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ON SIMPLE SPECTRAL MEASURES

BY

T.V. PANCHAPAGESAN

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UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS
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T. V. PANCHAPAGESAN [1]

Abstract

The notion of a simple spectral measure is introduced and several characterizations of a simple spectral measure are given. A classical result of von Neumann on the double commutant of the range of the resolution of the identity of a self-adjoint operator on a separable Hilbert space is generalized to spactral measures with the CGS-property. For such spectral measures some further characterizations of simplicity are obtained, which include one derived from the said generalization of von Neumann's result.

1. Introduction

For self-adjoint operators in separable Hilbert spaces the notion of simple spectrum was given by Stone [16] in terms of the multiplicity functions m_p and m_c associated with the operator, while it was given differently by Akhiezer and Glazman [1] in terms of the total multiplicity of the operator. On the other hand, for such operators T in arbitrary Hilbert spaces H, Wecken [17] and Plesner and Rohlin [13] defined the concept using the measures $\rho(x) = ||E(\cdot)x||^2$ and closed subspaces $[E(\delta)x : \delta \in \mathcal{B}(\sigma(T))]$, $x \in H$, where $E(\cdot)$ is the resolution of the identity of T. Later, in [15], Segal gave the concept for a bounded self-adjoint operator T in an arbitrary Hilbert space in terms of the $W^\star$-algebra generated by T. Though the concept of simple spectrum is defined differently, all these definitions are equivalent (see 3.1(b) and 5.1).

In [16] Stone studied the problem of unitary invariance of self-adjoint operators in separable Hilbert spaces, while Dunford and Schwartz [4] studied it for self-adjoint and bounded normal operators in such spaces. In our recent work [11] we extended their results to spectral measures with the CGS-property and in particular, to unbounded normal operators in separable Hilbert spaces. Using some rudiments of type I von Neumann algebras along with the spectral multiplicity theory of Halmos [5], we have also given in [8-12] a unified approach to the study of the above problem for arbitrary spectral measures and for spectral measures with the CGS-property. Making use of the results from these papers, we give here the notions of simple spectral measures

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and normal operators with simple spectra and obtain several characterizations of simplicity of a spectral measure $E(\cdot)$ (resp. of the spectrum of a normal operator in a Hilbert space H) when $E(\cdot)$ is arbitrary and when $E(\cdot)$ has the CGS-property (resp. when H is arbitrary and when H is separable). In this context, we generalize the theorem of von Neumann given in §129 of Riesz and Nagy [14] to spectral measures with the CGS-property (see Theorem 4.1) and deduce Lemma 1.2 of Segal [15] as a corollary of Theorem 5.1.

2. Preliminaries

In this section we fix notation and terminology. For convenience we recall some definitions and results from the literature, especially from [8-12].

$\mathcal{S}$ denotes a σ-algebra of subsets of a set $X(\neq \emptyset)$. H is a Hilbert space of arbitrary dimension (> 0) unless otherwise stated and $E(\cdot)$ is a spectral measure on $\mathcal{S}$ with values in projections of H. If $\mathcal{X} \subset H$, then $[\mathcal{X}]$ denotes the closed subspace spanned by $\mathcal{X}$. For $x \in H$ and $\mathcal{X} \subset H$, let $Z_E(x) = [E(\sigma)x : \sigma \in \mathcal{S}]$ and $Z_E(\mathcal{X}) = [E(\sigma)w : w \in \mathcal{X},\ \sigma \in \mathcal{S}]$. The orthogonal direct sum of closed subspaces of H and that of a family of Hilbert spaces are denoted by $\oplus$.

An isomorphism between two Hilbert spaces is an inner-product preserving onto linear map. An operator T on H is a linear transformation with domain and range in H and is not necessarily bounded.

Σ denotes the set of all finite (positive) measures on $\mathcal{S}$. For $x \in H$, $\rho(x)$ denotes the measure $||E(\cdot)x||^2$. For $\mu, \nu \in \Sigma$, we write $\mu << \nu$ (or $\nu >> \mu$) if $\mu(E) = 0$ whenever $\nu(E) = 0$.

