}$. Then there exists a finite orthogonal family $G_1,\dots,G_k$ of projections in $\mathcal{R}$ with $G_1 + G_2 + \dots + G_k = I$ and $\alpha \in \mathcal{D}$ such that

$$w_{G_j}(G_j\alpha(A)) \leq (\tfrac{3}{4})^n \; w(A), j= 1,2,\ldots,k.$$

Proof. We use induction on n. For n= 0, take k= 1, G_1=I and α the identity map on $\mathcal{R}$.

Suppose that for some n, suitable G_1, $G_2,\ldots,G_k$ and α have been found. For each j, $G_j\alpha(A)$ is a hermitian operator of $\mathcal{R}G_j$. By Lemma 3.5.2 there exist projections P_j and Q_j in ZG_j and U_j unitary in $\mathcal{R}G_j$ such that $P_jQ_j=0$, $P_j + Q_j= G_j$ ($\mathcal{R}G_j$ is a von Neumann algebra on $G_j(H)$ as $\mathcal{R}G_j$ is weakly closed. See the proof of Theorem 3.4.8 under case (i).) such that $w_{P_j}(\tfrac{1}{2}P_j(G_j\alpha(A) + U_jG_j\alpha(A)U_j^*)) \leq \tfrac{3}{4}\, w_{G_j}(G_j\alpha(A)) \leq (\tfrac{3}{4})^{n+1}\, w(A)$ by induction hypothesis.

Similarly, $w_{Q_j}(\tfrac{1}{2}Q_j(G_j\alpha(A) + U_jG_j\alpha(A)U_j^*))\leq (\tfrac{3}{4})^{n+1}\, w(A)$.

Then $U= \sum_1^k \oplus U_j$ is a unitary operator of $\mathcal{R}$ with $UG_j= U_j$.

For, $UU^*= \sum_1^k \oplus U_jU_j^*= \sum_1^k G_j= I$ and $U^*U = I$.

$UG_j= (\sum_1^k \oplus U_j)(0 + \ldots + G_j + \ldots)= U_jG_j$(For details see §4.1) $= U_j$.

With β , γ in $\mathcal{D}$ defined by $\beta(R)= \tfrac{1}{2}(R + URU^*)$, $(R \in \mathcal{R})$, $\gamma = \beta\alpha$ we have

$$\begin{aligned} P_j\gamma(A) &= \tfrac{1}{2}P_j(\alpha(A) + U\alpha(A)U^*) \\ &= \tfrac{1}{2}P_j(\sum_1^k G_i\alpha(A) + U(\sum_1^k G_i\alpha(A))U^*) \\ &= \tfrac{1}{2}P_j(G_j\alpha(A) + UG_j\alpha(A)U^*) \\ &= \tfrac{1}{2}P_j(G_j\alpha(A) + U_j\alpha(A)U_j^*) \end{aligned}$$

so that

$$w_{P_j}(P_j\alpha(A)) \leqq (\tfrac{3}{4})^{n+1}\ w(A)$$

and similarly,

$$w_{Q_j}(Q_j\gamma(A)) \leqq (\tfrac{3}{4})^{n+1}\ w(A),\ j = 1,2,\ldots,k.$$

Take $G_1 = P_1, \ldots, G_k = P_k$, $G_{k+1} = Q_1, \ldots, G_{2k} = Q_k$ and replace α by γ .Then the result holds for n + 1 and hence the lemma holds by the principle of finite induction.

Lemma 3.5.5. If A is a hermitian operator in R and $\varepsilon > 0$, then there exists $\alpha \varepsilon \mathcal{D}$ and $z \varepsilon Z$ such that $||\alpha(A) - z|| < \varepsilon$.

Proof. Let n(>0) be an integer such that $(\frac{3}{4})^n\ w(A) < \varepsilon$. Choose $\alpha, G_1, G_2, \ldots, G_k$ as in Lemma 3.5.4. With $a_j = m_{G_j}(G_j\alpha(A))$, we have

$$a_jG_j \leqq G_j\ \alpha(A) \leqq [a_j + w_{G_j}(G_j\alpha(A))]G_j$$

$$\leqq (a_j + \varepsilon\ )G_j$$

by Lemma 3.5.4 and by the fact that $m_G(AG) \leqq AG \leqq M_G(AG)$. Hence $0 \leqq G_j\alpha(A) - a_jG_j \leqq \varepsilon G_j$. Now summing up as j varies from 1 to k,

$$0 \leqq \alpha(A) - \sum_1^k a_jG_j \leqq \varepsilon\ I .$$

Hence $||\alpha(A) - \sum_1^k a_jG_j|| \leqq \varepsilon I.$

Taking $z = \sum_1^k a_j\ G_j$, we obtain the conclusion of the lemma.

Lemma 3.5.6. Let $A_1, A_2, \ldots, A_n$ be operators in R and $\varepsilon > 0$. Then there exists $\alpha \in \mathcal{D}$ and $z_1, z_2, \ldots, z_n \in Z$ such that $\| \alpha(A_j) - z_j \| < \varepsilon$ for $j=1,2,\ldots,n$.

Proof. We can assume that $A_1, A_2, \ldots, A_n$ are hermitian (if not, use real and imaginary parts, replacing n by 2n and ε by $\frac{1}{2}\varepsilon$).

We prove by induction on n. For n= 1, this holds by previous lemma. Suppose we have found $\beta \in \mathcal{D}$, $z_1, z_2, \ldots, z_{n-1}$ in Z so that $\|\beta(A_j) - z_j\| < \varepsilon$ for j= 1,2,...,n-1. By Lemma 3.5.5 applied to $\beta(A_n)$ there exists γ in $\mathcal{D}$ and z_n in Z such that $\| \gamma(\beta(A_n)) - z_n \| < \varepsilon$.

Also $\| \gamma(\beta(A_j)) - z_j \| = \| \gamma(\beta(A_j) - z_j) \| \leq \| \beta(A_j) - z_j \| < \varepsilon$, $j=1, 2,\ldots,n-1$, since $\|\gamma\| = 1$. Thus the result holds for n with $\alpha = \gamma\beta$ and $z_1, z_2, \ldots, z_n$. Then the lemma follows by the principle of finite induction.

Theorem 3.5.7. (The Dixmier approximation theorem). If $A_1, A_2, \ldots, A_n$ are operators in $\mathcal{R}$, then there exist $z_1, z_2, \ldots, z_n$ in Z and a sequence $(\alpha_m)_1^\infty$ in $\mathcal{D}$ such that

$$\lim_m \|\alpha_m(A_j) - z_j\| = 0 \quad \text{for } j= 1,2,\ldots,n.$$

Proof. By induction on m, we shall construct β_m in $\mathcal{D}$ and $z_1^{(m)}, \ldots, z_n^{(m)}$ in Z such that

$$(*) \quad \| \beta_m \beta_{m-1} \cdots \beta_1(A_j) - z_j^{(m)} \| \leq 2^{-m} \text{ for } j= 1,2,\ldots,n;\ m= 1,2,\ldots$$

The previous lemma gives the starting case m = 1, when applied to $A_1, A_2, \ldots, A_n$ with $\varepsilon = \frac{1}{2}$. It gives the 'set-up' from m to m + 1 when applied to $\beta_m \beta_{m-1} \cdots \beta_1(A_j)$, j= 1,2,...,n with $\varepsilon = 2^{-(m+1)}$. Thus the construction is possible for all $m \in \mathbb{N}$.

By (*), $\|z_j^{(m+1)} - z_j^{(m)}\| \leq \|z_j^{(m+1)} - \beta_{m+1}\beta_m \cdots \beta_1(A_j)\| +$

$$\|\beta_{m+1}\beta_m \cdots \beta_1(A_j) - z_j^{(m)}\|$$

$$\leq 2^{-(m+1)} + \|\beta_{m+1}(\beta_m \cdots \beta_1(A_j) - z_j^{(m)})\|$$

$$< 2^{-(m+1)} + 2^{-m} < \frac{4}{2^{m+1}} = 2^{-(m-1)}.$$

Thus for each j, $(z_j^{(m)})$ is a Cauchy sequence and hence converges in norm to some z_j in Z. Choose m_o such that $\|z_j - z_j^{(m)}\| < \varepsilon/2$ for j= 1,2,..,n, if $m \geq m_o$. Choose m_1 such that $2^{-m_1} < \frac{\varepsilon}{2}$. Take $m_2 = \max(m_o, m_1)$. For $m \geq m_2$ and j= 1,2,...,n, by (*) we have

$$\|\beta_m\beta_{m-1} \cdots \beta_2\beta_1(A_j) - z_j\| \leq \|\beta_m \cdots \beta_2\beta_1(A_j) - z_j^{(m)}\| + \|z_j^{(m)} - z_j\|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Take $\alpha_m = \beta_m\beta_{m-1} \cdots \beta_2\beta_1$. Then the theorem holds for $(\alpha_m)_1^\infty$.

Theorem 3.5.8. If R is a von Neumman algebra with centre Z and $A \in R$, then $\overline{Co}_R(A)$ meets Z.

Proof. By the above theorem, there exists a sequence $(\alpha_m)_1^\infty$ in $\mathcal{D}$ and $z \in Z$ such that $\lim_{m \to \infty} \|\alpha_m(A) - z\| = 0$.

But $\alpha_m(A) \in Co_R(A)$ by Lemma 3.5.(v) so that $\alpha_m(A) \to z$ implies $z \in \overline{Co_R(A)}$ (norm closure). Hence $\overline{Co}_R(A) \cap Z \neq \emptyset$.

Later we shall show that for finite von Neumann algebras R (see 6.4.11) $\overline{Co}_R(A) \cap Z$ is singleton for each A in the algebra R.

(See Corollary on p.254 of [1].)

We shall close this section with two more results, which have useful applications.

Proposition 3.5.9. Let A,B be in the von Neumann algebra $\mathcal{R}$. Then $\overline{Co}_{\mathcal{R}}(A+B) \cap Z \subset$ Norm closure of $(\overline{Co}_{\mathcal{R}}(A) \cap Z + \overline{Co}_{\mathcal{R}}(B) \cap Z)$.

Proof. Let $z \in \overline{Co}_{\mathcal{R}}(A + B) \cap Z$ and $\varepsilon > 0$. There exists an $\alpha \in \mathcal{D}$ such that

$$\|\alpha(A + B) - z\| < \varepsilon .$$

Consider $\alpha A, \alpha B$ and ε. Then by Theorem 3.5.8 there exists a $\beta \in \mathcal{D}$ and $z_1 \in \overline{Co}_{\mathcal{R}}(\alpha A) \cap Z \subset \overline{Co}_{\mathcal{R}}(A) \cap Z$ and $z_2 \in \overline{Co}_{\mathcal{R}}(\alpha B) \cap Z \subset \overline{Co}_{\mathcal{R}}(B) \cap Z$ such that

$$\|\beta(\alpha(A)) - z_1\| < \varepsilon \text{ and } \|\beta(\alpha(B)) - z_2\| < \varepsilon .$$

As $\|\beta\alpha(A + B) - z\| = \|\beta\{\alpha(A + B) - z\}\| \leq \|\alpha(A + B) - z\| < \varepsilon$, we have

$$\|z - (z_1 + z_2)\| = \|(z - \beta\alpha(A + B)) + \beta\alpha(A) - z_1 + \beta\alpha(B) - z_2\| \leq 3\varepsilon .$$

Hence the proposition.

Proposition 3.5.10. Let $A \in \mathcal{R}$ and $z \in Z$. Then $\overline{Co}_{\mathcal{R}}(zA) \cap Z$ is contained in $z(\overline{Co}_{\mathcal{R}}(A) \cap Z)$.

Proof. Let $R \in \overline{Co}_{\mathcal{R}}(zA) \cap Z$ and $\varepsilon > 0$. Then there is an $\alpha \in \mathcal{D}$ such that $\|\alpha(zA) - R\| < \varepsilon$. By Theorem 3.5.8, there is a $\beta \in \mathcal{D}$ and $R_1 \in \overline{Co}_{\mathcal{R}}(\alpha A) \cap Z \subset \overline{Co}_{\mathcal{R}}(A) \cap Z$ such that $\|\beta(\alpha A) - R_1\| < \varepsilon$.

Then $\|z\beta(\alpha A) - R\| = \|\beta\{\alpha(zA) - R\}\| < \varepsilon, \|z\beta(\alpha A) - zR_1\| \leq \|z\| \varepsilon$. Thus $\|R - zR_1\| \leq \varepsilon(1 + \|z\|)$. Thus $\overline{Co}_{\mathcal{R}}(zA) \cap Z \subset z\overline{(\overline{Co}_{\mathcal{R}}(A) \cap Z)} = z(\overline{Co}_{\mathcal{R}}(A) \cap Z)$, since $\overline{Co}_{\mathcal{R}}(A) \cap Z$ is norm closed.

Hence the proposition.

CHAPTER 4

ELEMENTARY CONSTRUCTIONS WITH VON NEUMANN ALGEBRAS

In this chapter we study in detail the von Neumann algebras that are obtained as the result of

(i) the direct sum of a given family of von Neumann algebras,

(ii) the reduction of a given von Neumann algebra,

(iii) the induction of a given von Neumann algebra, and finally

(iv) the tensor product of a finite family of von Neumann algebras.

4.1. Direct sum of a family of von Neumann algebras

Let $(H_\alpha)_{\alpha \varepsilon J}$ be a family of Hilbert spaces. We know that $H = \sum_{\alpha \varepsilon J} \oplus H_\alpha$, the direct sum or Hilbert sum of $(H_\alpha)_{\alpha \varepsilon J}$, is the Hilbert space of elements $x = (x_\alpha)_{\alpha \varepsilon J}$ with $x_\alpha \varepsilon H_\alpha$ for each $\alpha \varepsilon J$, $\sum_{\alpha \varepsilon J} \|x_\alpha\|^2 < \infty$, and with the inner product given by

$$[x,y] = \sum_{\alpha \varepsilon J} [x_\alpha, y_\alpha]$$

for $x = (x_\alpha)_{\alpha \varepsilon J}$, $y = (y_\alpha)_{\alpha \varepsilon J}$ in H.

Let $T_\alpha \in B(H_\alpha)$ for each $\alpha \in J$, with $\sup_{\alpha \in J} \| T_\alpha \| < \infty$. Then define $T: H \to H$ by $T(x_\alpha)_{\alpha \in J} = (T_\alpha x_\alpha)_{\alpha \in J}$. We denote T by $\sum_{\alpha \in J} \oplus T_\alpha$ and it is easy to check that $T \in B(H)$.

Theorem 4.1.1. If $(\mathcal{R}_\alpha)_{\alpha \in J}$ is a non-void family of von Neumann algebras, with $\mathcal{R}_\alpha$ on H_α for each $\alpha \in J$, then

$$\mathcal{R} = \left\{ \sum_{\alpha \in J} \oplus T_\alpha : \sup_{\alpha \in J} \| T_\alpha \| < \infty,\ T_\alpha \in \mathcal{R}_\alpha \right\}$$

is a von Neumann algebra on $H = \sum_{\alpha \in J} \oplus H_\alpha$. The operators $Q_\beta = \sum_{\alpha \in J} \oplus \delta_{\alpha\beta} I_\alpha$ form an orthogonal family of central projections in $\mathcal{R}$ and the centre of $\mathcal{R}$ is given by $Z = \{ \sum_{\alpha \in J} \oplus z_\alpha : \sup_{\alpha \in J} \| z_\alpha \| < \infty,\ z_\alpha \in \mathcal{Z}_\alpha$, the centre of $\mathcal{R}_\alpha \}$, where $\delta_{\alpha\beta} =$

$$\begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

and I_α is the identity operator on H_α. $\mathcal{R}$ is called the *direct sum* of $(\mathcal{R})_{\alpha \in J}$ and is denoted by $\sum_{\alpha \in J} \oplus \mathcal{R}_\alpha$.

Conversely, if $\mathcal{R}$ is a von Neumann algebra on H, $(Q_\alpha)_{\alpha \in A}$ is an orthogonal family of non-zero central projections in $\mathcal{R}$ with $\sum_{\alpha \in A} Q_\alpha = I$ and $H_\alpha = Q_\alpha H, \alpha \in A$, then there is an isomorphism $U: H \to \sum_{\alpha \in A} \oplus H_\alpha$ (onto) such that $U\mathcal{R}U^{-1} = \sum_{\alpha \in A} \oplus \mathcal{R}Q_\alpha$; i.e., $\mathcal{R}$ is spatially isomorphic to $\sum_{\alpha \in A} \oplus \mathcal{R}Q_\alpha$.

Proof. If $T = \sum_{\alpha \in J} \oplus T_\alpha$ and $S = \sum_{\alpha \in J} \oplus S_\alpha$, then clearly $T + S = \sum_{\alpha \in J} \oplus (T_\alpha + S_\alpha)$ and $TS = \sum_{\alpha \in J} \oplus T_\alpha S_\alpha$. Besides, $T^* = \sum_{\alpha \in J} \oplus T^*_\alpha$. For, if $x = (x_\alpha)_{\alpha \in J}$, $y = (y_\alpha)_{\alpha \in J}$ in $H = \sum_{\alpha \in J} \oplus H_\alpha$, then $[Tx, y] = \sum_{\alpha \in J} [T_\alpha x_\alpha, y_\alpha] =$

$\sum_{\alpha\,\varepsilon\, J}[x_\alpha, T^*_\alpha y_\alpha] = [x, (\sum_{\alpha\,\varepsilon\, J} \oplus T^*_\alpha)y]$ so that $T^* = \sum_{\alpha\,\varepsilon\, J} \oplus T^*_\alpha$.

It is clear that $R = \sum_{\alpha\,\varepsilon\, J} \oplus R_\alpha$ is a *-subalgebra of B(H). Let $T' \,\varepsilon$ B(H) with $T' \,\varepsilon\, R'$. Then T' commutes with $\sum_{\alpha\,\varepsilon\, J} \oplus \delta_{\alpha\beta} I_\alpha = Q_\beta$. Thus $T'Q_\beta = Q_\beta T'$ so that T' leaves H_β invariant. Let $T'_\beta = T'|H_\beta$. Then $(T'_\beta)_{\beta\,\varepsilon\, J}$ defines a bounded operator on H and $(\sum_{\beta\,\varepsilon\, J} \oplus T'_\beta)(x_\beta)_{\beta\varepsilon\, J} = (T'x_\beta)_{\beta\,\varepsilon\, J} = (T'x_\beta)_{\beta\varepsilon\, J} = \sum_{\beta\,\varepsilon\, J} \oplus Q_\beta T'x_\beta = T' \sum_{\beta\,\varepsilon\, J} \oplus Q_\beta x_\beta = T'x$ where $x = (x_\beta)_{\beta\,\varepsilon\, J}$. Thus $T' = \sum_{\beta\,\varepsilon\, J} \oplus T'_\beta$ and $(\sum_{\beta\,\varepsilon\, J} \oplus T'_\beta)(\sum_{\beta\,\varepsilon\, J} \oplus T_\beta) = \sum_{\beta\varepsilon\, J} \oplus T'_\beta T_\beta$; $(\sum_{\beta\,\varepsilon\, J} \oplus T_\beta)(\sum_{\beta\varepsilon\, J} + T'_\beta) = \sum_{\beta\,\varepsilon\, J} \oplus T_\beta T'_\beta$. As $T'T = TT'$, we conclude that $T_\beta T'_\beta = T'_\beta T_\beta$ for each $\beta\,\varepsilon\, J$. Hence, varying T_β in R_β with $T_\alpha = 0$ $(\alpha \neq \beta)$, $T'_\beta \,\varepsilon\, R'_\beta$. Thus $T' = \sum_{\beta\,\varepsilon\, J} \oplus T'_\beta$ $\varepsilon \sum_{\beta\,\varepsilon\, J} \oplus R'_\alpha$. i.e., $(\sum_{\alpha\,\varepsilon\, J} \oplus R_\alpha)' \subset \sum_{\alpha\,\varepsilon\, J} \oplus R'_\alpha$. The reverse inclusion is clear.