DEFINITION 2.1 ([11]). *$E(\cdot)$ is said to have the CGS-property in H if there is an utmost countable subset $\mathcal{X}$ of H such that $Z_E(\mathcal{X}) = H$.*

DEFINITION 2.2 ([11]). *Suppose $(x_i)_1^N$ is a finite or an infinite sequence of non-zero vectors in H such that (i) $H = \bigoplus_{i=1}^{N} Z_E(x_i)$, and (ii) $\rho(x_1) >> \rho(x_2) >> \ldots$. Then $H = \bigoplus_{i=1}^{N} Z_E(x_i)$ is called an OSD of H relative to $E(\cdot)$.*

PROPOSITION 2.1 ([11]). *$E(\cdot)$ has the CGS-property in H if and only if H admits an OSD relative to $E(\cdot)$. If $H = \bigoplus_{i=1}^{N} Z_E(x_i)$ is an OSD of H relative to $E(\cdot)$, then N is unique and N is called the OSD-multiplicity of $E(\cdot)$. When $N = \infty$, the OSD-multiplicity of $E(\cdot)$ is said to be countably infinite.*

DEFINITION 2.4 ([10]). *An OSD* $H = \overset{N}{\underset{i=1}{\oplus}} Z_E(x_i)$ *is called a uniform OSD if* $\rho(x_1) \equiv \rho(x_2) \equiv \ldots$, *where, for* $\mu, \nu \in \Sigma$, *we write* $\mu \equiv \nu$ *if* $\mu << \nu$ *and* $\nu << \mu$.

PROPOSITION 2.2 ([10]). *If H admits a uniform OSD relative to* $E(\cdot)$, *then every OSD of H relative to* $E(\cdot)$ *is a uniform OSD. In that case, the OSD-multiplicity of* $E(\cdot)$ *is referred to as the UOSD-multiplicity of* $E(\cdot)$.

DEFINITION 2.5 ([11]). *Suppose* $(\mu_i)_1^N$ *is a finite or an infinite sequence of non-null elements in* Σ *such that* $\mu_1 >> \mu_2 >> \ldots$. *If there is an isomophism* $U : H \to K = \overset{N}{\underset{i=1}{\oplus}} L_2(\mu_i)$ *such that*

$$UE(\cdot)U^{-1}(f_i) = (\chi_{(\cdot)} f_i), \; (f_i)_1^N \in K,$$

then U is called an OSR of H relative to $E(\cdot)$ *and N is called the OSR-multiplicity of* $E(\cdot)$ *- since it is the same for all OSRs of H relative to* $E(\cdot)$.

PROPOSITION 2.3 ([11]). *H has an OSR relative to* $E(\cdot)$ *if and only if* $E(\cdot)$ *has the CGS-property in H.*

DEFINITION 2.6 ([11]). *Suppose* $E(\cdot)$ *has the CGS-property in H. If there exists a finite dimensional generating subspace Y in H (that is,* $\dim Y < \infty$ *and* $Z_E(Y) = H$*), then the minimum of the dimensions of all generating subspaces of H is called the total multiplicity of* $E(\cdot)$. *If* $E(\cdot)$ *does not have any finite dimensional generating subspace, then the total multiplicity of* $E(\cdot)$ *is said to be countably infinite.*

PROPOSITION 2.4 ([11]). *Suppose* $E(\cdot)$ *has the CGS-property in H. Then its OSD-multiplicity, OSR-multiplicity and total multiplicity coincide.*

DEFINITION 2.7([11]). *Suppose X is a Hausdorff space and* $\mathcal{S} = \mathcal{B}(X)$, *the* σ*-algebra of all Borel subsets of X. Suppose* $E(\cdot)$ *has the CGS-property in H. We define* $p_E = \{t \in X : E(\{t\}) \neq 0\}$ *and* $c_E = X \backslash p_E$. *Let* $\mathcal{M}(E) = E(p_E)H$ *and* $\mathcal{N}(E) = E(c_E)H = H \ominus \mathcal{M}(E)$. *Let* $\mathcal{N}(E) = \overset{N}{\underset{i=1}{\oplus}} Z_E(y_i)$ *be an OSD of* $\mathcal{N}(E)$ *relative to* $E(\cdot)E(c_E)$.

The multiplicity function m_p *on X relative to* $E(\cdot)$ *is defined by* $m_p(t) = 0$ *if* $t \notin p_E$ *and* $m_p(t) = \dim E(\{t\})H$ *if* $t \in p_E$. *The multiplicity function* m_c *on X relative to* $E(\cdot)$ *is defined as follows:*

(i) $m_c(t) = 0$ *if* $\mathcal{N}(E) = \{0\}$, *or if* $\mathcal{N}(E) \neq \{0\}$ *and there exists an open set* $U \ni t$ *such that* $E(U)y_1 = 0$;

(ii) $m_c(t) = n \in \mathbb{N}$ *if* y_k *do exist for* $k = 1,2,\ldots,n$ *and for every open set* $U \ni t$, $E(U)y_k \neq 0$ *for* $k = 1,2,\ldots,n$ *while* $N = n$ *or* y_{n+1} *does exist and* $E(U)y_{n+1} = 0$ *for some open set* $U \ni t$;

(iii) $m_c(t) = \infty$ *if* $N = \infty$ *and for every open set* $U \ni t$, $E(U)y_k \neq 0$ *for each* $k \in \mathbb{N}$.

PROPOSITION 2.5 ([5]). *Let* $E(\cdot)$ *have the CGS-property in* H *and let* $\mathcal{S} = \mathcal{B}(X)$, *where* X *is a Hausdorff space. Then the total multiplicity of* $E(.)$ *is equal to* $sup_{t\in X}(m_p(t), m_c(t))$.

We now proceed to give some definitions and results from [5,8,9 and 12], along with some rudiments of von Neumann algebras from [3,7].

Let W be the von Neumann algebra generated by the range of $E(\cdot)$. The commutant of W is denoted by W'. If $W' = \Sigma_n \oplus W'Q_n$ is the type I_n direct sum decomposition of W' (Dixmier [3] uses the notation II and the terminology product), then the central projections $Q_n(\neq 0)$ are unique and $W'Q_n$ is of type I_n; in the sequel, Q_n will denote these non-zero central projections of W'. If P' is a projection in W', then the central support of P' is denoted by $C_{P'}$. As is customary in the theory of von Neumann algebras, a projection is also identified with its range.

Recall that a projection $P' \in W'$ is said to be an abelian projection if $P'W'P'$ is abelian. As observed in [9], the projection $P' \in W'$ is abelian if and only if P' is a row projection in the sense of Halmos [5] and the column $C(P')$ generated by P' in the sense of [5] is the same as the central support $C_{P'}$ of P'.

The abelian von Neumann algebra W is said to be maximal abelian if $W' = W$. The algebra W is said to have a generating vector $x \in H$ if $[Wx] = [Tx : T \in W] = H$ and a separating vector $y \in H$ if $Ty = 0$ for some $T \in W$ implies $T = 0$. A projection P' in W' said to be cyclic (in W') if there exists a vector $x \in H$ such that $[Wx] = P'$. A projection P in a von Neumann algebra $\mathcal{R}$ is said to be countably decomposable (in $\mathcal{R}$) if every orthogonal family of non-zero subprojections of P in $\mathcal{R}$ is utmost countable. The von Neumann algebra $\mathcal{R}$ is said to be countably generated if there exists an utomst countable set $\mathcal{X}$ of vectors in H such that $[Tx : T \in \mathcal{R}, x \in \mathcal{X}] = H$.

For the rest of the terminology in the theory of von Neumann algebras we follow Dixmier [3]. For an easily accessible account of von Neumann algebras the reader is referred to [7].

PROPOSITION 2.6 ([7, Lemma 3.3.9]). *Let P' be an abelian projection in W'. If $C_{P'}$ is countably decomposable in W, then P' is cyclic (in W').*

DEFINITION 2.8. *For a projection P in W, its multiplicity (resp. uniform multiplicity) in the sense of Halmos [5, pp.100-101] is referred to as its H-multiplicity (resp. UH-multiplicity) relative to $E(\cdot)$.*

As was observed in [9], Theorem 64.4 of Halmos [5] can be reformulated as follows:

PROPOSITION 2.7. *A projection P in W has UH-multiplicity $n > 0$ relative to $E(\cdot)$ if and only if there exists an orthogonal family $\{E'_\alpha\}_{\alpha \in J}$ of abelian projections in W' such that $C_{E'_\alpha} = P$, $\Sigma_{\alpha \in J} E'_\alpha = P$ and card. $J = n$; in other words, if and only if $W'P$ is of type I_n or, equivalently, if and only if $0 \neq P \leq Q_n$.*

DEFINITION 2.9 ([12]). *The multiplicity set M_E of $E(\cdot)$ is defined as the set of cardinals $\{n : Q_n \neq 0\}$.*

For $x \in H$, it is easy to verify that $[Wx] = Z_E(x)$.