Hence $(\sum_{\alpha\varepsilon J} \oplus R_\alpha)' = \sum_{\alpha\varepsilon J} \oplus R'_\alpha$.

Since R_α is a von Neumann algebra on H_α, by the double commutant theorem $R_\alpha = R''_\alpha$ for each $\alpha\varepsilon\, J$. Hence

$$\sum_{\alpha\varepsilon J} \oplus R_\alpha = \sum_{\alpha\varepsilon J} \oplus R''_\alpha = (\sum_{\alpha\varepsilon J} \oplus R'_\alpha)' \quad \text{(by the above argument)}$$

which is a von Neumann algebra on H, being the commutant of a *-subalgebra of B(H). The centre Z of $\sum_{\alpha\varepsilon J} \oplus R_\alpha = (\sum_{\alpha\varepsilon J} \oplus R_\alpha) \cap (\sum_{\alpha\varepsilon J} \oplus R_\alpha)'$

$$= (\sum_{\alpha\varepsilon J} \oplus R_\alpha) \cap (\sum_{\alpha\varepsilon J} \oplus R'_\alpha)$$

$$= \sum_{\alpha\varepsilon J} \oplus (R_\alpha \cap R'_\alpha) = \sum_{\alpha\varepsilon J} \oplus Z_\alpha .$$

Conversely, if R is a von Neumann algebra on H with centre Z and if Q_α is a central projection in R, then $RQ_\alpha = \{RQ_\alpha : R \varepsilon R\}$ is a von Neumann algebra on $H_\alpha = Q_\alpha H$. For, obviuosly, RQ_α is a τ_W-closed *-subalgebra of $B(H)$ and hence RQ_α is a von Neumann algebra on H_α. By abuse of notation we will denote this von Neumann algebra by RQ_α. Since $\sum_{\alpha\varepsilon A} Q_\alpha = I$, with $Q_\alpha Q_\beta = o$ for $\alpha \neq \beta$, $x \varepsilon H$ can be written as

$$x = \sum_{\alpha\varepsilon A} Q_\alpha x$$

and $\|x\|^2 = \sum_{\alpha\varepsilon A} \|Q_\alpha x\|^2$ and hence $(Q_\alpha x)_{\alpha\varepsilon A} \varepsilon \sum_{\alpha\varepsilon A} \oplus Q_\alpha H$. Recall $H_\alpha = Q_\alpha H, \alpha \varepsilon A$. Define the map

$$U : H \to \sum_{\alpha\varepsilon A} \oplus H_\alpha \text{ by}$$

$$Ux = (Q_\alpha x)_{\alpha\varepsilon A}.$$

Then, obviously, U is an isometry and if $(Q_\alpha y_\alpha)_{\alpha\varepsilon A} \varepsilon \sum_{\alpha\varepsilon A} \oplus H_\alpha$, so that $\sum_{\alpha\varepsilon A} \|Q_\alpha y_\alpha\|^2 < \infty$, then let $y = \sum_{\alpha\varepsilon A} Q_\alpha y_\alpha$ in H. This is possible since $\{Q_\alpha\}_{\alpha\varepsilon A}$ is an orthogonal family of projections and $\sum_{\alpha\varepsilon A} \|Q_\alpha y_\alpha\|^2 < \infty$. Now $Q_\beta y = Q_\beta (\sum_\alpha Q_\alpha y_\alpha) = Q_\beta y_\beta$ and hence $(Q_\alpha y_\alpha)_{\alpha\varepsilon A} = (Q_\alpha y)_{\alpha\varepsilon A}$, so that

$$(Q_\alpha y_\alpha)_{\alpha\varepsilon A} = Uy.$$

Thus U is an onto isometry and hence H and $\sum_{\alpha\varepsilon A} \oplus H_\alpha$ are isomorphic.

For $T_\alpha \varepsilon R$ with $\sup_{\alpha\varepsilon A} \|T_\alpha\| < \infty$, and x in H,

$$U^{-1}\Big(\sum_{\alpha\in A}\oplus T_\alpha Q_\alpha\Big)\, U\, x = U^{-1}\Big(\sum_{\alpha\in A}\oplus T_\alpha Q_\alpha\Big)(Q_\alpha x)_{\alpha\in A}$$

$$= U^{-1}(T_\alpha Q_\alpha x)_{\alpha\in A}$$

$$= \sum_{\alpha\in A} T_\alpha Q_\alpha x$$

so that $U^{-1}\big(\sum_{\alpha\in A}\oplus T_\alpha Q_\alpha\big)U = \sum_{\alpha\in A} T_\alpha Q_\alpha \in \mathcal{R}$ (4.1.1.1), as $\mathcal{R}$ is strongly closed. Thus the map $\psi : \sum_{\alpha\in A}\oplus T_\alpha Q_\alpha \to \sum_{\alpha\in A} T_\alpha Q_\alpha$ given by (4.1.1.1) is a *-isomorphism of $\sum_{\alpha\in A}\oplus \mathcal{R} Q_\alpha$ into $\mathcal{R}$. ψ is onto. For, since $\sum_{\alpha\in A} Q_\alpha = I$, and since $\mathcal{R}$ is strongly closed, $T = \sum_{\alpha\in A} TQ_\alpha$ for each T in $\mathcal{R}$. Further, TQ_α leaves $Q_\alpha H$ invariant as Q_α belongs to the centre of $\mathcal{R}$, $\sup_A \|TQ_\alpha\| \leq \|T\| < \infty$ and $U^{-1}\big(\sum_{\alpha\in A}\oplus TQ_\alpha\big)U = \sum_{\alpha\in A} TQ_\alpha = T$.

Hence $\sum_{\alpha\in A}\oplus \mathcal{R} Q_\alpha$ and $\mathcal{R}$ are spatially isomorphic (i.e., there exists an isomorphism $U: H \to \sum_{\alpha\in A}\oplus H_\alpha$ (onto) such that $U^{-1}\big(\sum_{\alpha\in A}\oplus \mathcal{R}Q_\alpha\big)U = \mathcal{R}$).

Remarks. Dixmier [1] uses $\pi_{\alpha\in A}\mathcal{R}_\alpha$ instead of $\sum_{\alpha\in A}\oplus \mathcal{R}_\alpha$ and calls it the product of $(\mathcal{R}_\alpha)_{\alpha\in A}$.

4.2. Reduction and induction

Throghout this section $\mathcal{R}$ is a von Neumann algebra acting on a Hilbert space H with centre $\mathcal{Z}$, E is a projection in $\mathcal{R}$ and M= E(H). Then $E\mathcal{R}E = \{EAE: A \in \mathcal{R}\}$ is a *-subalgebra of B(H) and τ_w-closed since $E\mathcal{R}E = \{A: A \in \mathcal{R}, EAE = A\}$. The restrictions $\{A|M: A \in E\mathcal{R}E\}$ form a von Neumann algebra acting on the Hilbert space M. By abuse of notation, we denote this von Neumann algebra by $E\mathcal{R}E$ and call it the *reduction* of $\mathcal{R}$ to M.

Lemma 4.2.1. If $A \in \mathcal{R}$ and $A' \in \mathcal{R}'$, then $AA' = 0$ if and only if $C_A C_{A'} = 0$.

Proof. If $C_A C_{A'} = 0$, then $AA' = AC_A C_{A'} A' = 0$.

Suppose conversely $AA' = 0$. Recall that $C_A = [RAx: R \in \mathcal{R}, x \in H]$. Since $A'(RAx) = RA'Ax = RAA'x = 0$ for each $R \in \mathcal{R}$, $x \in H$ and since A' is a bounded operator, $A'C_A = 0$. Hence $C_{A'} C_A = 0$; i.e., $C_A C_{A'} = 0$.

Remarks. In other words, for $A \in \mathcal{R}$, $A' \in \mathcal{R}'$ the following are equivalent:

(i) $AA' = 0$.

(ii) There is a central element $z \in Z$ such that $Az = 0$ and $A'z = A'$.

(ii)$\Rightarrow$ (i) clear. (i)$\Rightarrow$ (ii) if we take $z = C_{A'}$ and apply the above lemma.

This modified form of Lemma 4.2.1 is generalized to pairs of n operators from $\mathcal{R}$ and $\mathcal{R}'$ in Proposition 4.5.11.

For each $R' \in \mathcal{R}'$, the operator $R'E$ leaves M invariant and annihilates $M^{\perp}$ in H. The mapping

$$\Phi : R' \to R'E$$

is a *-homomorphism from $\mathcal{R}'$ into B(M). Thus the set $\{(R'E)|M : R' \in \mathcal{R}'\}$ is a *-subalgebra of B(M) and contains the identity $E|M$ on M. By abuse of notation we denote this algebra by $\mathcal{R}'E$ and call this the *induction of* $\mathcal{R}'$ on E or on M.

Theorem 4.2.2. $\mathcal{R}'E$ is a von Neumann algebra and $(\mathcal{R}'E)' = E\mathcal{R}E$.

Proof. We have already observed that $E\mathcal{R}E$ is a von Neumann algebra on M. Let $T \in B(M)$ such that $T \in (\mathcal{R}'E)'$. Let $S = T \circ E$. Then $S \in \mathcal{R}'' = \mathcal{R}$, for, $SR'y = (T\circ E)R'y = T\circ ER'y = T\circ R'Ey = R'ETEy = R'E(T\circ E)y = R'Sy$ for $R' \in \mathcal{R}'$ and for

$y \in H$. Hence, $T = ESE \in E\mathcal{R}E$. Thus $(\mathcal{R}'E)' \subset E\mathcal{R}E$. Conversely, for $R \in \mathcal{R}$ and $R' \in \mathcal{R}'$ we have $(ERE)(R'E) = (R'E)(ERE)$ and hence $(E\mathcal{R}E)' \supset (\mathcal{R}'E)$. Thus $E\mathcal{R}E = (\mathcal{R}'E)'$ (4.2.1.1).

Now let $T' \in B(M)$ such that $T' \in (E\mathcal{R}E)'$. We shall show that $T' \in \mathcal{R}'E$. If this is done, from (4.2.2.1) it follows that $\mathcal{R}'E$ is a von Neumann algebra as

$$\mathcal{R}'E \subset (\mathcal{R}'E)'' = (E\mathcal{R}E)' \subset \mathcal{R}'E \ ,$$

and as $\mathcal{R}'E$ is a $*$-subalgebra of $B(M)$ containing the identity.

Let $T' \in (E\mathcal{R}E)'$, which is a von Neumann algebra on M. First let T' be unitary. Since $C_E = [REx : R \in \mathcal{R},\ x \in H]$, the set $D = \{\sum_1^n R_i x_i : R_1, \ldots, R_n \in \mathcal{R};\ x_1, x_2, \ldots, x_n \in M\}$ is dense in $C_E(H)$. Define the linear transformation

$$\Phi : D \to C_E(H)$$

by

$$\Phi\left(\sum_{i=1}^n R_i x_i\right) = \sum_{i=1}^n R_i T' x_i .$$

Then Φ is norm preserving. For,

$$\left\|\Phi\left(\sum_{i=1}^n R_i x_i\right)\right\|^2 = \left\|\sum_{i=1}^n R_i T' x_i\right\|^2$$

$$= \sum_{i,j=1}^n [R_i T' x_i , R_j T' x_j]$$

$$= \sum_{i,j=1}^n [R_i E T' x_i ,\ R_j E T' x_j]$$

$$= \sum_{i,j=1}^n [E R_j^* R_i E T' x_i , T' x_j]$$

$$= \sum_{i,j=1}^{n} [T'ER_j^*R_iEx_i, T'x_j]$$

$$= \sum_{i,j=1}^{n} [ER_j^*R_iEx_i, T'^*T'x_j]$$

$$= \sum_{i,j=1}^{n} [ER_j^*R_iEx_i, x_j] \quad (\because T' \text{ is unitary})$$

$$= \sum_{i,j=1}^{n} [R_ix_i, R_jx_j]$$

$$= \|\sum_{i=1}^{n} R_ix_i\|^2.$$

Thus Φ can be extended uniquely to an isometry operator Φ' on $C_E(H)$. Define $\Psi = \Phi' \circ C_E$. Then $\Psi \varepsilon B(H)$ and $\Psi C_E = C_E\Psi = \Psi$.

For every $R \varepsilon \mathcal{R}$, and $x_i \varepsilon M$, we have

$$\Psi (R \sum_{i=1}^{n} R_ix_i) = \Phi'(\sum_{i=1}^{n} RR_ix_i) = \sum_{i=1}^{n} RR_iT'x_i$$

$$= R\sum_{i=1}^{n} R_iT'x_i = R\Psi(\sum_{i=1}^{n} R_ix_i).$$

Hence $\Psi RC_E = R\Psi C_E = R\Psi$ for each $R \varepsilon \mathcal{R}$. Then

$$\Psi R = \Psi R (C_E + (I - C_E))$$

$$= \Psi RC_E + \Psi R(I - C_E)$$

$$= R\Psi C_E + \Psi (I - C_E)R$$

$$= R\Psi.$$

Thus $\Psi \in \mathcal{R}'$. Now for $x \in M$,

$$T'x = \Psi(x) = \Psi Ex$$

and hence $T' = \Psi E \in \mathcal{R}'E$. Hence $(E\mathcal{R}E)' \subset (\mathcal{R}'E)$, since any operator in $(E\mathcal{R}E)'$ is a finite linear combination of unitary operators.

This completes the proof of the theorem.

Corollary 4.2.3.

(a) $\mathcal{R}'E$ is a B*-algebra of operators.

(b) The map $\Psi : \mathcal{R}'C_E \to \mathcal{R}'E : R'C_E \to R'C_E E = R'E$ is an isomorphism onto $\mathcal{R}'E$ and is norm preserving. Ψ gives an isomorphism of $\mathcal{R}'$ onto $\mathcal{R}'E$ if and only if $C_E = I$.

(c) The centre of $E\mathcal{R}E$ = the centre of $\mathcal{R}'E = ZE$.

Proof.

(a) is obvious, as $\mathcal{R}'E$ is a von Neumann algebra on $E(H)$.

(b) Ψ is injective, since $\Psi(R'C_E) = 0 \Rightarrow R'E = 0 \Rightarrow C_{R'}C_E = 0$ (by Lemma 4.2.1). Hence $R'C_E = R'C_{R'}C_E = 0$. Clearly, Ψ is a homomorphism. Ψ is onto. For, $R'E = (R'C_E)E$ for each $R' \in \mathcal{R}'$. Thus $R'E = \Psi(R'C_E)$. Therefore Ψ is an isomorphism of $\mathcal{R}'C_E$ onto $\mathcal{R}'E$. Ψ is an isometry by (a) and Theorem 1.5.12, as Ψ is an isomorphism between two B*-algebras. The last part of (b) is now clear.

(c) In view of the above theorem

centre of $E\mathcal{R}E$ = centre of $\mathcal{R}'E$.

By (b), centre of $\mathcal{R}'E = \Psi$(centre of $\mathcal{R}'C_E$). But the centre of $\mathcal{R}'C_E$ = $\mathcal{R}C_E \cap \mathcal{R}'C_E \supset (\mathcal{R}\cap\mathcal{R}')C_E = \mathcal{Z}C_E$. If $T \in \mathcal{R}C_E \cap \mathcal{R}'C_E$, then

$$T = R_1C_E = R_2'C_E, \quad R_1 \in \mathcal{R},\ R_2' \in \mathcal{R}'.$$

Then $T \in \mathcal{R} \cap \mathcal{R}'$ and $TC_E = T$ so that

$$T \in (\mathcal{R}\cap\mathcal{R}')C_E = \mathcal{Z}C_E.$$

Thus the centre of $\mathcal{R}'C_E = \mathcal{Z}C_E$. Consequently, the centre of $\mathcal{R}'E$ = $\Psi(\mathcal{Z}C_E) = \mathcal{Z}E$.

Recall by isomorphism and homomorphism we mean, respectively, *-isomorphism and *-homomorphism only.

Proposition 4.2.4. If $\mathcal{R}$ is a von Neumann algebra with countable generators and if its centre $\mathcal{Z}$ is countably decomposable, then $\mathcal{R}$ is isomorphic to a von Neumann algebra acting on a separable Hilbert space.

Proof. As $\mathcal{Z}$ is abelian and countably decomposable, by Theorem 3.3.11 $\mathcal{Z}$ has a separating vector x(say). Consider $[\mathcal{R}x] = E'$ in $\mathcal{R}'$. Then $C_{E'} = [R'E'y : R' \in \mathcal{R}', y \in H] = [R'Rx : R \in \mathcal{R}, R' \in \mathcal{R}'] = [\mathcal{Z}'x] = I$, by Lemma 3.3.10 and by the fact that x is a generating vector for $\mathcal{Z}'$. Hence by Corollary 4.2.3(b), $\mathcal{R}$ is isomorphic to $\mathcal{R}E'$. Since $\mathcal{R}$ has countable generators, the space $[\mathcal{R}x]$ is separable and hence the proposition.

In Theorem 3.4.2 a sufficient condition for $C_E \leq C_F$ to imply $E \precsim F$ was given. We give below another sufficient condition for this implication to hold.

Theorem 4.2.5. Let $\mathcal{R}$ be a von Neumann algebra and E and F be projections in

R. Let $C_F \leq C_E$. If ERE has a generating vector x and FRF has a separating vector y, then $F \precsim E$.

Proof. By the comparison theorem, there are central projections Z_1 and Z_2 with $Z_1 Z_2 = o$ and $Z_1 + Z_2 = I$ such that $EZ_1 \precsim FZ_1$ and $EZ_2 \succ FZ_2$.