DEFINITION 2.10 ([5]). *For $\mu \in \Sigma$, let $C(\mu)$ be the projection on the closed subspace $\{x \in H : \rho(x) << \mu\}$. When $\mu = \rho(x)$, $C(\mu)$ is denoted by $C(x)$. (Note that Halmos [5] uses $C(x)$ to denote $C_{[Wx]}$. But, by Theorem 66.2 of [5], $C(\rho(x)) = C_{[Wx]}$ and hence we can use $C(x)$ to denote $C(\rho(x))$.)*

The multiplicity and uniform multiplicity $u_E(\mu)$ of $\mu \in \Sigma$ with respect to $E(\cdot)$ are defined as on p.106 of Halmos [5].

PROPOSITION 2.8 ([12, Lemma 3.2]). *If $\mu \in \Sigma$ has uniform multiplicity $u_E(\mu) > 0$, then $C(\mu)$ has UH-multiplicity $u_E(\mu)$ relative to $E(\cdot)$.*

DEFINITION 2.11 ([8]). *The total H-multiplicity of $E(\cdot)$ is defined as the supremum of the H-multiplicities of all projections in W (in the order topology of cardinals). The total H-multiplicity of a normal operator T, whether bounded or not, is defined as that of its resolution* *of the identity.*

The following result is due to Theorem 64.2 of [5].

PROPOSITION 2.9 ([8]). *The total H-multiplicity of $E(\cdot)$ is equal to* $\sup\{n : n \in M_E\}$. *If $E(\cdot)$ has the CGS-property in H, then its total multiplicity coincides with its total H-multiplicity. For a normal operator T on a seperable Hilbert space, its total multiplicity is the same as its total H-multiplicity.*

3. Characterizations of Simple Spectral Measures

We define a simple spectral measure and a normal operator with simple spectrum. Using the definitions and results given in Section 2 (after Proposition 2.5) we obtain several characterizations of simple spectral measures (resp. of normal operators with simple spectra). These characterizations include in particular those given in Theorem 9.1 of Brown [2]. Finally, we deduce that the different definitions of a self-adjoint operator with simple spectrum given by Wecken [17] and Segal [15] coincide with our definition and thus they are one and the same.

DEFINITION 3.1. *$E(\cdot)$ is said to be simple if its total H-multiplicity is one. A normal operator T (not necessarily bounded) on H is said to have simple spectrum if its resolution of the identity is simple.*

THEOREM 3.1. *Let $E(\cdot)$ be an arbitrary spectral measure on $\mathcal{S}$ with values in projections of H. Then the following statements are equivalent:*

(i) *$E(\cdot)$ is simple.*

(ii) $M_E = \{1\}$.

(iii) *W is maximal abelian.*

(iv) $Z_E(x) = C(x)$ *for each* $x \in H$.

(v) *There do not exist a pair of vectors x ($\neq 0$) and y in H such that* $\rho(x) \equiv \rho(y)$ *and* $Z_E(x) \perp Z_E(y)$.

(vi) *There do not exist orthogonal nontrivial reducing closed subspaces M and N for $E(\cdot)$ such that $E(\cdot)|M$ and $E(\cdot)|N$ are unitarily equivalent.*

(vii) *The identity operator has UH-multiplicity one.*

(viii) *If* $\mathcal{J}_E = \{\rho(x) : x \in H\}$, *then* $\mathcal{J}_1 = \{\rho(x) : C(x) \leq Q_1\} = \mathcal{J}_E$.

(ix) *Every non-zero projection in* W *has UH-multiplicity one.*

(x) *If* $Z_E(x) \perp Z_E(y)$, *then* $\rho(x) \perp \rho(y)$.

(xi) *If* P' *is a projection that commutes with the range of* $E(\cdot)$, *then* $P' \in W$.

(xii) *For each* $\mu \in \Sigma$, $u_E(\mu) = 0$ *or 1 and there is some* $\mu \in \Sigma$ *with* $u_E(\mu) = 1$.

(xiii) W *is spatially isomorphic to the algebra of all multiplications of bounded measurable functions on the Hilbert space* $L_2(\Omega, \mathcal{R}, \mu)$ *for some appropriate localizable measure space* $(\Omega, \mathcal{R}, \mu)$ *in the sense of Segal [15].*

Proof.

We shall show that the statements (i) to (ix) are equivalent; then we shall prove (iv) $\Leftrightarrow$ (x); (iii) $\Leftrightarrow$ (xi); (ix) $\Leftrightarrow$ (xii) and finally, (iii) $\Leftrightarrow$ (xiii).

(i) $\Rightarrow$ (ii) by Proposition 2.9.

(ii) $\Rightarrow$ (iii) Since $M_E = \{1\}$, $Q_1 = I$ and hence, by Proposition 2.7 there exists an abelian projection $E' \in W'$ such that $C_{E'} = I$. On the other hand, by Proposition I.2.2 of [3] the abelian algebra $E'W'E'$ is isomorphic to $W'C_{E'} = W'$ and thus W' is abelian. Therefore, $W = W'$ and hence (iii) holds.

(iii) $\Rightarrow$ (iv) For $x \in H$, by Theorem 66.2 of [5] we have $C(x) = C_{[Wx]}$ and by Corollary 2 of Proposition I.1.7 of [3], $C_{[Wx]} = [W'x]$. Since $W = W'$ by (iii), it follows that $Z_E(x) = [Wx] = [W'x] = C_{[Wx]} = C(x)$. Thus (iv) holds.

(iv) $\Rightarrow$ (v) Suppose $x \in H, x \neq 0$ and $Z_E(x) \perp Z_E(y)$ for some $y \in H$. Then by (iv) we have $C(x)C(y) = 0$ and hence by Theorem 65.1 of [5], $\rho(x) \perp \rho(y)$. Since $\rho(x) \neq 0$, $\rho(x)$ is not equivalent to $\rho(y)$. This proves that (iv) implies (v).

(v) $\Rightarrow$ (vi) Let M and N be as in (vi) and let $F(\cdot) = E(\cdot)\,|M$ and $G(\cdot) = E(\cdot)\,|N$. If $F(\cdot)$ and $G(\cdot)$ are unitarily equivalent, then there is an isomorphism $U : M \to N$ such that $UF(\cdot)U^{-1} = G(\cdot)$. Let $x \in M, x \neq 0$ and let $y = Ux$. Then $y \neq 0$, $\rho(x) = \|E(\cdot)x\|^2 = \|F(\cdot)x\|^2$, $\rho(y) = \|E(\cdot)y\|^2 = \|G(\cdot)y\|^2$ and

$$\rho(x) = \|F(\cdot)x\|^2 = \|U^{-1}G(\cdot)Ux\|^2 = \|G(\cdot)y\|^2 = \rho(y).$$

Since M and N are reduced by $E(\cdot)$ and $M \perp N$, it follows that $Z_E(x) \perp Z_E(y)$. As $x \neq 0$ and $\rho(x) \equiv \rho(y)$, this contradicts the hypothesis (v) and hence (v) implies (vi).

(vi) $\Rightarrow$ (vii) Suppose (vii) does not hold. Then by Proposition 2.7 there exists $n \in M_E$ with $n > 1$. Let $x \in Q_n H$, $x \neq 0$. Then by Theorem 58.2 of [5], $P = C_{[Wx]}$ is countably decomposable in W. As $0 \neq P \leq Q_n$, again by Proposition 2.7 the projection P has UH-multiplicity n and therefore there exist abelian projections E_1', E_2' in W' such that $E_1'E_2' = 0$ and $C_{E_1'} = C_{E_2'} = P$. Since W is abelian and P is countably decomposable in W, by Proposition 2.6 there exist vectors y, z in H such that $[Wy] = E_1'$ and $[Wz] = E_2'$. Then $M = E_1'H$ and $N = E_2'H$ are orthogonal nontrivial reducing closed subspaces for $E(\cdot)$. Moreover, by Theorem 66.2 of [5], $C(y) = C_{[Wy]} = C_{E_1'} = P = C_{E_2'} = C_{[Wz]} = C(z)$ and consequently, by Theorem 65.2 of [5] we have $\rho(y) \equiv \rho(z)$. Therefore, by Theorem 60.1 of [5], $E(\cdot)|M$ and $E(\cdot)|N$ are unitarily equivalent. This contradicts the hypothesis (vi) and hence (vii) holds.

(vii) $\Rightarrow$ (viii) Since I has UH-multiplicity one, by Proposition 2.7 the central projection $Q_1 = I$ and hence (viii) holds.

(viii) $\Rightarrow$ (ix) Since $\mathcal{J}_1 = \mathcal{J}_E$, it follows that $Q_1 = I$ and consequently, by Proposition 2.7 the statement (ix) holds.

(ix) $\Rightarrow$ (i) Obvious from Definition 3.1.