(a) If P, Q are projections in R with $P \sim Q$ and PRP has a generating vector w, then QRQ has Uw as a generating vector, where $U^*U = P$, $UU^* = Q$, $U \in R$. In fact,

$R'P$ is spatially isomorphic to $R'Q$. For, if $UP = V$, then $U^*Q = V^{-1}$ considering $V: P(H) \to Q(H)$. Now $VR'P\,V^{-1} = UPR'P\,U^*\,Q = R'UPU^*Q = R'Q.Q = R'Q$, since $UPU^* = Q$. Hence the respective commutants of PRP and QRQ are spatially isomorphic. Hence Uw is a generating vector for QRQ.

(b) If P,Q are projections in R with $P \leq Q$, $C_P = C_Q$ and $R'P$ has a separating vector $w \in P$, then $R'Q$ also has w as a separating vector.

For, since $C_P = C_Q$, by Corollary 4.2.3(b) $R'P$ and $R'Q$ are isomorphic, under the map $R'P \to R'Q$. $R'Qw = 0 \Rightarrow R'QPw = 0$ ($\because w \in P$) $\Rightarrow R'Pw = 0$ ($\because P \leq Q$) $\Rightarrow R'P = 0$ ($\because$ w is a separating vector for $R'P$) $\Rightarrow R'Q = o$ (under isomorphism). Hence (b) holds.

Proceeding with the proof of the theorem, let $EZ_1 \sim E_1 Z_1 \leq FZ_1$. Since ERE has a generating vector x, $EZ_1 R EZ_1$ has $Z_1 x$ as a generating vector and by (a) $UZ_1 x$ is a generating vector for $E_1 Z_1 R E_1 Z_1$, where $U^*U = EZ_1$ and $UU^* = E_1 Z_1$, $U \in R$. Further, $C_{EZ_1} = C_{E_1 Z_1} \leq C_{FZ_1}$; but, by hypothesis, $C_F \leq C_E$. Hence by (b), as $UZ_1 x$ is a separating vector for $R'E_1 Z_1$, it is a separating vector for $R'FZ_1$. Thus $UZ_1 x$ is a generating vector for $FZ_1 R FZ_1$. Therefore

$[(FZ_1 \mathcal{R} FZ_1)(UZ_1x)] = FZ_1 \supset [FZ_1 \mathcal{R} FZ_1y]$. Hence by Theorem 3.4.6 $[(\mathcal{R}'FZ_1)(UZ_1x)] \succsim [\mathcal{R}'FZ_1y]$.(4.2.5.1)

By hypothesis, y is a separating vector for $F\mathcal{R}F$ and hence for its subalgebra $FZ_1 \mathcal{R} FZ_1$. Hence y is a generating vector for $\mathcal{R}'FZ_1$ and therefore $[(\mathcal{R}'FZ_1)y] = FZ_1$ (4.2.5.2). As observed earlier, UZ_1x is a generating vector for $E_1Z_1\mathcal{R}E_1Z_1$. Now (4.2.5.1) and (4.2.5.2) imply that $FZ_1 \precsim [(\mathcal{R}'FZ_1)UZ_1x] = [(\mathcal{R}'FZ_1)E_1UZ_1x](\because UZ_1x \in E_1Z_1) = [\mathcal{R}'E_1Z_1UZ_1x] \leq E_1Z_1 \sim EZ_1$. Thus $FZ_1 \precsim EZ_1$. But $EZ_1 \precsim FZ_1$. Hence $EZ_1 \sim FZ_1$. As $EZ_2 \succ FZ_2$, it follows that $F \precsim E$.

Lemma 4.2.6. Let $\mathcal{R}$ be a von Neumann algebra over H and x a separating vector for $\mathcal{R}$. If there is a generating vector for $\mathcal{R}$, then $[\mathcal{R}x] \sim I$.

Proof. As x is a separating vector for $\mathcal{R}$, x is a generating vector for $\mathcal{R}'$. Hence $[\mathcal{R}'x] = I$. By Proposition 3.3.5, $C_{[\mathcal{R}x]} = C_{[\mathcal{R}'x]} = I$. Let $[\mathcal{R}x] = E'$ be in $\mathcal{R}'$. Clearly, $[E'\mathcal{R}'E'x] = [E'\mathcal{R}'x] = E'[\mathcal{R}'x] = E'I = E'$. Thus x is a generating vector for $E'\mathcal{R}'E'$. If y is a generating vector for $\mathcal{R}$, then it is separting for $\mathcal{R}'$. By applying Theorem 4.2.5 to E', I in $\mathcal{R}'$, we have $I \sim E'$, since $I \leq C_{E'}$, x is a generating vector for $E'\mathcal{R}'E'$ and y is a separating vector for $\mathcal{R}'(=I\mathcal{R}'I)$. Hence $E' \sim I$.

Corollary 4.2.7. If there exist a generating vector x and a separating vector y for a von Neumann algebra $\mathcal{R}$, then there is a vector w which is both generating and separating for $\mathcal{R}$.

Proof. Let $E' = [\mathcal{R}y]$. Then by Lemma 4.2.6, $E' \sim I$. Hence there exists a partial isometry U' in $\mathcal{R}'$ such that $U'^*U' = E'$, $U'U'^* = I$. Then as y is a generating vector for $\mathcal{R}'$, y is a generating vector for $E'\mathcal{R}'E'$. Hence y is a separating vector for $\mathcal{R}E'$

Also, $[\mathcal{R}E'y]= E'[\mathcal{R}y]= E'$. Thus y is both generating and separating for $E'\mathcal{R}'E'$ and hence by (a) and (b) in the proof of Theorem 4.2.5, $U'y = w$ is both generating and separating for $\mathcal{R}'$ and hence for $\mathcal{R}$.

4.3. Finite tensor products of Hilbert spaces

Definition 4.3.1. Let $H_1, H_2, \ldots, H_N$ be a finite sequence of Hilbert spaces. Let Φ be a conjugate-multilinear functional on $\prod_{n=1}^{N} H_n$. Then Φ is said to be of Hilbert class if

(a) $\Phi(x^{(1)}, \ldots, x^{(N)})$ is separately continuous in the variable $x^{(n)} \in H_n$, $1 \leq n \leq N$, when other variable are fixed; and

(b) $\sum_{\alpha_1, \ldots, \alpha_N} |\Phi(x^{(1)}_{\alpha_1}, x^{(2)}_{\alpha_2}, \ldots, x^{(N)}_{\alpha_N})|^2 < \infty$,

where for each $1 \leq n \leq N$, $(x^{(n)}_{\alpha_n})_{\alpha_n \in J_n}$ is an orthonormal basis for the Hilbert space H_n.

The following lemma shows that the property (b) in the above definition is independent of the particular choice of orthonormal bases used in the definition.

Lemma 4.3.2. Let H_n be Hilbert spaces, $1 \leq n \leq N$, and let $(x^{(n)}_{\alpha_n})_{\alpha \in J_n}$, $(y^{(n)}_{\alpha})_{\alpha \in J_n}$ be a pair of orthonormal bases for H_n. Let Φ and ψ be a pair of conjugate multilinear functionals defined on $\prod_{n=1}^{N} H_n$. Then the following hold:

(i) If, for each n, $\Phi(x^{(1)}, \ldots, x^{(N)})$ is separately continuous in the variable $x^{(n)} \in H_n$ when the other variables are fixed, then

$$\sum_{\alpha_1, \ldots, \alpha_N} |\Phi(x^{(1)}_{\alpha_1}, \ldots, x^{(N)}_{\alpha_N})|^2 = \sum_{\alpha_1, \ldots, \alpha_N} |\Phi(y^{(1)}_{\alpha_1}, \ldots, y^{(N)}_{\alpha_N})|^2.$$

(Thus if Φ is of Hilbert class relative to the bases $(x^{(n)}_{\alpha_n})_{\alpha_n \in J_n}$, (n = 1,2,...,N), it is of Hilbert class relative to the bases $(y^{(n)}_{\alpha_n})_{\alpha_n \in J_n}$ (n= 1,2,...,N)).

(ii) If Φ and Ψ are both of Hilbert class, the series

$$[\Phi,\Psi]= \sum_{\alpha_1,\ldots,\alpha_N} \Phi(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N}) \overline{\Psi(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N})}$$

converges absolutely and we also have

$$[\Phi,\Psi]= \sum_{\alpha_1,\ldots,\alpha_N} \Phi(y^{(1)}_{\alpha_1},\ldots,y^{(N)}_{\alpha_N}) \overline{\Psi(y^{(1)}_{\alpha_1},\ldots,y^{(N)}_{\alpha_N})}.$$

Proof.

(i) First, we shall prove the result for N=2. For fixed $x^{(2)}$, $\Phi(.,x^{(2)})$ is a continuous conjugate-linear functional on H_1 and hence by the Riesz representation theorem, there exists a unique vector $X(x^{(2)})$ in H_1 with $\|\Phi(.,x^{(2)})\| = \|X(x^{(2)})\|$ such that

$$\Phi(.,x^{(2)}) = [X(x^{(2)}),.] \qquad (4.3.2.1)$$

where [.,.] is the inner-product of H_1. Thus by the Parseval identity

$$\sum_{\alpha_1}|\Phi(x^{(1)}_{\alpha_1}, x^{(2)})|^2 = \sum_{\alpha_1}|[x^{(1)}_{\alpha_1}, X(x^{(2)})]|^2$$

$$= \|X(x^{(2)})\|^2$$

$$= \sum_{\alpha_1}|[y^{(1)}_{\alpha_1}, X(x^{(2)})]|^2$$

$$= \sum_{\alpha_1}|\Phi(y^{(1)}_{\alpha_1}, x^{(2)})|^2$$

where we use the fact that $(x^{(1)}_{\alpha_1})_{\alpha_1 \in J_1}$ and $(y^{(1)}_{\alpha_1})_{\alpha_1 \in J_1}$ are orthonormal bases in H_1 and the identity (4.3.2.1). Thus

$$\sum_{\alpha_1} |\Phi(x_1^{(1)}, x^{(2)})|^2 = \sum_{\alpha_1} |\Phi(y_{\alpha_1}^{(1)}, x^{(2)})|^2. \qquad (4.3.2.2)$$

Then from (4.3.2.2) it follows that

$$\sum_{\alpha_1,\alpha_2} |\Phi(x_{\alpha_1}^{(1)}, x_{\alpha_2}^{(2)})|^2 = \sum_{\alpha_1,\alpha_2} |\Phi(y_{\alpha_1}^{(1)}, x_{\alpha_2}^{(2)})|^2. \qquad (4.3.2.3)$$

Arguing with $\Phi(x^{(1)}, .)$, we similarly have

$$\sum_{\alpha_2} |\Phi(x^{(1)}, x_{\alpha_2}^{(2)}|^2 = \sum_{\alpha_2} |\Phi(x^{(1)}, y_{\alpha_2}^{(2)})|^2 \qquad (4.3.2.4)$$

and hence

$$\sum_{\alpha_2} |\Phi(y_{\alpha_1}^{(1)}, x_{\alpha_2}^{(2)})|^2 = \sum_{\alpha_2} |\Phi(y_{\alpha_1}^{(1)}, y_{\alpha_2}^{(2)})|^2.$$

Consequently, we obtain

$$\sum_{\alpha_1,\alpha_2} |\Phi(y_{\alpha_1}^{(1)}, x_{\alpha_2}^{(2)})|^2 = \sum_{\alpha_1,\alpha_2} |\Phi(y_{\alpha_1}^{(1)}, x_{\alpha_2}^{(2)})|^2. \qquad (4.3.2.5)$$

From (4.3.2.3) and (4.3.2.5) we obtain $\sum_{\alpha_1,\alpha_2} |\Phi(x_{\alpha_1}^{(1)}, x_{\alpha_2}^{(2)})|^2 = \sum_{\alpha_1,\alpha_2} |\Phi(y_{\alpha_1}^{(1)}, y_{\alpha_2}^{(2)})|^2$. With the modification that $\Phi(.,x^{(2)})$ is replaced by $\Phi(.,x^{(2)},\ldots,x^{(N)})$ and $\Phi(x^{(1)},.)$ is replaced by $\Phi(x^{(1)},.,x^{(3)},\ldots,x^{(N)})$ so that $X(x^{(2)})$ is replaced by $X(x^{(2)},\ldots,x^{(N)})$ and $X(x^{(1)})$ by $X(x^{(1)}, x^{(3)},\ldots,x^{(N)})$, we obtain

$$\sum_{\alpha_1,\ldots,\alpha_N} |\Phi(x_{\alpha_1}^{(1)}, x_{\alpha_2}^{(2)},\ldots,x_{\alpha_N}^{(N)})|^2 = \sum_{\alpha_1,\ldots,\alpha_N} |\Phi(y_{\alpha_1}^{(1)}, y_{\alpha_2}^{(2)}, x_{\alpha_3}^{(3)},\ldots,x_{\alpha_N}^{(N)})|^2.$$

Now replacing the argument above inductively as often as necessary, we obtain (i).

(ii) $\Sigma|\Phi(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N})\Psi(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N})| \leq \frac{1}{2}\{\Sigma|\Phi(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N})|^2 + \Sigma|\Psi(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N})|^2\} < \infty$

and this implies the absolute convergence of the series defining $[\Phi,\Psi]$. Now writing, $\Psi(x^{(1)},\ldots,x^{(N)}) = [Y(x^{(2)},\ldots,x^{(N)}), x^{(1)}]$ by the Riesz representation theorem, where $Y(x^{(2)},\ldots,x^{(N)})$ is a vector in H_1 and using the fact that $\{x^{(1)}_{\alpha_1}\}_{\alpha_1 \in J_1}$ is an orthonormal basis in H_1, we have

$$\sum_{\alpha_1}\Phi(x^{(1)}_{\alpha_1}, x^{(2)},\ldots,x^{(N)})\overline{\Psi(x^{(1)}_{\alpha_1}, x^{(2)},\ldots,x^{(N)})} =$$

$$= \sum_{\alpha_1}[X(x^{(2)},\ldots,x^{(N)}), x^{(1)}_{\alpha_1}][x^{(1)}_{\alpha_1}, Y(x^{(2)},\ldots,x^{(N)})]$$

$$= [X(x^{(2)},\ldots,x^{(N)}), Y(x^{(2)},\ldots,x^{(N)})]$$

$$= \sum_{\alpha_1}[X(x^{(2)},\ldots,x^{(N)}), y^{(1)}_{\alpha_1}][y^{(1)}_{\alpha_1}, Y(x^{(2)},\ldots,x^{(N)})]$$

$$= \sum_{\alpha_1}\Phi(y^{(1)}_{\alpha_1}, x^{(2)},\ldots,x^{(N)})\overline{\Psi(y^{(1)}_{\alpha_1}, x^{(2)},\ldots,x^{(N)})}.$$

Thus

$$[\Phi,\Psi] = \sum_{\alpha_1,\ldots,\alpha_N}\Phi(x^{(1)}_{\alpha_1}, x^{(2)}_{\alpha_2},\ldots,x^{(N)}_{\alpha_N})\overline{\Psi(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N})} =$$

$$= \sum_{\alpha_1,\ldots,\alpha_N}\Phi(y^{(1)}_{\alpha_1}, x^{(2)}_{\alpha_2},\ldots,x^{(N)}_{\alpha_N})\overline{\Psi(y^{(1)}_{\alpha_1}, x^{(2)}_{\alpha_2},\ldots,x^{(N)}_{\alpha_N})}.$$

Arguing inductively, we obtain the result (ii).

Definition 4.3.3. Let Φ,Ψ be two conjugate-multilinear functionals on $\prod_{n=1}^{N} H_n$, both of Hilbert class. Let $\alpha \in \mathbb{C}$. Then we define

$(\Phi+\Psi)(x^{(1)},\ldots,x^{(N)}) = \Phi(x^{(1)},\ldots,x^{(N)}) + \Psi(x^{(1)},\ldots,x^{(N)})$, $(\alpha\Phi)(x^{(1)},\ldots,x^{(N)}) = \alpha.\Phi(x^{(1)},\ldots,x^{(N)})$.

Theorem 4.3.4. The set of all conjugate multilinear functionals of Hilbert class on $H = \pi_1^N H_n$ is a Hilbert space under the operations in Definition 4.3.3 and the inner product $[\cdot,\cdot]$ given by (ii) of Lemma 4.3.2.

Proof. Clearly, $\Phi+\Psi$ and $\alpha\Phi$ are conjugate multilinear when Φ and Ψ are so. Let $\{x_\alpha^{(n)}\}_{\alpha\varepsilon J_n}$ be an orthonormal basis in H_n, $1 \leq n \leq N$. Then

$$\Sigma|(\Phi+\Psi)(x_{\alpha_1}^{(1)},\ldots,x_{\alpha_N}^{(N)})|^2 \leq 2\Sigma\{|\Phi(x_{\alpha_1}^{(1)},\ldots,x_{\alpha_N}^{(N)})|^2 + |\Psi(x_{\alpha_1}^{(1)},\ldots,x_{\alpha_N}^{(N)})|^2\} < \infty .$$

Hence $\Phi+\Psi$ is of Hilbert class with Φ and Ψ. Clearly, $\alpha\Phi$ is of Hilbert class if Φ is.

It is easy to verify that

$$[\Phi,\Psi] = \Sigma\Phi(x_{\alpha_1}^{(1)},\ldots,x_{\alpha_N}^{(N)})\,\overline{\Psi(x_{\alpha_1}^{(1)},\ldots,x_{\alpha_N}^{(N)})}$$

has all the properties of an inner-product excepting that $[\Phi,\Phi] = 0 \Rightarrow \Phi = 0$.

To prove that $[\Phi,\Phi] = 0 \Rightarrow \Phi = 0$, let $(x^{(1)},\ldots,x^{(N)})$ be an arbitrary vector in H with each $x^{(i)} \neq 0$. Without loss, we can assume the vector to be such that $\|x^{(n)}\| = 1, 1 \leq n \leq N$. Since any vector of unit norm can be incorporated into an orthonormal basis it follows that $[\Phi,\Phi] = 0$ implies $\Phi(x^{(1)},\ldots,x^{(N)}) = 0$.

Now if $(y^{(1)},\ldots,y^{(N)})$ is an arbitrary element in H, $y^{(i)} \neq 0$ for each i, then $\Phi(y^{(1)},\ldots,y^{(N)}) = (\pi_1^N \|y^{(i)}\|)\,\Phi\left(\frac{y^{(1)}}{\|y^{(1)}\|},\ldots,\frac{y^{(N)}}{\|y^{(N)}\|}\right) = 0$. Since Φ is conjugate multilinear, $\Phi(y^{(1)},\ldots,y^{(N)}) = 0$ if any $y^{(i)} = 0$. Thus $\Phi = 0$.

To show that the set of all conjugate multilinear functionals of Hilbert class is complete under the norm induced by $[.,.]$, let $\{\Phi_k\}_k$ be a Cauchy sequence. Then

$$|\Phi_k(x^{(1)},\ldots,x^{(N)}) - \Phi_\ell(x^{(1)},\ldots,x^{(N)})|^2 \leqq \|\Phi_k - \Phi_\ell\|^2 \prod_1^N \|x^{(i)}\|$$

since, if $x^i \neq 0$, $\frac{x^{(i)}}{\|x^{(i)}\|}$ can be extended to an orthonormal basis in H_i, $(1 \leqq i \leqq N)$. If $x^{(i)} = 0$, the inequality reduces to equality to 0.

Thus for fixed $(x^{(1)},\ldots,x^{(N)}) \in H$, $\{\Phi_k(x^{(1)},\ldots,x^{(N)})\}_k$ is a Cauchy sequence of complex numbers and hence converges to a unique number $\Phi(x^{(1)},\ldots,x^{(N)})$ (say). The conjugate multilinearity of Φ_k for each k clearly implies that Φ is also conjugate multilinear on H.

$$\Phi(.,\ x^{(2)},\ldots,x^{(N)}) = \lim_k \Phi_k(.,\ x^{(2)},\ldots,x^{(N)}). \qquad (4.3.4.1)$$

If $f_o(x) = \lim_k f_k(x)$, $x \in H_1$, f_k linear functionals on H_1, then $|f_k(x) - f_o(x)| < \varepsilon$ if $k \geqq k_o(x)$, so that $\sup_k |f_k(x)| \leqq M(x)$, for each $x \in H_1$. Hence by the uniform boundedness principle, $\sup_k \|f_k\| \leqq K < \infty$, when f_k are further continuous.

Thus from (4.3.4.1) we have $\|\Phi(.,x^{(2)},\ldots,x^{(N)})\| < K'$ and

$$\sup_k \|\Phi_k(.,x^{(2)},\ldots,x^{(N)})\| < K' \text{ for some finite } K'.$$

$|\Phi(x^{(1)} - x_o^{(1)},\ x^{(2)},\ldots,x^{(N)})| \leqq K'\|x^{(1)} - x_o^{(1)}\| \to 0$ as $x^{(1)} \to x_o^{(1)}$ in H_1. Thus Φ has the property (a) of 4.3.1.

Then for each N-tuple $M_1,\ldots,M_N$ of finite sets in respective index sets

$$\sum_{\alpha_1 \in M_1} \cdots \sum_{\alpha_N \in M_N} |\Phi(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N})|^2 = \lim_{k\to\infty} \sum_{\alpha_1 \in M_1} \cdots \sum_{\alpha_N \in M_N} |\Phi_k(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N})|^2$$

$$\leqq \lim_{k\to\infty} \sum_{\alpha_1 \in J_1} \cdots \sum_{\alpha_N \in J_N} |\Phi_k(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N})|^2 = \lim_k \|\Phi_k\|^2 = L \text{ (say)} \qquad \text{since}$$

$\|\Phi_k - \Phi_\ell\|^2 < \varepsilon^2$, $k, \ell \geqq k_o$, implies that $\lim_{k\to\infty} \|\Phi_k\|^2$ exists and is finite. Hence Φ is of Hilbert class.

Finally,

$$\sum_{\alpha_1 \in M_1} \cdots \sum_{\alpha_n \in M_N} |\Phi(x^{(1)}_{1}, \ldots, x^{(N)}_{\alpha_N}) - \Phi_k(x^{(1)}_{\alpha_1}, \ldots, x^{(N)}_{\alpha_N})|^2 =$$

$$= \lim_{\ell\to\infty} \sum_{\alpha_1 \in M_1} \cdots \sum_{\alpha_N \in M_N} |\Phi_\ell(x^{(1)}_{\alpha_1}, \ldots, x^{(N)}_{\alpha_N}) - \Phi_k(x^{(1)}_{\alpha_1}, \ldots, x^{(N)}_{\alpha_N})|^2$$

$\leqq \lim_{\ell\to\infty} ||\Phi_\ell - \Phi_k||^2 \leqq \varepsilon^2$ if $k \geqq n_o$, for arbitrary finite subsets $M_1, \ldots, M_n$ of the respective index sets.

Thus $||\Phi - \Phi_k|| \leqq \varepsilon$ if $k \geqq n_o$.

Hence $\{\Phi_k\}$ converges to Φ in the norm $||.||$ induced by [.,.] of 4.3.2(ii).

Definition 4.3.5.

(a) The Hilbert space of Lemma 4.3.4 is called the tensor product of $H_1, \ldots, H_N$ and is denoted by $H_1 \otimes H_2 \otimes \cdots \otimes H_N$ or by $\overset{N}{\underset{1}{\pi}} \otimes H_n$.

(b) The symbol $\overset{N}{\underset{1}{\pi}} \otimes Z_n$ denotes the conjugate multilinear functional Ψ on $\overset{N}{\underset{1}{\pi}} H_n$, defined by $\Psi(x^{(1)}, \ldots, x^{(N)}) = [Z_1, x^{(1)}] \ldots [Z_N, x^{(N)}]$, where $Z_i \in H_i (1 \leqq i \leqq N)$. We also write $Z_1 \otimes \cdots \otimes Z_N$ for $\overset{N}{\underset{n=1}{\pi}} \otimes Z_n$.

Lemma 4.3.6. Let H_n, $Z_n (1 \leqq n \leqq N)$ be as un Definition 4.3.5. Let $Z'_n \in H_n, 1 \leqq n \leqq N$, $\alpha \in \mathbb{C}$. Then:

(i) $\overset{N}{\underset{1}{\pi}} \otimes Z_n \in \overset{N}{\underset{1}{\pi}} \otimes H_n$ and $||\overset{N}{\underset{1}{\pi}} \otimes Z_n|| = \overset{N}{\underset{1}{\pi}} ||Z_n||$.

(ii) $[\overset{N}{\underset{1}{\pi}} \otimes Z_n, \overset{N}{\underset{1}{\pi}} \otimes Z'_n] = \overset{N}{\underset{1}{\pi}} [Z_n, Z'_n]$.

(iii) $Z_1 \otimes Z_2 \otimes \cdots \otimes Z_n + Z_1 \otimes Z_2 \cdots \otimes Z_{i-1} \otimes Z'_i \otimes Z_{i+1} \otimes \cdots \otimes Z_n =$

$$= Z_1 \otimes Z_2 \otimes \cdots \otimes Z_{i-1} \otimes (Z_i + Z'_1) \otimes Z_{i+1} \otimes \cdots \otimes Z_N.$$

(iv) $Z_1 \otimes Z_2 \otimes \cdots \otimes \alpha Z_i \otimes \cdots \otimes Z_N = \alpha(Z_1 \otimes \cdots \otimes Z_N)$.

Proof.

(i) Obviously, $\overset{N}{\underset{i=1}{\pi}} \otimes Z_i$ is a conjugate multilinear functional having the property (a) of Definition 4.3.1. To show that it has also property (b) of this definition, we argue as follows. If a $Z_n = 0$, clearly $\overset{N}{\underset{n=1}{\pi}} \otimes Z_n = 0$ and (i) holds. Hence let $Z_n \neq 0$, $1 \leqq n \leqq N$. Then choose an orthonormal basis $(x^{(n)}_{\alpha_n})_{\alpha_n \in J_n}$ containing $\frac{Z_n}{\|Z_n\|}$ in H_n, $1 \leqq n \leqq N$. Then let $x^{(n)}_{\beta_n} = \frac{Z_n}{\|Z_n\|}$.

$$\sum_{\alpha_1,\ldots,\alpha_N} \left| \overset{N}{\underset{1}{\pi}} [Z_n, x^{(n)}_{\alpha_n}] \right|^2$$

$$= \overset{N}{\underset{n=1}{\pi}} |[Z_n, x^{(n)}_{\beta_n}]|^2 = \overset{N}{\underset{n=1}{\pi}} \|Z_n\|^2 < \infty .$$

Hence $\overset{N}{\underset{1}{\pi}} \otimes Z_n \in \overset{N}{\underset{1}{\pi}} \otimes H_n$ and $\|\overset{N}{\underset{1}{\pi}} \otimes Z_n\| = \overset{N}{\underset{1}{\pi}} \| Z_n\|$.

(ii) Let $\{x^{(n)}_{\alpha_n}\}_{\alpha_n \in J_n}$ be an orthonormal basis of H_n, $(1 \leqq n \leqq N)$. Then

$$[\overset{N}{\underset{1}{\pi}} \otimes Z_n, \overset{N}{\underset{1}{\pi}} \otimes Z'_n]$$

$$= \sum_{\alpha_1,\ldots,\alpha_N} (\overset{N}{\underset{1}{\pi}} \otimes Z_n)(x_{\alpha_1}^{(1)},\ldots,x_{\alpha_N}^{(N)}) \overline{(\overset{N}{\underset{1}{\pi}} \otimes Z'_n)(x_{\alpha_1}^{(1)},\ldots,x_{\alpha_N}^{(N)})}$$

$$= \sum_{\alpha_1,\ldots,\alpha_N} \overset{N}{\underset{1}{\pi}} [Z_n, x_{\alpha_n}^{(n)}] [x_{\alpha_n}^{(n)}, Z'_n]$$

$$= \overset{N}{\underset{n=1}{\pi}} \left(\sum_{\alpha_n \in J_n} [Z_n, x_{\alpha_n}^{(n)}] [x_{\alpha_n}^{(n)}, Z'_n]\right) \text{(due to absolute convergence)}$$

$$= \pi_1^N [Z_n, Z'_n].$$

Hence (ii) holds.

(iii) and (iv) easily follow from Definition 4.3.5.

This completes the proof of the lemma.

4.4. Finite tensor products of von Neumann algebras

Definition 4.4.1. Let H_n, $(1 \leqq n \leqq N)$, be Hilbert spaces and $A_n \varepsilon B(H_n)$, $(1 \leqq n \leqq N)$. If $\Phi \varepsilon \pi_1^N \otimes H_n$, then the linear transformation A defined on the set of conjugate multilinear functionals Φ by the equation

$$(A\Phi)(x^{(1)},\ldots,x^{(N)}) = \Phi(A_1^* x^{(1)},\ldots A_N^* x^{(N)})$$

is denoted by the symbol $\pi_{n=1}^N \otimes A_n$ or by $A_1 \otimes \ldots \otimes A_N$.

Lemma 4.4.2. Let H_n, $(1 \leqq n \leqq N)$, be Hilbert spaces and $Z_n \varepsilon H_n$, $(1 \leqq n \leqq N)$.

Let A_n, $A'_n \varepsilon B(H_n)$ and $\alpha \varepsilon \mathbb{C}$. Then:

(i) $\pi_1^N \otimes A_n$ is a bounded operator on $\pi_1^N \otimes H_n$ and $\| \pi_1^N \otimes A_n \| = \pi_1^N \| A_n \|$.

(ii) $(\pi_1^N \otimes A_n)(\pi_1^N \otimes A'_n) = \pi_1^N \otimes (A_n A'_n)$.

(iii) $(\pi_1^N \otimes A_n)^* = \pi_1^N \otimes A_n^*$.

(iv) $(\pi_1^N \otimes A_n)(\pi_1^N \otimes Z_n) = \pi_1^N \otimes (A_n Z_n)$.

(v) $A_1 \otimes \ldots \otimes A_{i-1} \otimes A_i \otimes A_{i+1} \otimes \ldots \otimes A_N +$

$$+ A_1 \otimes \dots \otimes A_{i-1} \otimes A_i' \otimes A_{i+1} \otimes \dots \otimes A_N$$

$$= A_1 \otimes A_2 \dots \otimes A_{i-1} \otimes (A_i + A_i') \otimes A_{i+1} \otimes \dots \otimes A_N .$$

(vi) $A_1 \otimes \dots \otimes \alpha A_i \otimes \dots \otimes A_N = \alpha(A_1 \otimes \dots \otimes A_i \otimes \dots \otimes A_N)$.

Proof. We shall prove the lemma in the following order (ii), (iv), (i), (iii) , (v) and (vi).

(ii) $$(\pi_1^N \otimes A_n)(\pi_1^N \otimes A_n') \, \Phi \, (x^{(1)},\dots, x^{(N)})$$

$$= (\pi_1^N \otimes A_n) \, \Psi \, (x^{(1)},\dots, x^{(N)}) \quad \text{(say)}$$

$$= \Psi(A_1^* x^{(1)},\dots,A_N^* x^{(N)}) \qquad \text{(by Definition 4.4.1)}$$

$$= (\pi_1^N \otimes A_n') \, \Phi \, (A_1^* x^{(1)},\dots,A_N^* x^{(N)})$$

$$= \Phi(A_1'^* A_1^* x^{(1)},\dots,A_N'^* A_N^* x^{(N)})$$

$$= \Phi((A_1 A_1')^* x^{(1)},\dots,(A_N A_N')^* x^{(N)}) = \pi_1^N \otimes (A_n A_n') \, \Phi \, (x^{(1)},\dots,x^{(N)})$$

and hence (ii) holds.

(iv) $$(\pi_1^N \otimes A_n)(\pi_1^N \otimes Z_n)(x^{(1)},\dots,x^{(N)})$$

$$= (\pi_1^N \otimes A_n) \, \Phi \, (x^{(1)},\dots,x^{(N)}) \text{ (say)}$$

$$= \Phi(A_1^* x^{(1)},\dots A_N^* x^{(N)})$$

$$= [Z_1, A_1^* x^{(1)}] \dots [Z_N, A_N^* x^{(N)}]$$

$$= [A_1 Z_1, x^{(1)}] \dots [A_N Z_N, x^{(N)}]$$

$$= (\pi_1^N \otimes (A_n Z_n))(x^{(1)},\ldots,x^{(N)}).$$

Hence (iv) holds.

(i) From (ii) and by finite induction,

$$\pi_1^N \otimes A_n = (A_1 \otimes I \otimes \ldots \otimes I)\ (I \otimes A_2 \otimes I \otimes \ldots \otimes I) \ldots$$
$$(I \otimes \ldots \otimes I \otimes A_N).$$

To show that $\pi_1^N \otimes A_n$ is a bounded operator on $\pi_1^N \otimes H_n$, thus it suffices to show that $A_1 \otimes I \ldots \otimes I$ is a bounded operator.

For $\Phi \varepsilon \pi_1^N \otimes H_n$,

$$(A_1 \otimes I \otimes \ldots \otimes I)\Phi\ (x^{(1)},\ldots x^{(N)}) = \Phi\ (A_1^* x^{(1)}, x^{(2)},\ldots x^{(N)}).$$

On the other hand, $\Phi(x^{(1)}, x^{(2)},\ldots,x^{(N)}) = [X_{\Phi(x^{(2)},\ldots,x^{(N)})}, x^{(1)}]$ for fixed vectors $x^{(2)},\ldots,x^{(N)}$, by the Riesz representation theorem. Thus

$$(A_1 \otimes I \otimes \ldots \otimes I)\ \Phi\ (x^{(1)},\ldots,x^{(N)}) = \Phi(A_1^* x^{(1)}, x^{(2)},\ldots,x^{(N)})$$
$$= [X_{\Phi(x^{(2)},\ldots,^{(N)})}, A_1^* x^{(1)}]$$
$$= [A_1 X_{\Phi(x^{(2)},\ldots,x^{(N)})}, x^{(1)}]. \qquad (4.4.2.1)$$

Now, $\|\Phi\|^2 = \Sigma|\Phi(x_{\alpha_1}^{(1)},\ldots,x_{\alpha_N}^{(N)})|^2$, where $(x_{\alpha_i})_{\alpha_i \varepsilon J_i}$ is an orthonomal basis in H_i.

$$\|\Phi\|^2 = \sum_{\alpha_1,\ldots,\alpha_n} |[X_{\Phi(x_{\alpha_2}^{(2)},\ldots,x_{\alpha_N}^{(N)})}, x_{\alpha_1}^{(1)}]|^2. \qquad (4.4.2.2)$$

Thus $\| (A_1 \otimes I \otimes \ldots \otimes I)\Phi \|^2 = \sum_{\alpha_2,\ldots,\alpha_N} \sum_{\alpha_1} | [A_1 X_{\Phi(x^{(2)}_{\alpha_2},\ldots,x^{(N)}_{\alpha_N})}, x^{(1)}_{\alpha_1}] |^2$

$$= \sum_{\alpha_2,\ldots,\alpha_N} \| A_1 X_{\Phi(x^{(2)}_{\alpha_2},\ldots,x^{(N)}_{\alpha_N})} \|^2$$

$$\leq \| A_1 \|^2 \sum_{\alpha_2,\ldots,\alpha_N} \| X_{\Phi(x^{(2)}_{\alpha_2},\ldots,x^{(N)}_{\alpha_N})} \|^2$$

$$= \| A_1 \|^2 \, \| \Phi \|^2 \text{ by } (4.4.2.2).$$

Hence $(A_1 \otimes I \times \ldots \otimes I)\,\Phi$ is of Hilbert class and $\| A_1 \otimes I \times \ldots \otimes I \| \leq \| A_1 \|$. Thus

$$\| \prod_1^N \otimes A_n \| \leq \prod_1^N \| A_n \|. \qquad (4.4.2.3)$$

To prove the reverse inequality, let $o < \varepsilon < 1$. Let $x_n \varepsilon H_n$ be a unit vector such that $\| A_n x_n \| \geq (1 - \varepsilon) \| A_n \|$.

Then by Lemma 4.3.6 and (iv)

$$\| \prod_1^N \otimes A_n)(\prod_1^N \otimes x_n) \| = \| \prod_1^N \otimes A_n x_n \|$$

$$= \prod_1^N \| A_n x_n \|$$

$$\geq (1 - \varepsilon)^N \prod_1^N \| A_n \|$$

while $\prod_1^N \otimes x_n$ is a unit vector. This shows that

$$\| \prod_1^N \otimes A_n \| \geq \prod_1^N \| A_n \|. \qquad (4.4.2.3)$$

Then by (4.4.2.3) and(4.4.2.4), $\| \prod_1^N \otimes A_n \| = \prod_1^N \| A_n \|$.