Thus the statements (i) to (ix) are equivalent.

(iv) $\Rightarrow$ (x) Suppose $Z_E(x) \perp Z_E(y)$. Then by (iv) we have $C(x)C(y) = 0$ and hence, by Theorem 65.2 of [2], $\rho(x) \perp \rho(y)$. Thus (x) holds.

(x) $\Rightarrow$ (iv) Suppose (iv) does not hold. Then there exists $x \in H$ such that $Z_E(x) \subset C(x)$ and $Z_E(x) \neq C(x)$. If $y \in C(x) \ominus Z_E(x)$, $y \neq 0$, then clearly $Z_E(x) \perp Z_E(y)$ and $C(y) \subset C(x)$. But, by (x) we have $\rho(x) \perp \rho(y)$ and consequently, by Theorem 65.2 of [5], $C(x)C(y) = 0$. In other words, $C(x)C(y) = C(y) = 0$ and hence $y = 0$, a contradiction. Hence (x) implies (iv).

(iii) $\Rightarrow$ (xi) If P' is a projection commuting with the range of $E(\cdot)$, then $P' \in W'$. Since $W' = W$ by (iii), $P' \in W$ and hence (xi) holds.

(xi) $\Rightarrow$ (iii) Since W' is the von Neumann algebra generated by all projections in W', the hypothesis (xi) implies that $W' \subset W$. Since W is abelian, it then follows that $W = W'$ and hence (iii) holds.

(ix) $\Rightarrow$ (xii) Let $x \in H, x \neq 0$. Then by (ix), $C(x)$ has UH-multiplicity one. Let $0 \neq \nu << \rho(x)$, $\nu \in \Sigma$. Then by Theorem 65.3 of [5] there exists $y \in C(x)H$ such that $\nu = \rho(y)$ so that $C(\nu) = C(y)$ and hence by (ix), $C(\nu)$ has UH-multiplicity one. This shows that $u_E(\rho(x)) = 1$. If $u_E(\mu) \neq 0$ for some $\mu \in \Sigma$, then for every $\nu \in \Sigma$ with $0 \neq \nu << \mu$ we have $C(\nu) \neq 0$ and hence the hypothesis (ix) implies that $u_E(\mu) = 1$. Thus (xii) holds.

(xii) $\Rightarrow$ (ix) Let P be a non-zero projection in W. Let $\{x_\alpha\}_{\alpha \in J}$ be an orthogonal family of non-zero vectors in H such that $\{[W'x_\alpha]\}_{\alpha \in J}$ is a maximal orthogonal family of subprojections of P. Let $E_\alpha = [W'x_\alpha]$. By maximality, $\sum_{\alpha \in J} E_\alpha = P$. Moreover, by Corollary 2 of Proposition I.1.7 of [3] and by Theorem 66.2 of [5] we have $E_\alpha = C_{[Wx_\alpha]} = C(x_\alpha)$. If $0 \neq \nu << \rho(x_\alpha)$, $\nu \in \Sigma$, then as in the proof of (ix) $\Rightarrow$ (xii) we have $C(\nu) \neq 0$ and similarly, $C(w) \neq 0$ for $0 \neq w << \nu$, $w \in \Sigma$ so that $u_E(\nu) \neq 0$. Consequently, by (xii) we have $U_E(\nu) = 1$. This shows that $\rho(x_\alpha)$ has uniform multiplicity one for each $\alpha \in J$. Then by Proposition 2.8, $C(x_\alpha)$ has UH-multiplicity one and consequently, by Theorem 64.3 of [5] we conclude that P has UH-multiplicity one and hence (ix) holds.

(iii) $\Leftrightarrow$ (xiii) by Theorem 1 of Segal [15].

COROLLARY 3.1. *If T is a normal operator on H, then T has simple spectrum if and only if its resolution of the identity $E(\cdot)$ on $\mathcal{S} = \mathcal{B}(\sigma(T))$ satisfies anyone of the equivalent conditions of Theorem 3.1.*

REMARKS 3.1.