(iii) If $(A_1 \otimes I \otimes .. \otimes ... \otimes I)^* = (A_1^* \otimes I \otimes ... \otimes I)$, etc., then

$$(A_1 \otimes \cdots \otimes A_N)^* = (\prod_1^N (I \otimes \cdots \otimes A_n \otimes I \otimes \cdots \otimes I)^*$$

$$= \prod_1^N (I \otimes \cdots \otimes A_n^* \otimes I \otimes \cdots \otimes I)$$

$$= A_1^* \otimes A_2^* \otimes \cdots \otimes A_N^*.$$

Thus it suffices to show that

$$(A_1 \otimes I \otimes \cdots \otimes I)^* = A_1^* \otimes I \otimes \cdots \otimes I.$$

Now, for Φ, Ψ in $\prod_1^N \otimes H_n$, and $A = A_1 \otimes I \otimes \cdots \otimes I$ we have

$$[A\Phi,\Psi] = \sum_{\alpha_1,\dots,\alpha_N} (A\Phi)(x_{\alpha_1}^{(1)},\dots,x_{\alpha_N}^{(N)}) \overline{\Psi(x_{\alpha_1}^{(1)},\dots,x_{\alpha_N}^{(N)})}$$

$$= \sum_{\alpha_2,\dots,\alpha_N} \sum_{\alpha_1} [X_{A\Phi}(x_{\alpha_2}^{(2)},\dots,x_{\alpha_N}^{(N)}), x_{\alpha_1}^{(1)}] \overline{[X_\Psi(x_{\alpha_2}^{(2)},\dots,x_{\alpha_N}^{(N)}), x_{\alpha_1}^{(1)}]}$$

(in the notation of the proof of (i) above)

$$= \sum_{\alpha_2,\dots,\alpha_N} [X_{A\Phi}(x_{\alpha_2}^{(2)},\dots,x_{\alpha_N}^{(N)}), X_\Psi(x_{\alpha_2}^{(2)},\dots,x_{\alpha_N}^{(N)})]$$

$$= \sum_{\alpha_2,\dots,\alpha_N} [A_1 X_\Phi(x_{\alpha_2}^{(2)},\dots x_{\alpha_N}^{(N)}), X_\Psi(x_{\alpha_2}^{(2)},\dots,x_{\alpha_N}^{(N)})] \text{ (by (4.4.2.1))}$$

$$= \sum_{\alpha_2,\dots,\alpha_N} [X_\Phi(x_{\alpha_2}^{(2)},\dots,x_{\alpha_N}^{(N)}), A_1^* X_\Psi(x_{\alpha_2}^{(2)},\dots,x_{\alpha_N}^{(N)})]$$

$$= \sum_{\alpha_1,\dots,\alpha_N} [X_\Phi(x_{\alpha_2}^{(2)},\dots,x_{\alpha_N}^{(N)}), x_{\alpha_1}^{(1)}][x_{\alpha_1}^{(1)}, A_1^* X_\Psi(x_{\alpha_2}^{(2)},\dots,x_{\alpha_N}^{(N)})]$$

$$= \sum_{\alpha_1,\dots,\alpha_N} \Phi(x_{\alpha_1}^{(1)}, \dots x_{\alpha_N}^{(N)}) \overline{\Psi(A_1 x_{\alpha_1}^{(1)}, x_{\alpha_2}^{(2)},\dots,x_{\alpha_N}^{(N)})}$$

$$= [\Phi, (A_1^* \otimes I \otimes \cdots \otimes I)\Psi].$$

Thus $(A_1 \otimes I \otimes \ldots \otimes I)^* = A_1^* \otimes I \otimes \ldots \otimes I.$

(v) $\{(A_1 \otimes \ldots \otimes A_N) + (A_1 \otimes \ldots \otimes A_i' \otimes \ldots \otimes A_N)\}\Phi(x^{(1)}, \ldots x^{(N)})$

$= \Phi(A_1^* x^{(1)}, \ldots (A_i^* + A_i'^*)x^{(i)}, \ldots, A_N^* x^N)$ (since Φ is additive in each variable)

$= (A_1 \otimes \ldots \otimes (A_i + A_i') \otimes \ldots \otimes A_N)\ \Phi(x^{(1)}, \ldots, x^{(N)}).$

Hence (v) holds.

(vi) We have to make use of 4.4.1 and the fact that each $\Phi \in \overset{N}{\underset{I}{\pi}} \otimes H_n$ is conjugate-multilinear.

This completes the proof of the lemma.

Lemma 4.4.3. Let H_n, $(1 \leqq n \leqq N)$, be Hilbert spaces and let $\{x_{\alpha_n}^{(n)}\}_{\alpha_n \in J_n}$ be an orthonormal basis for H_n. Then the set $\{x_{\alpha_1}^{(1)} \otimes \ldots \otimes x_{\alpha_N}^{(N)}\}_{\alpha_j \in J_j,\ 1 \leqq j \leqq N}$ of vectors is an orthonormal basis for $\overset{N}{\underset{1}{\pi}} \otimes H_n$. In particular, $\overset{N}{\underset{1}{\pi}} \otimes H_n$ is separable if each H_n is separable; $\overset{N}{\underset{1}{\otimes}} H_n = [x^{(1)} \otimes \ldots \otimes x^{(N)} : x^{(i)} \in H_i,\ 1 \leqq i \leqq N]$.

Proof. $[x_{\alpha_1}^{(1)} \otimes \ldots \otimes x_{\alpha_N}^{(N)},\ x_{\alpha_1'}^{(1)} \otimes \ldots \otimes x_{\alpha_N'}^{(N)}]$

$= [x_{\alpha_1}^{(1)}, x_{\alpha_1'}^{(1)}] \ldots [x_{\alpha_N}^{(N)}, x_{\alpha_N'}^{(N)}]$

$= 0$ if $\alpha_i^{(i)} \neq \alpha_i'^{(i)}$, $(i = 1, 2 \ldots, N)$,

and $\qquad \| x_{\alpha_1}^{(1)} \otimes \ldots \otimes x_{\alpha_N}^{(N)} \| = \overset{N}{\underset{1}{\pi}} \| x_{\alpha_i}^{(i)} \| = 1.$

Thus these vectors form an orthonormal system. We have to prove completeness. If it is not complete, then there is a non-zero vector $\Phi \in \overset{N}{\underset{1}{\pi}} \otimes H_n$ such that $[\Phi, x^{(1)}_{\alpha_1} \otimes \ldots \otimes x^{(N)}_{\alpha_N}] = 0$ for all $\alpha_1,\ldots,\alpha_N$. (4.4.3.1)

But,

$$[\Phi, x^{(1)}_{\alpha_1} \otimes \ldots \otimes x^{(N)}_{\alpha_N}]$$

$$= \sum_{\alpha_1',\ldots,\alpha_N'} \Phi(x^{(1)}_{\alpha_1'},\ldots,x^{(N)}_{\alpha_N'}) \overline{(x^{(1)}_{\alpha_1} \otimes \ldots \otimes x^{(N)}_{\alpha_N}) \, (x^{(1)}_{\alpha_1'},\ldots,x^{(N)}_{\alpha_N'})}$$

$$= \sum_{\alpha_1',\ldots,\alpha_N'} \Phi(x^{(1)}_{\alpha_1'},\ldots,x^{(N)}_{\alpha_N'}) \prod_1^N [x^{(i)}_{\alpha_i'}, x^{(i)}_{\alpha_i}]$$

$$= \Phi\,(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N}). \qquad (4.4.3.2)$$

Now (4.4.3.1) and (4.4.3.2) together imply that

$$\|\Phi\|^2 = \sum_{\alpha_1,\ldots,\alpha_N} |\Phi(x^{(1)}_{\alpha_1},\ldots,x^{(N)}_{\alpha_N})|^2 = 0; \quad \text{i.e.}\ \Phi = 0.$$

Definition 4.4.4. Let H_n, $(1 \leqq n \leqq N)$, be Hilbert spaces and $\mathcal{R}_n$, $(1 \leqq n \leqq N)$, von Neumann algebras on H_n, respectively. The smallest von Neumann algebra of operators on $\prod_1^N \otimes H_n$ containing the set $\mathfrak{M} = \{A_1 \otimes A_2 \otimes \ldots \otimes A_N: A_i \varepsilon \mathcal{R}_i, i=1,1,\ldots,N\}$ of operators will be denoted by $\prod_1^N \otimes \mathcal{R}_i$ or, by the symbol, $\mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \ldots \otimes \mathcal{R}_N$ and will be called the tensor product of the von Neumann algebras $\mathcal{R}_1,\ldots,\mathcal{R}_N$. Since the algebra generated by $\mathfrak{M}$ is a *-subalgebra of $\mathcal{B}(\prod_1^N \otimes H_n)$ with identity by 4.4.2, it is obvious that $\prod_1^N \otimes \mathcal{R}_i = \mathfrak{M}'' =$ weak closure of $\mathfrak{M}$. (See 2.3.11.)

Lemma 4.4.5. Let H_n be a Hilbert space and $\mathcal{B}_n = B(H_n)$ for $1 \leqq n \leqq N$. Then $\mathcal{B} = \prod_1^N \otimes \mathcal{B}_n$ is the algebra of all bounded operators on $H = \prod_1^N \otimes H_n$.

Proof. Since $\mathcal{B}$ is a von Neumann algebra, if we show that $\mathcal{B}' = \mathbb{C}$, then $\mathcal{B} = \mathcal{B}'' = B(H)$, as the commutant of $\mathbb{C}$ is $B(H)$. Suppose $T \in \mathcal{B}'$. Let $\{x^{(n)}_{\alpha_n}\}_{\alpha_n \in J_n}$ be an orthonormal basis of H_n, $(1 \leq n \leq N)$. Let $m_1,\ldots,m_N$ and $k_1,\ldots,k_N$ be two given N-*tuples* of elements from the indexing sets of the orthonormal bases. Let $A_n \in B(H_n)$ such that $A_n x^{(n)}_{m_n} = x^{(n)}_{m_n}$, $A_n x^{(n)}_{\alpha_n} = 0$ if $\alpha_n \neq m_n$, $(1 \leq n \leq N)$.

Let $B_n \in B(H_n)$ such that $B_n x^{(n)}_{m_n} = x^{(n)}_{k_n}$, $B_n x^{(n)}_{\alpha_n} = 0$ if $\alpha_n \neq m_n$, $(1 \leq n \leq N)$.

Then T commutes with $A_1 \otimes \ldots \otimes A_N$ and $B_1 \otimes \ldots \otimes B_n$.

Now,
$$(A_1 \otimes \ldots \otimes A_N)(x^{(1)}_{m_1} \otimes \ldots \otimes x^{(N)}_{m_N}) = A_1 x^{(1)}_{m_1} \otimes \ldots \otimes A_N x^{(N)}_{m_N}$$
$$= x^{(1)}_{m_1} \otimes \ldots \otimes x^{(N)}_{m_N}$$

and $A_1 \otimes \ldots \otimes A_N$ maps all the other vectors of the basis $x^{(1)}_{\alpha_1} \otimes \ldots \otimes x^{(N)}_{\alpha_N}$ onto zero. Thus
$$(A_1 \otimes \ldots \otimes A_N)\, T\, (x^{(1)}_{m_1} \otimes \ldots \otimes x^{(N)}_{m_N})$$
$$= T(A_1 \otimes \ldots \otimes A_N)(x^{(1)}_{m_1} \otimes \ldots \otimes x^{(N)}_{m_N})$$
$$= T(x^{(1)}_{m_1} \otimes \ldots \otimes x^{(N)}_{m_N})$$

and $(A_1 \otimes \ldots \otimes A_N)\, T\, (x^{(1)}_{\alpha_1} \otimes \ldots \otimes x^{(N)}_{\alpha_N}) = 0$ if $\alpha_i \neq m_i$, $(1 \leq i \leq N)$.

If $T(x^{(1)}_{m_1} \otimes \ldots \otimes x^{(N)}_{m_N}) = \Sigma\, C_{\alpha_1,\ldots,\alpha_N}\, x^{(1)}_{\alpha_1} \otimes \ldots \otimes x^{(N)}_{\alpha_N}$, then
$$T(x^{(1)}_{m_1} \otimes \ldots \otimes x^{(N)}_{m_N}) = (A_1 \otimes \ldots \otimes A_N)(\Sigma\, C_{\alpha_1,\ldots,\alpha_N}\, x^{(1)}_{\alpha_1} \otimes \ldots \otimes x^{(N)}_{\alpha_N}) =$$
$$= C_{m_1,\ldots,m_N}\, x^{(1)}_{m_1} \otimes \ldots \otimes x^{(N)}_{m_N}.$$

Thus
$$T(x^{(1)}_{m_1} \otimes \ldots \otimes x^{(N)}_{m_N}) = C_{m_1,\ldots,m_N}\, x^{(1)}_{m_1} \otimes \ldots \otimes x^{(N)}_{m_N}. \qquad (4.4.5.1)$$

Similarly, we have in general

$$T(x^{(1)}_{\alpha_1} \otimes \cdots \otimes x^{(N)}_{\alpha_N}) = C_{\alpha_1,\ldots,\alpha_N} x^{(1)}_{\alpha_1} \otimes \cdots \otimes x^{(N)}_{\alpha_N}. \quad (4.4.5.2)$$

Consequently, by (4.4.5.1)

$$(B_1 \otimes \cdots \otimes B_N)\, T\, (x^{(1)}_{m_1} \otimes \cdots \otimes x^{(N)}_{m_N})$$
$$= C_{m_1,\ldots,m_N} x^{(1)}_{k_1} \otimes \cdots \otimes x^{N}_{k_n}.$$

But, $(B_1 \otimes \cdots \otimes B_N)T = T(B_1 \otimes \cdots \otimes B_N)$ and hence

$$(B_1 \otimes \cdots \otimes B_N)\, T\, (x^{(1)}_{m_1} \otimes \cdots \otimes x^{(N)}_{m_N}) = T(B_1 \otimes \cdots \otimes B_N)(x^{(1)}_{m_1} \otimes \cdots \otimes x^{(N)}_{m_N})$$
$$= T(x^{(1)}_{k_1} \otimes \cdots \otimes x^{(N)}_{k_N})$$
$$= C_{k_1,\ldots,k_N} x^{(1)}_{k_1} \otimes \cdots \otimes x^{(N)}_{k_n}$$

by (4.4.5.2).

Thus $C_{m_1,\ldots,m_N} = C_{k_1,\ldots,k_N}$ and this holds for all $m_i \in J_i$, $k_i \in J_i$ ($i = 1,2\ldots,N$) and hence the constants $C_{\alpha_1,\ldots,\alpha_N}$ = some C, so that $T = CI$.

This completes the proof.

Lemma 4.4.6. For each n, ($1 \leqq n \leqq N$), let H_n be a Hilbert space; let $\mathcal{B}_n^{(o)}$ be dense in $\mathcal{B}_n$ in the weak operator topology for $1 \leqq n \leqq N$. Let T be an operator on $H = \pi_1^N \otimes H_n$ such that T commutes with each operator of the form $I \otimes \cdots \otimes I \otimes A_n^{(o)} \otimes I \cdots \otimes I$, $A_n^{(o)} \in \mathcal{B}_n^{(o)}$, ($1 \leqq n \leqq N$). Then T commutes with all operators of the form $I \otimes \cdots \otimes A_n \otimes I \otimes \cdots \otimes I$ with $A_n \in \mathcal{B}_n$. Consequently, $T \in (\pi_1^N \otimes \mathcal{B}_n)'$.

Proof. We suppose, for simplicity of notation, that n = 1. The details of the remaining cases follow exactly on similar lines.

Let $x^{(n)}, y^{(n)}$ be in H_n, $(1 \leq n \leq N)$. If $x_k^{(1)} \to x^{(1)}$ in H_1, then

$$\|(x_k^{(1)} - x^{(1)}) \otimes x^{(2)} \dots \otimes x^{(N)}\|$$

$$= \|x_k^{(1)} - x^{(1)}\| \ \|x^{(2)} \otimes \dots \otimes x^{(N)}\| \to 0 \text{ as } k \to \infty .$$

Hence $\|T(x_k^{(1)} \otimes \dots \otimes x^{(N)}) - T(x^{(1)} \otimes \dots \otimes x^{(N)})\|$

$$\leq \|T\| \ \|x_k^{(1)} - x^{(1)}\| \dots \|x^{(N)}\| \to 0 \tag{4.4.6.1}$$

as $k \to \infty$. Then $[T(x_k^{(1)} \otimes \dots \otimes x^N), y_k^{(1)} \otimes \dots \otimes y^{(N)}] \to [T(x^{(1)} \otimes \dots \otimes x^N), y^{(1)} \otimes \dots \otimes y^{(N)}]$ if $x_k^{(1)} \to x^{(1)}$, $y_k^{(1)} \to y^{(1)}$, since $\sup_k \|y_k^{(1)}\| < \infty$.

Also by (4.4.6.1) there exists a bounded operator $S(x^{(2)},\dots,x^{(N)}; y^{(2)},\dots, y^{(N)})$ on H_1 such that

$$[T(x^{(1)} \otimes \dots \otimes x^{(N)}), y^{(1)} \otimes \dots \otimes y^{(N)}]$$

$$= [S(x^{(2)},\dots x^{(N)}; y^{(2)},\dots y^{(N)}) \, x^{(1)}, y^{(1)}]. \tag{4.4.6.2}$$

Since $T(A_1 \otimes I \otimes \dots \otimes I) = (A_1 \otimes I \otimes \dots \otimes I)T$ for each $A_1 \in \mathcal{B}_1^{(o)}$,

$$[T(A_1 \otimes I \otimes \dots \otimes I)(x^{(1)} \otimes \dots \otimes x^{(N)}), y^{(1)} \otimes \dots \otimes y^{(N)}]$$

$$= [S(x^{(2)},\dots x^{(N)}; y^{(2)},\dots,y^{(N)})A_1 \, x^{(1)}, y^{(1)}]$$

$$= [T(x^{(1)} \otimes \dots \otimes x^{(N)}), (A_1^* y^{(1)} \otimes y^{(2)} \otimes \dots \otimes y^{(N)}]$$

$$= [S(x^{(2)},\dots,x^{(N)}; y^{(2)},\dots,y^{(N)}) \, x^{(1)}, A_1^* y^{(1)}]$$

$$= [A_1 \, S(x^{(2)},\dots,x^{(N)}; y^{(2)},\dots,y^{(N)}) \, x^{(1)}, y^{(1)}].$$

Thus $[S(x^{(2)},\dots,x^{(N)}; y^{(2)},\dots,y^{(N)}) \, A_1 \, x^{(1)}, y^{(1)}]$

$= [A_1 \, S(x^{(2)},\ldots,x^{(N)}; y^{(2)}\ldots,y^{(N)}) \, x^{(1)}, y^{(1)}]$ for all $x^{(1)}, y^{(1)}$ in H_1. Hence

$S(x^{(2)},\ldots,x^{(N)}; y^{(2)},\ldots,y^{(N)})A_1 = A_1 \, S(x^{(2)},\ldots,x^{(N)}; y^{(2)},\ldots,y^{(N)})$ for all $A_1 \, \varepsilon \mathcal{B}_1^{(o)}$. For $A \varepsilon \mathcal{B}_1$, let $A = \lim_\alpha A_\alpha$ (weakly), $A_\alpha \, \varepsilon \, \mathcal{B}_1^{(o)}$. Then

$$[A \, S(x^{(2)},\ldots x^{(N)}; y^{(2)},\ldots y^{(N)}) \, x^{(1)}, y^{(1)}]$$

$$= \lim_\alpha [A_\alpha \, S(x^{(2)},\ldots,x^{(N)}; y^{(2)},\ldots y^{(N)}) \, x^{(1)}, y^{(1)}]$$

$$= \lim_\alpha [S(x^{(2)},\ldots,x^{(N)}; y^{(2)},\ldots,y^{(N)}) \, A_\alpha \, x^{(1)}, y^{(1)}]$$

$$= [S(x^{(2)},\ldots,x^{(N)}; y^{(2)},\ldots,y^{(N)}) \, Ax^{(1)}, y^{(1)}]$$

for all $x^{(1)}, y^{(1)}$ in H_1. Hence

$$AS(x^{(2)},\ldots,x^{(N)}; y^{(2)},\ldots,y^{(N)}) = S(x^{(2)},\ldots,x^{(N)}; y^{(2)},\ldots,y^{(N)})A \quad (4.4.6.3)$$

for all $A \, \varepsilon \, \mathcal{B}_1$.

Using (4.4.6.2), again for $A \varepsilon \, \mathcal{B}_1$,

$$[T(A \otimes I \otimes \ldots \otimes I)(x^{(1)} \otimes \ldots \otimes x^{(N)}), y^{(1)} \otimes y^{(2)} \otimes \ldots \otimes y^{(N)}]$$

$$= [T(Ax^{(1)} \otimes x^{(2)} \otimes \ldots \otimes x^{(N)}), y^{(1)} \otimes y^{(2)} \otimes \ldots \otimes y^{(N)}]$$

$$= [S(x^{(2)},\ldots,x^{(N)}; y^{(2)},\ldots,y^{(N)}) \, Ax^{(1)}, y^{(1)}]$$

$$= [S(x^{(2)},\ldots,x^{(N)}; y^{(2)},\ldots,y^{(N)}) \, x^{(1)}, A^* y^{(1)}] \quad \text{(by 4.4.6.3)}$$

$$= [T(x^{(1)} \otimes x^{(2)} \ldots \otimes x^{(N)}), \; A^* y^{(1)} \otimes y^{(2)} \otimes \ldots \otimes y^{(N)}]$$

$$= [(A \otimes I \otimes \ldots \otimes I)T(x^{(1)} \otimes \ldots \otimes x^{(N)}), y^{(1)} \otimes \ldots \otimes y^{(N)}].$$

Since $\{ x^{(1)}_{\alpha_1} \otimes \cdots \otimes x^{(N)}_{\alpha_N} \}$, $\alpha_i \in \mathcal{J}_i$, $(1 \leq i \leq N)$, is an orthonormal basis for $\overset{N}{\underset{1}{\pi}} \otimes H_n$, the above equality implies that

$$T(A \otimes I \otimes \cdots \otimes I) = (A \otimes I \otimes \cdots \otimes I)T$$

for all $A \in \mathcal{B}_1$.

Now, by Lemma 4.4.2 (ii) and by the above part,

$$\begin{aligned} T(A_1 \otimes \cdots \otimes A_N) &= T(\overset{N}{\underset{1}{\pi}} (I \otimes I \otimes \cdots \otimes A_n \otimes \cdots \otimes I)) \\ &= \overset{N}{\underset{1}{\pi}} (I \otimes \cdots \otimes A_n \otimes \cdots \otimes I)T \\ &= (A_1 \otimes \cdots \otimes A_N)T, \ A_i \in \mathcal{B}_i, \end{aligned}$$

and hence T commutes with the von Neumann algebra $\overset{N}{\underset{i=1}{\pi}} \otimes \mathcal{B}_i$. (See Definition 4.4.4.) Thus $T \in (\overset{N}{\underset{i=1}{\pi}} \otimes \mathcal{B}_i)'$.

Corollary 4.4.7. Let H_i, $(1 \leq i \leq N)$, be Hilbert spaces; $\mathcal{R}_i$, $(1 \leq i \leq N)$, be von Neumann algebras of operators on H_i and let M_i generate the von Neumann algebra $\mathcal{R}_i$. If $\mathcal{R}$ is the von Neumann algebra generated by the operators of the form $T_1 \otimes I \cdots \otimes I, \ldots, I \otimes \cdots \otimes I \otimes \cdots \otimes \cdots \otimes T_N$ with $T_i \in M_i$, then

$$\mathcal{R} = \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_N.$$

Proof. Let N_i be the *-algebra generated by $M_i \cup M_i^*$ with identity, $(1 \leq i \leq N)$. Then $\mathcal{R}_i$ is the closure of N_i in the weak operator topology by Corollary 2.3.11. Now $I \otimes \cdots \otimes T_i \otimes \cdots \otimes I \in \mathcal{R}$, $T_i \in N_i$, $(1 \leq i \leq N)$.

Let S be the von Neumann algebra generated by

$$\Sigma = \{ I \otimes \cdots \otimes T_i \otimes \cdots \otimes I, T_i \in N_i, 1 \leq i \leq N \}.$$

Since N_i is dense in $\mathcal{R}_i$ in the weak operator topology, by the above lemma and by the double commutant theorem, $S = \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_N$. Thus $\mathcal{R} \supset \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_N$, as $\mathcal{R}$ contains Σ obviously. That $\mathcal{R} \subset \mathcal{R}_1 \otimes \cdots \otimes \mathcal{R}_N$ is clear. Hence the corollary.

Theorem 4.4.8. Let $\mathcal{R}_n$ be a von Neumann algebra on H_n, $(1 \leq n \leq N)$, H_n a Hilbert space. Then $\mathcal{R} = \prod_1^N \otimes \mathcal{R}_n$ is a factor if and only if each $\mathcal{R}_n$ is a factor.

Proof. If $B_1 \in \mathcal{R}_1$ belongs to the centre Z_1 of $\mathcal{R}_1$, then, by Lemma 4.4.2 (ii), $B_1 \otimes I \otimes \cdots \otimes I = B'$ and its adjoint commute with every operator of the form $I \otimes \cdots \otimes A_i \otimes \cdots \otimes I, \ldots, A_1 \in \mathcal{R}_i$, $1 \leq i \leq N$, and hence with the von Neumann algebra $\mathcal{R}$. Thus B' and B'^* belong to the centre Z of $\mathcal{R}$. Hence, if $\mathcal{R}_1$ is not a factor, then $\mathcal{R}$ is not a factor. Similarly, if $\mathcal{R}_i$ is not a factor for some i, $(1 \leq i \leq N)$, then $\mathcal{R}$ is not a factor. Thus $\mathcal{R}$ is a factor only if each $\mathcal{R}_n$, $(1 \leq n \leq N)$, is a factor.

Suppose, conversely, each $\mathcal{R}_n$ is a factor. Let T be an operator on $\prod_1^N \otimes H_n$, lying in Z, the centre of $\mathcal{R}$. Then T commutes with every operator in $\mathcal{R}$ and every operator in $\mathcal{R}'$. Thus, in particular, T commutes with every operator of the form

$$I \otimes \cdots \otimes A_i \otimes \cdots \otimes \cdots \otimes I, \quad A_i \in \mathcal{R}_i,$$

$$I \otimes \cdots \otimes A_i' \otimes \cdots \otimes I, \qquad A_i' \in \mathcal{R}_i', \ (1 \leq i \leq N).$$

Let $\tilde{\mathcal{R}}_i$ be the *-algebra of all linear combinations of products $A'_i A_i$, $A_i \in \mathcal{R}_i$, $A_i' \in \mathcal{R}_i'$. Then T commutes with all operators of the form $I \otimes \cdots \otimes A_i \otimes \cdots \otimes I$, for all $A_i \in \tilde{\mathcal{R}}_i$. The weak closure of $\tilde{\mathcal{R}}_i$ is a von Neumann algebra on H_i and is $\tilde{\mathcal{R}}_i''$. Hence $T \in (\prod_1^N \otimes \tilde{\mathcal{R}}_i'')'$, by Lemma 4.4.6.

If $S \in \tilde{R}_i'$, then S commutes with each $A_i \in R_i$ and each $A_i' \in R_i'$. Thus $S \in R_i \cap R_i'$ so that $\tilde{R}_i' \subset R_i \cap R_i'$. Clearly, $R_i \cap R_i' \subset \tilde{R}_i'$. Thus $\tilde{R}_i' = R_i \cap R_i'$ so that $\tilde{R}_i'' = (R_i \cap R_i')' = (\mathbb{C}I)' = B(H_i)$. Hence by Lemma 4.4.5, $\overset{N}{\underset{1}{\pi}} \otimes \tilde{R}_i'' = B(\overset{N}{\underset{1}{\pi}} \otimes H_i)$.

Thus $T \in B(\overset{N}{\underset{1}{\pi}} \otimes H_i)' = \mathbb{C}I$ so that the centre Z of $R = \mathbb{C}I$. Hence R is a factor.

Corollary 4.4.9. If Z_i is the centre of the von Neumann algebra R_i, $(1 \leqq i \leqq N)$, on H_i, and if Z is the centre of $= \overset{N}{\underset{i}{\pi}} \otimes R_i$, then

$$Z_1 \otimes \dots \otimes Z_N \subset Z.$$

If $Z = \mathbb{C}I$, then equality holds.

Note: The equality holds always, without any restriction on R_i; i. e., $\overset{N}{\underset{1}{\pi}} \otimes Z_i = Z$. This will be proved later in §5.9. (See Corollary 5.9.11)

Lemma 4.4.10. Let R_i be von Neumann algebras of operators on Hilbert spaces H_i for $1 \leqq i \leqq N$. Let $R = R_1 \otimes \dots \otimes R_N$. Then R is countably decomposable if and only if each R_i, $(1 \leqq i \leqq N)$, is countably decomposable.

Proof. Let each R_i be countably decomposable. Then R_i' has a countable generating set X_i, $(i = 1,2,\dots,N)$, by Lemma 3.3.9. Let $X_i = \{x_n^{(i)}\}_{n=1}^{\infty}$. Consider $Y = \{x_{n_1}^{(1)} \otimes \dots \otimes x_{n_N}^{(N)}\}$ where $n_1,\dots,n_N$ are positive integers. Then, as R_i' has X_i as a generating set, the closed subspace $K =$

$$[A_1' x_{n_1}^{(1)} \otimes A_2' x_{n_2}^{(2)} \otimes \dots \otimes A_N' x_{n_N}^{(N)} : x_{n_i}^{(i)} \varepsilon X_i, 1 \leqq i \leqq N]$$

$= \overset{N}{\underset{1}{\pi}} \otimes H_i$. Otherwise, there exists a vector $x^{(1)} \otimes \dots \otimes x^{(N)} \neq 0$ in $\overset{N}{\underset{1}{\pi}} \otimes H_i$ orthogonal to K. Then, for $x_{n_i}^{(i)} \varepsilon X_i$, $(1 \leqq i \leqq N)$, and $A_i' \varepsilon R_i'$,

$$0 = [A_1' x_{n_1}^{(1)} \otimes \dots \otimes A_N' x_{n_N}^{(N)}, x^{(1)} \otimes \dots \otimes x^{(N)}]$$

$$= [A_1' x_{n_1}^{(1)}, x^{(1)}] \dots [A_N' x_{n_N}^{(N)}, x^{(N)}].$$

Consequently, as $[R_i' X_i] = H_i$, for i= 1,2,...,N, we have

$$0 = [y^{(1)}, x^{(1)}] \dots [y^{(N)}, x^{(N)}] \text{ for arbitrary } y^{(i)} \varepsilon H_i.$$

Then $\quad 0 = \|x^{(1)}\|^2 \dots \|x^{(N)}\|^2$, taking $y^{(i)} = x^{(i)}$.

Thus $\quad x^{(1)} \otimes \dots \otimes x^{(N)} = 0$, a contradiction.

Thus $R_1' \otimes \dots \otimes R_N'$ has Y as a generating set which is utmost countable. But, as $R' \supset R_1' \otimes \dots \otimes R_N'$, R is countably decomposable. (Indeed $R' = R_1' \otimes \dots \otimes R_N'$. This will be proved later in §5.9).

Conversly, let $R = R_1 \otimes \dots \otimes R_N$ be countable decomposable. Let $(E_\alpha)_{\alpha \varepsilon A_i}$ be an orthogonal family of projections in R_i. Then it is easy to verify that $\{ I \otimes \dots \otimes E_\alpha \otimes \dots \otimes I : \alpha \varepsilon A_i \}$ is an orthogonal family of projections in R and hence by hypothesis on R, A_i is at most countable. Hence R_i, $(1 \leqq i \leqq N)$, are coubtably decomposable.

4.5. Matrix representation for operators on $H_1 \otimes H_2$.

Lemma 4.5.1. Let H_1 and H_2 be two Hilbert spaces with $(e_i)_{i \varepsilon J}$ an orthonormal basis in H_2. Then $H_1 \otimes H_2$ is isomorphic to $\underset{J}{\Sigma} \oplus H_1$. Conversely, if H =

$\sum_{i \varepsilon J} \oplus H_i$, where H_i are closed subspaces of H which are pairwise orthogonal and each of which is isomorphic to some fixed Hilbert space H_1, then H is isomorphic to $H_1 \otimes H_2$, where $H_2 = L^2_{\mathbb{C}}(J) = \{(\lambda_i)_{i \varepsilon J}, \lambda_i \varepsilon \mathbb{C}, \sum_{i \varepsilon J} |\lambda_i|^2 < \infty\}$.

Proof. Since $(e_i)_{i \varepsilon J}$ is an orthonormal basis in H_2, obviously H_2 is isomorphic to $L^2_{\mathbb{C}}(J)$. Define $U_i : H_1 \to H_1 \otimes H_2$ by $U_i(x) = x \otimes e_i$. Then clearly, U_i is linear by (iii) and (iv) of Lemma 4.3.6. Further, U_i is inner-product preserving, as

$$[U_i x, U_i y] = [x \otimes e_i, y \otimes e_i] = [x,y]\,[e_i, e_i] = [x,y]$$

since $\|e_i\| = 1$. Thus U_i is an isomorphism of H_1 onto $U_i H_1$. Being U_i isometric and H_1 complete, $U_i H_1$ is a closed subspace of $H_1 \otimes H_2$. Call $U_i H_1 = H_i$. Since $[U_i x, U_j y] = [x \otimes e_i, y \otimes e_j] = [x,y][e_i, e_j] = 0$ if $i \neq j$, $\{H_i\}_{i \varepsilon J}$ is an orthogonal family of closed subspaces in $H_1 \otimes H_2$ and each of them is isomorphic to H_1. Let $K = \sum_{i \varepsilon J} \oplus H_i$. If $K \neq H_1 \otimes H_2$, let $z \neq 0$ be an element in $H_1 \otimes H_2$, orthogonal to K. If $\{x_\alpha\}_{\alpha \varepsilon A}$ is an orthonormal basis in H_1, then by hypothesis, $z \perp \{x_\alpha \otimes e_i\}_{(\alpha, i) \varepsilon A \times J}$. But by Lemma 4.4.3, this means $z = 0$, a contradiction. Thus $K = H_1 \otimes H_2$.

Now define the mapping U as follows:

$$U: \sum_J \oplus H_1 \to \sum_{i \varepsilon J} \oplus H_i$$

$$U\{(x_i)_{i \varepsilon J}\} = \sum_{i \varepsilon J} x_i \otimes e_i, \quad x_i \varepsilon H_1 \text{ for } i \varepsilon J.$$

Then, clearly, U is a linear isometry and onto. Hence U is an isomorphism and thus $H_1 \otimes H_2$ is isomorphic to $\sum_J \oplus H_1$.

Conversely, if $H = \sum_{i \varepsilon J} \oplus H_i$, let $x = \sum_{i \varepsilon J} x_i$, $x_i \varepsilon H_i$, be the unique representation of $x \varepsilon H$. If U_i is the isomorphism of H_i onto H_1, then define U: $H \to H_1 \otimes H_2$ by $Ux = \sum_{i \varepsilon J} (U_i x_i) \otimes e_i$, $\{e_i\}_{i \varepsilon J}$ an orthonormal basis in $H_2 = L^2_{\mathbb{C}}$ (J). Since $\sum_{i \varepsilon J} \|(U_i x_i) \otimes e_i\|^2 = \sum_{i \varepsilon J} \|x_i\|^2 < \infty$, $\sum_{i \varepsilon J} (U_i x_i) \otimes e_i \; \varepsilon H_1 \otimes H_2$ and $\|Ux\|^2 = \|\sum_{i \varepsilon J} (U_i x_i) \otimes e_i\|^2 = \sum_{i \varepsilon J} \|x_i\|^2 = \|x\|^2$, since $[e_i, e_j] = \delta_{i,j}$. Clearly, U is linear. U is onto. For, if $z \perp U(H)$, and if $(x_\alpha)_{\alpha \varepsilon A}$ is an orthonormal basis in H_1, then $x_\alpha \otimes e_i = U(0, \ldots, U_i^{-1} x_\alpha, 0 \ldots)$ as U_i is onto for each i. Thus $z \perp (x_\alpha \times e_i)_{(\alpha, i) \varepsilon A \times J}$ and hence z = 0 by Lemma 4.4.3. Thus U is an isomorphism of H onto $H_1 \otimes H_2$, where $H_2 = L^2_{\mathbb{C}}(J)$.

This completes the proof of the lemma.

Notation. Throughout this section and the next one the following notation is used. $H = H_1 \otimes H_2 = \sum_{i \varepsilon J} \oplus H_i$ (identified by the isomorphism in Lemma 4.5.1) where each H_i is isomorphic to H_1 and J is the index set of an orthonormal basis $\{e_i\}$ in H_2. If we define $U_i : H_1 \to H$ by $U_i(x) = x \otimes e_i$, then U_i is an isomorphism of H_1 with $U_i H = H_i$ (say). Then $U_i x = x \otimes e_i$ is identified with $(x_j)_{j \varepsilon J}$, where $x_i = x \otimes e_i$ and $x_j = o$ for $j \neq i$. We define the linear transformation $U_i^* : H \to H_1$ as follows: $U_i^*(H \ominus H_i) = o$ and $U_i^*(x \otimes e_i) = x$. Then $U_i^* U_i = I$ on H_1 and $U_i U_i^* = P_{H_i} = P_i$ (say) on H.