(a) *As noted in the paragraph prior to Remarks 3.7 of [12], the statement* (viii) *of Theorem 3.1 is the same as* (i) *of Theorem 9.1 of Brown [2]. Moreover, the equivalence among the statements* (iv), (vi), (viii), (x) *and* (xi) *of Theorem 3.1 have already been established in the said theorem of [2]. However, we include here more characterizations and our proof is based on von Neumann algebras and the results of Halmos [5]. The present study also brings out clearly how von Neumann algebras play a key role in the unitary invariance problem. Such a unified treatment is absent in the work of Brown [2].*

(b) *By the equivalence of the statements* (i) *and* (v) *(resp.* (i) *and* (iii)) *of Theorem 3.1, a self-*

adjoint operator T on an arbitrary Hilbert space H has simple spectrum if and only if it does so in the sense of Wecken [17] or Plesner and Rohlin [13] (resp. in the sense of Segal [15]).

4. Generalization of a Theorem of von Neumann

If T is a self-adjoint operator on a separable Hilbert space, then every bounded operator A that commutes with all the operators commuting with the resolution of the identity of T is given by

$$A = \int_{\sigma(T)} f(\lambda) dE(\lambda)$$

for some bounded complex Borel function f on $\sigma(T)$. This result is due to von Neumann (see Theorem XVII.3.22 of [4] or see p.351, Section 129 of [14]). Presently, we generalize the above theorem to spectral measures with the CGS-property in H.

Let $\mathcal{L}(H)$ denote the C^*-algebra of all bounded operators on H.

THEOREM 4.1. *Suppose $E(\cdot)$ has the CGS-property in H. Then a bounded operator A commutes with every bounded operator that commutes with the range of $E(\cdot)$ if and only if A is of the form*

$$A = \int_X f dE$$

for some bounded $\mathcal{S}$-measurable complex function f on X. Consequently, W coincides with $\mathcal{F} = \{S(f) : f \in B(\mathcal{S})\}$, where $B(\mathcal{S}) = \{f : X \to \mathbb{C},\ f\ \mathcal{S}\text{-measurable and bounded}\}$ and

$$S(f) = \int_X f dE\ ,\ f \in B(\mathcal{S}).$$

Moreover, the set of all projections in W coincides with the range of $E(\cdot)$.

Proof. Clearly, the condition is sufficient. Since $E(\cdot)$ is strongly countably additive, the range $\mathbf{E}$ of $E(\cdot)$ is a σ-complete Boolean algebra of projections in H. If $\mathcal{R}$ is the linear span of $\mathbf{E}$, then $\mathcal{R}$ is a *-subalgebra of $\mathcal{L}(H)$ containing the identity. As $E(\cdot)$ has the CGS-property in H, by Lemma XVII.3.21 of [4] $\mathbf{E}$ is a complete Boolean algebra of projections.Therefore, by Corollary XVII.3.17 of [4], $\bar{\mathcal{R}}^{\tau_w} = \bar{\mathcal{R}}^{\tau_n}$, where $^{-\tau_w}$ denotes the closure in the weak operator topology τ_w in $\mathcal{L}(H)$ and $^{-\tau_n}$ denotes the closure in the uniform operator topology τ_n in $\mathcal{L}(H)$. Moreover, by Lemma XVII.3.6 of [4] the set of all projections in $\bar{\mathcal{R}}^{\tau_w}$ coincides with $\mathbf{E}$. On the other hand, by Corollary 1 of Theorem I.3.2 of [3] we have $W = \bar{\mathcal{R}}^{\tau_w}$. Thus $W = \bar{\mathcal{R}}^{\tau_n}$.

Let $\mathcal{F} = \{S(f) : f \in B(\mathcal{S})\}$. By Corollary X.2.9 of [4], $\mathcal{F}$ is a *-subalgebra of $\mathcal{L}(H)$ containing the identity and is closed in the uniform operator topology τ_n. As $\mathcal{R} \subset \mathcal{F}$ and $W = \bar{\mathcal{R}}^{\tau_n}$, it follows that $W \subset \mathcal{F}$. On the other hand, every element of $\mathcal{F}$ is obtained as the limit of a sequence of elements from $\mathcal{R}$ in the uniform operator topology (see p.892 of [4]) and thus we conclude that $W = \mathcal{F}$.

Let us now suppopse that $A \in \mathcal{L}(H)$ commutes with every bounded operator that commutes with the range of $E(\cdot)$. Then $A \in W'' = W$ and hence the condition is also necessary.

The following corollary is immediate from the proof of the above theorem.