Let $T \varepsilon B(H)$. Define $T_{ij} = U_i^* T U_j \varepsilon B(H_1)$. Given $T \varepsilon B(H)$, naturally T_{ij} are well defined by T for $i, j \varepsilon J$. Moreover, for $T \varepsilon B(H)$ and $x \varepsilon H$ we have

$$\|Tx\|^2 = \sum_{i\in J} \|P_i Tx\|^2 = \sum_{i\in J} \|U_i^* P_i Tx\|^2 = \sum_{i\in J} \|U_i^* Tx\|^2 =$$

$$= \sum_{i\in J} \Big\| \sum_{j\in J} U_i^* T P_j x\Big\|^2 = \sum_{i\in J} \Big\| \sum_{j\in J} T_{ij} U_j^* x\Big\|^2$$

since $T_{ij} = U_i^* T U_j$, $\sum_{j\in J} P_j = I$, T is bounded and $U_i^* Tx = U_i^* P_i Tx$. Consequently,

$$\sum_{i\in J} \Big\| \sum_{j\in J} T_{ij} U_j^* x\Big\|^2 = \|Tx\|^2 \leqq \|T\|^2 \, \|x\|^2$$

for $x \in H$ and

$$\|T\| = \inf \{ C: \sum_{i\in J} \Big\| \sum_{j\in J} T_{ij} U_j^* x\Big\|^2 \leqq C^2 \|x\|^2, \ x \in H\} \ . \ (*)$$

Moreover,

$$Tx = T(\sum_{j\in J} P_j x) = \sum_{j\in J} TP_j x = \sum_{i,j\in J} P_i T P_j x$$

$$= \sum_{i,j\in J} U_i U_i^* T U_j U_j^* x = \sum_{i\in J} \sum_{j\in J} U_i T_{ij} U_j^* x \ (**)$$

for $x \in H$. Thus (T_{ij}), with $T_{ij} = U_i^* T U_j$, determines T uniquely by (**).

Motivated by (*), we say that a matrix (T_{ij}) of operators on H_1 is bounded if there is a constant C such that

$$\sum_{i\in J} \Big\| \sum_{j\in J} T_{ij} U_j^* x\Big\|^2 \leqq C^2 \|x\|^2 \qquad (***)$$

for all $x \in H = H_1 \otimes H_2$. Then we define

$$\|(T_{ij})\| = \inf\{C : C \text{ as in } (***)\}.$$

Affirmation. Given a bounded matrix (T_{ij}) of operators on H_1, there exists a unique operator $T \in B(H)$ such that $Tx = \sum_{i\in I} \sum_{j\in J} U_i T_{ij} U_j^* x$, $x \in H$ and $\|T\| = \|(T_{ij})\|$. Moreover, $T_{ij} = U_i^* T U_j$, $i,j \in J$.

By (***),

$$S_i x = \sum_{j\in J} T_{ij} U_j^* x$$

exists for each $x \in H$ and $S_i : H \to H_1$ is a bounded linear transformation with $\|S_i\| \leq \|(T_{ij})\| = M$ (say). Again, by (***), $Tx = \sum_{i \in J} U_i S_i x$ exits for each $x \in H$ and $T : H \to H$ is a bounded operator with $\|T\| \leq M$. Now, for fixed $i_o, j_o \in J$ we have

$$P_{i_o} Tx = U_{i_o} S_{i_o} x = U_{i_o} (\sum_{j \in J} T_{i_o j} U_j^*) x, \; x \in H$$

since $U_i S_i x \in H_i$ and $P_{i_o} H_i = o$ for $i \neq i_o$. Thus

$$P_{i_o} TU_{j_o} x = U_{i_o} T_{i_o j_o} U_{j_o}^* U_{j_o} x, \; x \in H_1$$

i.e., $U_{i_o} (U_{i_o}^* TU_{j_o}) x = U_{i_o} T_{i_o j_o} U_{j_o}^* U_{j_o} x, \; x \in H_1$.

Since $U_{i_o} : H_1 \to H$ is injective, it follows that

$$U_{i_o}^* TU_{j_o} = T_{i_o j_o}, \; i_o, j_o \in J$$

and hence the matrix (T_{ij}) corresponds to $T \in B(H)$. Moreover, by (*) we also have $\|T\| = \|(T_{ij})\|$.

Thus by the foregoing discussion we conclude that there is an one-to-one correspondence between the operators T in $B(H) = B(H_1 \otimes H_2)$ and the collection of all bounded matrices (T_{ij}) of operators on H_1, the correspondence $T \leftrightarrow (T_{ij})$ being given by (**). Moreover, $\|T\| = \|(T_{ij})\|$ and $T_{ij} = U_i^* T U_j$, $i, j \in J$.

Thus $T \in B(H_1 \otimes H_2)$ can be represented by a bounded matrix (T_{ij}) of operators on H_1 and conversely. By abuse of notation we shall write $T = (T_{ij})$ with $T_{ij} = U_i^* TU_j$, $i, j \in J$.

If H_2 is finite dimensional, i.e., if J is finite, then any (T_{ij}) of operators on H_1 defines an operator in $B(H_1 \otimes H_2)$. But, when J is infinite, this does not hold as (T_{ij}) has to be bounded to define an operator in $B(H_1 \otimes H_2)$.

following matrix representations:

$$T_1 \otimes I_{H_2} = (\delta_{ij}\, T_1),\; I_{H_1} \otimes T_2 = (\lambda_{ij}\, I_{H_1}),$$

$$T_1 \otimes T_2 = (\lambda_{ij}\, T_1),$$ where $\lambda_{ij} \in \mathbb{C}$ and the matrix (λ_{ij}) is bounded.

Proof. Let $[T_2 e_j, e_i] = \lambda_{ij}$. Then $T_2 e_j = \sum_{i \in J} \lambda_{ij} e_i$, for $x \in H_1$

$$U_i^* (T_1 \otimes T_2)\, U_j\, x = U_i^* (T_1 \otimes T_2)\, (x \otimes e_j)$$

$$= U_i^* (T_1 x \otimes T_2 e_j)$$

$$= U_i^* (T_1 x \otimes (\sum_{k \in J} \lambda_{kj} e_k))$$

$$= U_i^* (\sum_{k \in J} \lambda_{kj} (T_1 x \otimes e_k)) = \lambda_{ij} T_1 x\,.$$

Thus $(T_1 \otimes T_2)_{ij} = \lambda_{ij}\, T_1$. If $T_1 = I_{H_1}$, then $I_{H_1} \otimes T_2 = (\lambda_{ij}\, I_{H_1})$. If $T_2 = I_{H_2}$, then $\lambda_{ij} = \delta_{ij}$. Hence $T_1 \otimes I_{H_2} = (\delta_{ij}\, T_1)$. (λ_{ij}) is bounded by the discussion preceding the lemma.

Lemma 4.5.4. $T \in B(H_1 \otimes H_2)$ commutes with $U_i U_j^*$ for all $i, j \in J$ if and only if T is of the form $T = T_1 \otimes I_{H_2}$ for some $T_1 \in B(H_1)$.

Proof. Let T commute with all $U_i U_j^*$. Let $\alpha \in J$. Then $T_{ij} = U_i^* T U_j = U_\alpha^* U_\alpha U_i^* T U_j = U_\alpha^* T U_\alpha U_i^* U_j = 0$ if $i \neq j$. And $T_{ij} = U_\alpha^* T U_\alpha U_i^* U_j = T_{\alpha\alpha}$ if $i = j$. Thus $T_{ii} = T_{\alpha\alpha}$ for all i. Let $T_1 = T_{\alpha\alpha}$. Then, clearly, $T = (\delta_{ij} T_1)$ and hence $T = T_1 \otimes I_{H_2}$ by Lemma 4.5.3.

Conversely, writing $U_i U_j^*$ in matrix, we have

$$U_\alpha^*\, U_i\, U_j^*\, U_\beta = \delta_{\alpha i}\, \delta_{j\beta} I_{H_1} \tag{4.5.4.1}$$

Thus, if $(S_{\alpha\beta}) = U_i U_J^*$, then $S_{\alpha\beta} = \delta_{\alpha i}\delta_{j\beta} I_{H_1}$. (4.5.4.2)

Since $T_1 \otimes I_{H_2} = (\delta_{ij}\, T_1)$,

$$(T_1 \otimes I_{H_2})\, U_i U_J^* = (\sum_r \delta_{\alpha r}\, T_1\, S_{r\beta})_{\alpha,\beta}$$

$$= (T_1 S_{\alpha\beta})_{\alpha,\beta}\ ;$$

$$(U_i U_J^*)\ (T_1 \otimes I_{H_2}) = (\sum_r S_{\alpha r}\, \delta_{r\beta}\, T_1)_{\alpha,\beta}$$

$$= (S_{\alpha\beta} T_1)_{\alpha,\beta}$$

But, for all α, β in J, $S_{\alpha\beta} T_1 = T_1 S_{\alpha\beta}$ by (4.5.4.2). Hence $T_1 \otimes I_{H_2}$ commutes with $U_i U_J^*$ for all $i, j \in J$.

Definition 4.5.5. If K is a subset of $B(H_1)$, $\mathcal{M}(K)$ will be the set of all $T = (T_{ij})$ in $B(H_1 \otimes H_2)$ with $T_{ij} \in K$ for $i, j \in J$ with card $.J$ = dimension of H_2, and $\mathcal{D}(K)$ will be the set of all $\{ T = (\delta_{ij}\ S) : S \in K \}$.

Lemma 4.5.6.

(a) $\mathcal{D}(K)' = \mathcal{M}(K')$ where $K' = (KUK^*)'$; (b) $\mathcal{D}(K)'' = \mathcal{D}(K'')$.

If $\{o, I_{H_1}\} \subset K$, then

(c) $\mathcal{M}(K)' = \mathcal{D}(K')$ and

(d) $\mathcal{M}(K)'' = \mathcal{M}(K'')$.

Proof. Suppose $T \in B(H)$, where $H = H_1 \otimes H_2$. Let $T = (T_{ij})$. If $T \in \mathcal{D}(K)'$, then for $S \in K \cup K^*$.

$$(T_{ij})(\delta_{ij} S) = (\delta_{ij} S)(T_{ij}).$$

i.e., $\sum_\alpha T_{i\alpha} \delta_{\alpha j} S = \sum_\alpha \delta_{i\alpha} S T_{\alpha j}$

i.e., $T_{ij} S = S T_{ij}$ for all $i, j \in J$, $S \in K \cup K^*$.

Thus $T_{ij} \in K'$ and hence $T \in M(K')$. Conversely, if T belongs to $M(K')$, by retracing the above steps we obtain $T \in \mathcal{D}(K)'$. Thus (a) holds.

Next we shall prove (c) and then deduce (b) and (d). Replacing K by K' in (a) we have $\mathcal{D}(K')' = M(K'') \supset M(K)$. Thus $\mathcal{D}(K') = \mathcal{D}(K')'' \subset M(K)'$. (4.5.6.1)

If $K \supset \{0, I_{H_1}\}$, let $T = (T_{ij})$, $T \in M(K)'$. First we have $M(K)' \subset \mathcal{D}(K)' = M(K')$ by (a), as $\mathcal{D}(K) \subset M(K)$ (when $\{o, I_{H_1}\} \subset K$). Thus $T_{ij} \in K'$, for all $i, j \in J$. Since $K \supset \{o, I_{H_1}\}$, from the matrix representation of $U_i U_j^*$ (see 4.5.4.1) it is clear that $U_i U_j^* \in M(K)$ for all $i, j \in J$. As $T \in M(K)'$, T commutes with $U_i U_j^*$ for all $i, j \in J$. Hence, by Lemma 4.5.4, T is of the form $T = T_1 \otimes I_{H_2}$, whose matrix representation is $(\delta_{ij} T_1)$, with $T_1 \in K'$. Thus $T \in \mathcal{D}(K')$. Therefore, $M(K)' \subset \mathcal{D}(K')$, (4.5.6.2). Clearly (c) follows from (4.5.6.1) and (4.5.6.2).

As $\mathcal{D}(K)' = M(K')$ by (a) and as $K' \supset \{o, I_{H_1}\}$, replacing K by K' in (c) we obtain $M(K')' = \mathcal{D}(K'')$.

Then, by (a), $\mathcal{D}(K'') = M(K')' = \mathcal{D}(K)''$ and hence (b) holds.

Finally, by (c) and (a) we have $M(K)'' = (M(K)')' = (\mathcal{D}(K'))' = M(K'')$ as $K \supset \{o, I_{H_1}\}$. Thus (d) holds.

Corollary 4.5.7. If R is a von Neumann algebra, then

$$\mathcal{D}(R)' = \mathcal{M}(R'); \ \mathcal{D}(R)'' = \mathcal{D}(R);$$

$$\mathcal{M}(R)' = \mathcal{D}(R'); \ \mathcal{M}(R)'' = \mathcal{M}(R).$$

In particular, $\mathcal{D}(R)$ and $\mathcal{M}(R)$ are von Neumann algebras.

Definition 4.5.8. Let R be a von Neumann algebra of operators on a Hilbert space H_1 and let Φ be the map $T \to T \otimes I_{H_2}$ of R into $R \otimes \mathbb{C}_{H_2}$, where H_2 is another Hilbert space. Then by Lemma 4.5.3 $\{T \otimes I_{H_2}: T \in R\} = \mathcal{D}(R)$ and is a von Neumann algebra (by Corollary 4.5.7 or by direct verification). Besides, $\Phi(R) = R \otimes \mathbb{C}_{H_2}$ and clearly Φ is an isomorphism of R onto $R \otimes \mathbb{C}_{H_2}$. Φ is called the *amplification* of R onto $R \otimes \mathbb{C}_{H_2}$, acting on $H_1 \otimes H_2$. (Notation: $\mathbb{C}_{H_2} = \mathbb{C}I_{H_2}$.)

Lemma 4.5.9. The operators of $B(H_1 \otimes H_2)$ which commute with operators of the form $I_{H_1} \otimes T_2$, $T_2 \in B(H_2)$, are of the form $T_1 \otimes I_{H_2}$, $T_1 \in B(H_1)$, and conversely.

Proof. Let $K = \mathbb{C}_{H_1}$. Then $\{ I_{H_1} \otimes T_2 \colon T_2 \in B(H_2)\} = \mathcal{M}(K)$ by Lemma 4.5.3. Thus $\mathcal{M}(K)' = \mathcal{D}(K') = \mathcal{D}(B(H_1))$ by Corollary 4.5.7. Thus by Lemma 4.5.3, operators that commute with $I_{H_1} \otimes T_2$ are of the form $T_1 \otimes I_{H_2}$, $T_1 \in B(H_1)$.

Conversely, with $K = B(H_1)$, $T_1 \otimes I_{H_2} \in \mathcal{D}(B(H_1))$ by Lemma 4.5.3. Since by Corollary 4.5.7, $(\mathcal{D}(B(H_1)))' = \mathcal{M}(\mathbb{C}_{H_1})$, operators that commute with all $T_1 \otimes I_{H_2}$ are of the form $I_{H_1} \otimes T_2$, $T_2 \in B(H_2)$, by Lemma 4.5.3.

Theorem 4.5.10. Let R be a von Neumann algebra of operators on the Hilbert space H_1 and $\mathbb{C}_{H_2}$ the algebra of scalar operators $\mathbb{C}I_{H_2}$ on H_2. Then:

(i) $R \otimes \mathbb{C}_{H_2} = \mathcal{D}(R)$;

(ii) $R \otimes B(H_2) = M(R)$;

(iii) $R \otimes \mathbb{C}_{H_2} = (R' \otimes B(H_2))'$. Consequently, $(R \otimes B(H_2))' = R' \otimes \mathbb{C}_{H_2}$. (Thus the commutation theorem for tensor products of two von Neumann algebras holds if one of them is $B(H_2)$ or $\mathbb{C}_{H_2}$. The validity of the commutation theorem in the general case is dealt with in §5.9).

Proof. (i) follows from Lemma 4.5.3. (iii) $R' \otimes B(H_2)$ = The von Neumann algebra generated by $\{R' \otimes \mathbb{C}_{H_2}, \mathbb{C}_{H_1} \otimes B(H_2)\}$.

Hence $(R' \otimes B(H_2))' = (R' \otimes \mathbb{C}_{H_2})' \cap (\mathbb{C}_{H_1} \otimes B(H_2))'$.

But, $(\mathbb{C}_{H_1} \otimes B(H_2))' = B(H_1) \otimes \mathbb{C}_{H_2}$ by Lemma 4.5.9.

Thus $(R' \otimes B(H_2))' = (R' \otimes \mathbb{C}_{H_2})' \cap (B(H_1) \otimes \mathbb{C}_{H_2})$. Now, an operator $T_1 \otimes I_{H_2}$ commutes with $R' \otimes \mathbb{C}_{H_2}$ if and only if $T_1 \in R'' = R$. This can be easily seen by using matrix representation (see 4.5.3). Thus $(R' \otimes \mathbb{C}_{H_2})' \cap (B(H_1) \otimes \mathbb{C}_{H_2}) = R \otimes \mathbb{C}_{H_2}$. Consequently, $(R' \otimes B(H_2))' = R \otimes \mathbb{C}_{H_2}$, so that $(R \otimes B(H_2))' = R' \otimes \mathbb{C}_{H_2}$. This proves (iii).

(ii) $(R \otimes B(H_2))' = (R' \otimes \mathbb{C}_{H_2})$ (by (iii))

$= \mathcal{D}(R')$ (by Lemma 4.5.3)

so that $R \otimes B(H_2) = \mathcal{D}(R')' = M(R'') = M(R)$ by Corollary 4.5.7.

This proves (ii).

The above theorem has the following elegant application. The following proposition generalizes Lemma 4.2.1.

Proposition 4.5.11. Let R be a von Neumann algebra of operators on a Hilbert space H with centre Z. Let $T_{ij} \varepsilon R$, $T'_{ij} \varepsilon R'$, for i,j = 1,2,...,n. The following conditions are equivalent:

(i) $\sum_{k=1}^{n} T_{ik} T'_{kj} = 0$ for i, j = 1,2...,n.

(ii) There exist z_{ij}, (i,j = 1,2,...,n), in Z such that

$$\sum_{k=1}^{n} T_{ik} z_{kj} = 0, \quad \sum_{k=1}^{n} z_{ik} T'_{kj} = T'_{ij} \text{ for } i,j=1,2,...,n.$$

Proof. (ii) $\Longrightarrow$ (i) For,

$$\sum_k T_{ik} T'_{kj} = \sum_k T_{ik}(\Sigma_\ell z_{k\ell} T'_{\ell j})$$

$$= \sum_\ell (\sum_k T_{ik} z_{k\ell})T'_{\ell j}$$

$$= 0 \quad , i,j=1,2...,n.$$

(i) $\Longrightarrow$ (ii) Let $K = \sum_1^n \oplus H$. Then we shall identity K with $H \otimes L^2_{\mathbb{C}}(J)$, card. J = n. Then $S \varepsilon B(K)$ has the matrix representation $S = (S_{ij})$ i,j=1,2...,n, $S_{ij} \varepsilon B(H)$. By Theorem 4.5.10, $T = (T_{ij})_{i,j=1,2...,n} \varepsilon R \otimes B(H_2)$, where $H_2 = L^2_{\mathbb{C}}(J)$ and $T' = (T'_{ij})_{i,j=1,2...,n} \varepsilon R' \otimes B(H_2)$. By hypothesis (i), TT' = 0. Let P = {projections $E' \varepsilon R' \otimes B(H_2)$: TE' = o }. Then $P \neq \Phi$, as $[T'(K)] \varepsilon P$. Let E'_0 be the supremum of the members of P. Then $E'_0 [T'(K)] = [T'(K)]$ and $E'_0 T' = T'$.

Since E'_o can be obtained as the strong limit of a net of projections from $\mathcal{P}$, TE'_o = 0. Thus, if $E'_o = (z_{ij})_{i,j=1,2\ldots,n}$ with $z_{ij} \in \mathcal{R}'$ then $TE'_o = 0$ means $\sum_{k=1}^{n} T_{ik} z_{kj} = 0$, $i,j=1,2\ldots,n$ and $E'_o T' = T'$ means $\sum_{k=1}^{n} z_{ik} T'_{kj} = T'_{ij}$ $i,j=1,1\ldots,n$. The proof will be complete if we show that $z_{ij} \in \mathcal{R}$, as they are already in $\mathcal{R}'$. For this it suffices to show that for every hermitian operator R' in $\mathcal{R}'$, $z_{ij} R' = R' z_{ij}$, $i,j=1,2\ldots,n$. Let $S' = R' \otimes I_{H_2}$. Clearly, S' is hermitian. $TS'E'_o = S'TE'_o = 0$, since $S' \in \mathcal{R}' \otimes \mathbb{C}_{H_2} = (\mathcal{R} \otimes B(H_2))'$ by 4.5.10 (iii). Hence $[S'E'_o(K)] \in \mathcal{P}$. Thus, $[S'E'_o(K)] \subset E'_o(K)$ so that $E'_o S' E'_o = S'E'_o$. Taking adjoints, $E'_o S' E'_o = S'E'_o$. Thus $E'_o S' = S'E'_o$. With the matrix representation of S', this means $z_{ij} R' = R' z_{ij}$, $i,j=1,2\ldots,n$. Thus, $z_{ij} \in \mathcal{R}$.

This completes the proof.

4.6. Some spatial isomorphism theorems for Neumann algebras.

Let $H_2 = L^2_{\mathbb{C}}(J)$, J an index set. Let $H = H_1 \otimes H_2$, where H_1 is a Hilbert space. Let $P_i: H \to H_i = \{x \otimes e_i : x \in H_1\}$. Then $P_i \in \mathbb{C}_{H_1} \otimes B(H_2)$, as its matrix representation is given by $P_{i,\alpha,\beta} = U^*_\alpha P_i U_\beta = U^*_\alpha U_i U^*_i U_\beta$

$$= \delta_{i\alpha} \delta_{i\beta} I_{H_1}$$

Thus P_i belongs to $\mathcal{R} \otimes B(H_2) = (\mathcal{R}' \otimes \mathbb{C}_{H_2})'$ for any von Neumann algebra $\mathcal{R}$ on H_1. The partial isometries $U_j U^*_i = U_{ij}$, which admit P_i as initial projection and P_j as final projection, also belong to $\mathbb{C}_{H_1} \otimes B(H_2)$ (by their matrix representation) and hence to $R \otimes B(H_2) = (R' \otimes \mathbb{C}_{H_2})'$.

Theorem 4.6.1.

(i) $P_i\ (R \otimes B(H_2))P_i$ is spatially isomorphic to R and $(R' \otimes \mathbb{C}_{H_2})P_i$ is spatially isomorphic to R', where R is a von Neumann algebra on H_1.

(ii) Let R be a von Neumann algebra on a Hilbert space K, $(E_i)_{i\varepsilon J}$ be an orthogonal family of equivalent projections in R with $\sum_{i\varepsilon J} E_i = I$. Let $\alpha \varepsilon J$ and let $H_1 = E_\alpha(K)$, and $H_2 = L^2_{\mathbb{C}}(J)$. Then R is spatially isomorphic to the von Neumann algebra $(E_\alpha R\, E_\alpha) \otimes B(H_2)$ on $H_1 \otimes H_2$ and R' is spatially isomorphic to $R' E_\alpha \otimes \mathbb{C}_{H_2}$.

Proof.

(i) By Theorem 4.5.10 (i), $R' \otimes \mathbb{C}_{H_2} = \mathcal{D}(R')$. The isomorphism U_i maps H_1 onto H_i, so that if $T \varepsilon R'$, then $U_i T U_i^*$ has the matrix representation $(T_{\alpha\beta})$, where $T_{\alpha\beta} = U_\alpha^* U_i T U_i^* U_\beta = \delta_{\alpha i}\delta_{i\beta}\, T$.

$$\text{Thus } U_i T U_i^* = (\delta_{i\alpha}\ \delta_{i\beta} T)$$
$$= (\delta_{\alpha\beta} T)(\delta_{i\alpha}\ \delta_{i\beta} I_{H_1})$$
$$= (\delta_{i\alpha}\delta_{i\beta}\ I_{H_1})(\delta_{\alpha\beta} T)$$

so that $U_i T U_i^* = (T \otimes I_{H_2})P_i = P_i(T \otimes I_{H_2})$ by (4.5.4.2)

where $P_i : H \to H_i$ is the projection on H_i.

Restricting U_i^* to H_i, which is then U_i^{-1},

$$U_i T U_i^{-1} = (T \otimes I_{H_2})P_i = P_i(T \otimes I_{H_2}).$$

Thus $\mathcal{R}'$ is spatially isomorphic to $(\mathcal{R}' \otimes \mathbb{C}_{H_2})P_i$. Hence its commutant $\mathcal{R}'' = \mathcal{R}$ is spatially isomorphic to

$$((\mathcal{R}' \otimes \mathbb{C}_{H_2})P_i)' = P_i(\mathcal{R}' \otimes \mathbb{C}_{H_2})'P_i \qquad \text{(by Theorem 4.2.2)}$$

$$= P_i(\mathcal{R} \otimes B(H_2))P_i \qquad \text{(by Theorem 4.5.10 (iii)).}$$

(ii) Let $H_1 = E_\alpha(K)$, with $\alpha \in J$ fixed. As $E_\alpha \sim E_i$, $i \in J$, there exist partial isometries $U_{\alpha i}$ and $U^*_{\alpha i}$ in $\mathcal{R}$ such that

$$U_{\alpha i}: E_\alpha(K) \to E_i(K)$$

$$U^*_{\alpha i} : E_i(K) \to E_\alpha(K).$$

Clearly, $U^*_{\alpha i}| E_i(K)$ is an isomorphism of $E_i(K)$ onto $E_\alpha(K)$. Call the isomorphism V_i. Thus $V_i: E_i(K) \to E_\alpha(K)$.

Since $\sum_{i\in J} E_i = I$, each x in K has the unique representation $x = \sum_{i\in J} E_i x$. Let $\{e_i\}_{i\in J}$ be an orthonormal basis in H_2. Let $H_i = \{x \otimes e_i : x \in H_1\}$. Then by Lemma 4.5.1, $H_1 \otimes H_2 = \Sigma \oplus H_i$. Thus

$$K = \sum_{i\in J} \oplus E_i(K) \overset{V}{\to} \sum_{i\in J} \oplus H_i = H_1 \otimes H_2$$

where $V(\Sigma E_i x) = \sum_{i\in J} (V_i E_i x) \otimes e_i$ for $x \in K$. Clearly, V is an isomorphism of K onto $H_1 \otimes H_2$.

Let $T \in \mathcal{R}$, and let (T_{ij}) be the matrix representation of $VTV^{-1} \in B(H_1 \otimes H_2)$. Then $T_{ij} = U_i^*(VTV^{-1})U_j$ where $U_j : H_1 = (E_\alpha(K)) \to H_j$. Thus $U_j = VU_{\alpha j}E_\alpha|H_1$ and $U_i^* = U^*_{\alpha i} V^{-1}$. Consequently,

$$T_{ij} = U^*_{\alpha i}V^{-1}(VTV^{-1})VU_{\alpha j}E_\alpha$$
$$= (U^*_{\alpha i}TU_{\alpha j})E_\alpha$$
$$= E_\alpha(U^*_{\alpha i}TU_{\alpha j})E_\alpha \quad (\text{since } U^*_{\alpha i}: K \to E_\alpha(K))$$
$$\varepsilon\ E_\alpha R E_\alpha$$

as $U^*_{\alpha i}$, $U_{\alpha j}$, T are in R. Thus $VTV^{-1} \varepsilon (E_\alpha R E_\alpha) \otimes B(H_2)$, so that $VRV^{-1} \subset (E_\alpha RE_\alpha) \otimes B(H_2)$.(4.6.1.1)

Let $T' \varepsilon R'$. Then the matrix representation (T'_{ij}) of $VT'V^{-1}$ is given by

$$T'_{ij} = U^*_i (VT'V^{-1})U_j = U^*_{\alpha i}V^{-1}VT'V^{-1}VU_{\alpha j}E_\alpha$$
$$= U^*_{\alpha i}T'U_{\alpha j}E_\alpha = T'U^*_{\alpha i}U_{\alpha j}E_\alpha$$
$$= \delta_{ij}T'E_\alpha$$

so that $VT'V^{-1} = (\delta_{ij}\ T'E_\alpha)$. Hence $VT'V^{-1} \varepsilon R'E_\alpha \otimes \mathbb{C}_{H_2}$ so that $VR'V^{-1} \subset R'E_\alpha \otimes \mathbb{C}_{H_2}$. Thus $(VR'V^{-1})' \supset (R'E_\alpha \otimes \mathbb{C}_{H_2})'$. Hence by Theorems 4.2.2 and 4.5.10, $VRV^{-1} \supset (E_\alpha R E_\alpha) \otimes B(H_2)$. Now (4.6.1.2)

(4.6.1.1) and (4.6.1.2) together give $VRV^{-1} = E_\alpha RE_\alpha \otimes B(H_2)$ and hence R is spatially isomorphic to $E_\alpha RE_\alpha \otimes B(H_2)$. Consequently, R' is spatially isomorphic to $R'E_\alpha \otimes \mathbb{C}_{H_2}$. This proves (ii).

Theorem 4.6.2. Let R and S be von Neumann algebras on Hilbert spaces H_1 and H_2, respectively. Let $E \varepsilon R$ and $F \varepsilon S$ be projections with ranges M and N, respectively. Then $E \otimes F$ is a projection in $R \otimes S$ and has its range $(E \otimes F)(H_1 \otimes H_2) = M \otimes N$. Further,

$$(E \otimes F)(R \otimes S)(E \otimes F) = ERE \otimes FSF,$$

$$(R' \otimes S')(E \otimes F) = R'E \otimes S'F.$$

Proof. $E \otimes F$ is hermitian and idempotent by Lemma 4.4.2. Hence $E \otimes F$ is a projection in $R \otimes S$. If $\{x_\alpha^{(1)}\}_{\alpha \varepsilon J_1}$, $\{x_\alpha^{(2)}\}_{\alpha \varepsilon J_2}$ are orthonormal bases in M and N, respectively, then $\{x_\alpha^{(1)} \otimes x_\beta^{(2)}\}_{(\alpha,\beta) \varepsilon J_1 \times J_2}$ is an orthonormal basis in $M \otimes N$ by Lemma 4.4.3. Now $(E \otimes F)(x_\alpha^{(1)} \otimes x_\beta^{(2)}) = E x_\alpha^{(1)} \otimes F x_\beta^{(2)} = x_\alpha^{(1)} \otimes x_\beta^{(2)}$ and hence $(E \otimes F)$ leaves $M \otimes N$ pointwise invariant. Also, if x and y are in H_1 and H_2, respectively, with $x \perp M$ or $y \perp N$, then $(E \otimes F)(x \otimes y) = E x \otimes F y = 0$. These facts together with Lemma 4.4.3 give that $(E \otimes F)(H_1 \otimes H_2) = M \otimes N$.

Since $R \otimes S$ is generated by the set of all operators of the form $R_1 \otimes S_1 + R_2 \otimes S_2 + \dots + R_n \otimes S_n$, with $R_i \varepsilon R$, $S_i \varepsilon S (i=1,2,\dots,n)$, $(E \otimes F)(R \otimes S)(E \otimes F)$ is generated by the collection of all operators of the form $(E \otimes F)(\sum_1^n R_i \otimes S_i)(E \otimes F) = \sum_{i=1}^n (ER_iE) \otimes (FS_iF)$ which belong to $ERE \otimes FSF$. Thus $(E \otimes F)(R \otimes S)(E \otimes F) \subset ERE \otimes FSF$. Conversely, $\sum_{i=1}^n (ER_iE) \otimes (FS_iF) = (E \otimes F)(\sum_1^n R_i \otimes S_i)(E \otimes F) \varepsilon (E \otimes F)(R \otimes S)(E \otimes F)$ and hence $ERE \otimes FSF \subset (E \otimes F)(R \otimes S)(E \otimes F)$. Similarly, $(R' \otimes S')(E \otimes F) = R'E \otimes S'F$.

This completes the proof of the theorem.

Theorem 4.6.3. Let H_i, $i \varepsilon J$, K_j, $j \varepsilon A$, be Hilbert spaces with R_i, S_j von Neuman algebras on H_i, K_j, respectively, for all $i \varepsilon J$, $j \varepsilon A$. Let $R = \sum_{i \varepsilon J} \oplus R_i$ on $H = \sum_{i \varepsilon J} \oplus H_i$ and $S = \sum_{j \varepsilon A} \oplus S_j$ on $K = \sum_{j \varepsilon A} \oplus K_j$. Then $H \otimes K$ is canonically identified with $\sum_{(i,j)\varepsilon J \times A} \oplus (H_i \otimes K_j)$ and $R \otimes S$ is identified with $\sum_{(i,j)\varepsilon J \times A} \oplus (R_i \otimes S_j)$.

Proof. Let $\{x^i_\alpha\}_{\alpha\varepsilon J_i}$ be a complete orthonormal set in $H_i, i \ \varepsilon \ J$ and $\{y^j_\beta\}_{\beta\varepsilon A_j}$ be a complete orthonormal set in K_j, $j \ \varepsilon \ A$. Then $\{X^i_\alpha = (\delta_{\alpha\beta} \ x^i_\alpha)_{\alpha,\beta\varepsilon J_i}\}$, $i\varepsilon J$, $\alpha\varepsilon J_i$ is clearly an orthonormal basis in H and $\{Y^j_\alpha = (\delta_{\alpha\beta} \ y^i_\alpha)_{\alpha,\beta\varepsilon A_j}\}$, $j \ \varepsilon \ A$, $\alpha \ \varepsilon \ A_j$ is an orthonormal basis in K. Then by Lemma 4.4.3 $\{X^i_\alpha \otimes Y^j_\beta : i \ \varepsilon \ J, \ j \ \varepsilon \ A; \ \alpha \ \varepsilon \ J_i,$ $\beta \ \varepsilon \ A_j\}$ is an orthonormal basis in $H \otimes K$. For fixed $i \ \varepsilon \ J$, $j \ \varepsilon \ A$, $\{x^i_\alpha \otimes y^j_\beta : \alpha \ \varepsilon J_i,$ $\beta \ \varepsilon \ A_j\}$ is an orthonormal basis in $H_i \otimes K_j$. Hence, if $U: \sum_{(i,j)\varepsilon J \times A} \oplus (H_i \otimes K_j) \to H$ is given by $U(\delta_{ii'} \delta_{jj'} \ x^i_\alpha \otimes y^j_\beta) = X^i_\alpha \otimes Y^j_\beta$, $\alpha \ \varepsilon \ J_i$, $\beta \ \varepsilon \ A_j$, $i,i' \ \varepsilon \ J$, $j,j' \varepsilon \ A$ and is extended linearly and continuously to all elements, then

$$[U(\delta_{i\ell} \delta_{jm} \ x^i_\alpha \otimes \ y^j_\beta), \ U(\delta_{i'\ell} \delta_{j'm} \ x^{i'}_{\alpha'} \otimes \ y^{j'}_{\beta'}]$$
$$= [X^i_\alpha \otimes Y^j_\beta \ , \ X^{i'}_{\alpha'} \otimes Y^{j'}_{\beta'} \ ]$$
$$= [X^i_\alpha, \ X^{i'}_{\alpha'}][Y^j_\beta, \ Y^{j'}_{\beta'}] = \ \delta_{\alpha\alpha'} \delta_{\beta\beta'} \ \delta_{ii'} \delta_{jj'}.$$

Thus U preserves orthonormality of the basis vectors. Clearly, U is onto. Hence U transforms the given orthonormal basis of $\sum \oplus (H_i \otimes K_j)$ onto an orthonormal basis of $H \otimes K$. Hence U is an isomorphism. (This isomorphism, which is so natural is called the canonical isomorphism.)

Let $E_i : H \to H_i$, $F_j : K \to K_j$, $(i \ \varepsilon \ J, \ j \ \varepsilon \ A)$, be the canonical projections. Then, by Theorem 4.6.2, $E_i \otimes F_j$ is a projection on $H \otimes K$ with range $H_i \otimes K_j$. Since $H \otimes K = \sum_{(i,j)\in J \times A} \oplus (H_i \otimes K_j)$ on identification, $\sum_{(i,j)\in J\times A} E_i \otimes F_j = I$. Since $E_i \otimes F_j$ is the identity on $H_i \otimes K_j$, $E_i \otimes F_j \in R_i \otimes S_j$ (considering the restriction of $E_i \otimes F_j$ on $H_i \otimes K_j$). Since $R = \Sigma \oplus R_i$, $E_i = \sum_{j\in J} \oplus \delta_{ij} I_{H_j} \in R'$ and $F_j = \sum_{\ell\in A} \oplus \delta_{j\ell} I_{K_\ell} \in S'$. Also these operators belong to R and S respectively. Hence $E_i \otimes F_j$ is in Z, the centre of $R \otimes S$, as $Z_1 \otimes Z_2 \subset Z$ by Corollary 4.4.8, where Z_1 is the centre of R and Z_2 that of S. Thus we have:

(i) $E_i \otimes F_j \in Z$.

(ii) $\{E_i \otimes F_j\}_{(i,j)\in J \times A}$ is an orthogonal family of projections in $R \otimes S$.

(iii) $\sum_{(i,j)\in J \times A} E_i \otimes F_j = I$.

Hence by the converse part of Lemma 4.1.1, $R \otimes S$ is spatially isomorphic to

$\sum_{(i,j)\in J \times A} \oplus (R \otimes S)(E_i \otimes F_j) = \sum_{(i,j)\in J \times A} \oplus (RE_i \otimes SF_j)$ (by Theorem 4.6.2).

But $R = \sum_{i\in J} \oplus R_i$, $E_i = \sum_{j\in J} \oplus \delta_{ij} I_{H_j}$, so that $RE_i = \sum_{j\in J} \oplus \delta_{ij} R_j$. Thus RE_i is spatially isomorphic to R_i. Hence $R \otimes S$ is spatially isomorphic to

$\sum_{(i,j)\in J \times A} \oplus (R_i \otimes S_j)$.

This completes the proof of the theorem.

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* Books recommended for further reading.

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