COROLLARY 4.1. *If $E(\cdot)$ has the CGS-property in H, then the range of $E(\cdot)$ is a complete Boolean algebra of projections in H. Moreover, the range of $E(\cdot)$ coincides with the collection of all projections in W, the von Neumann algebra generated by $E(\cdot)$.*

5. Characterizations of Simple Spectral Measures with the CGS-property

Using the definitions and results given in Section 2 upto Proposition 2.5 and the results in Sections 3 and 4 we give several characterizations of a simple spectral measure $E(\cdot)$ with the CGS-property in H. We then deduce in Remarks 5.1 that a self-adjoint operator on a separable Hilbert space has simple spectrum if and only if it does so in the sense of Stone [16] (resp. in the sense of Akhiezer and Glazman [1]). We also obtain Lemma 1.2 of Segal [15] as a corollary of the equivalence of the statements (vi) and (ix) of Theorem 5.1.

THEOREM 5.1. *Let $E(\cdot)$ be a spectral measure on $\mathcal{S}$ with the CGS-property in H. Then the following statements are equivalent. Moreover, they are equivalent to each one of the statements of Theorem 3.1.*

(i) *$E(\cdot)$ is simple.*

(ii) *$E(\cdot)$ has total multiplicity one.*

(iii) *$E(\cdot)$ has OSD-multiplicity one.*

(iv) *$E(\cdot)$ has UOSD-multiplicity one.*

(v) *$E(\cdot)$ has OSR-multiplicity one.*

(vi) *W has a generating vector.*

(vii) *W has a generating-separating vector.*

(viii) *Every projection in W' is cyclic.*

(ix) *There exists a measure $\mu \in \Sigma$ such that W is spatially isomorphic to the algebra of multiplications by bounded $\mathcal{S}$-measurable functions on $L_2(X, \mathcal{S}, \mu)$.*

(x) *Suppose X is a Hausdorff space and $\mathcal{S} = \mathcal{B}(X)$. Then $m_p(t) = 0$ or 1 and $m_c(t) = 0$ or 1 for each $t \in X$ and there exists $t_o \in X$ such that $max(m_p(t_o), m_c(t_o)) = 1$.*

Proof. In the light of Theorem 3.1 it suffices to prove the equivalence of the statements (i) to (x).

Since $E(\cdot)$ has the CGS-property in H, the equivalence of the statements (i), (ii), (iii) and (v) is immediate from Propositions 2.4 and 2.9. When $E(\cdot)$ has OSD-multiplicity one, trivially every OSD of H relative to $E(\cdot)$ is a uniform OSD and consequently, by Proposition 2.2 the statements (iii) and (iv) are equivalent. Thus the statements (i) to (v) are equivalent.

Now we shall prove (i) $\Leftrightarrow$ (vi); (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (vi); (vi) $\Leftrightarrow$ (ix) and (ii) $\Leftrightarrow$ (x).

(i) $\Rightarrow$ (vi) As $E(\cdot)$ has the CGS-property in H, W is countably generated. Thus, by Corollary to Proposition I.2.6 of [3], W has a separating vector and hence W' has a generating vector. If (i) holds, then by Theorem 3.1 W is maximal abelian so that W $(=W')$ has a generating vector.

(vi) $\Rightarrow$ (i) Let W have a generating vector x. Then $[Wx] = H$ and hence by Corollary 2 of Proposition I.6.4 of [3], W is maximal abelian. Thus $E(\cdot)$ is simple by Theorem 3.1.

(vi) $\Rightarrow$ (vii) As shown above, (vi) implies that W is maximal abelian and hence by Corollary to Proposition I.1.5 of [3] W has a generating-separating vector.

(vii) $\Rightarrow$ (viii) If $[Wx] = H$, then for each projection $P' \in W'$ we have $P' = P'[Wx] = [WP'x]$ and hence P' is cyclic. Thus (viii) holds.

(viii) $\Rightarrow$ (vi) By the hypothesis (viii), the identity operator as a member of W' is cyclic and hence these exists $x \in H$ such that $[Wx] = I$. Thus (vi) holds.

(vi) $\Rightarrow$ (ix) Let x be a generating vector of W and let $\rho(x) = \mu$. Then by Theorem 60.1 of [5] there exists an isomorphism $U : Z_E(x) \to L_2(X, \mathcal{S}, \mu) = K$ (say) such that

$$UE(\cdot)U^{-1}f = \chi_{(\cdot)}f, \ f \in K. \tag{1}$$

Let $B(\mathcal{S}) = \{f : X \to \mathbb{C},\ f\ \mathcal{S}\text{-measurable and bounded}\}$. For $g \in B(\mathcal{S})$, we define $M_g f = gf,\ f \in K$. Clearly, $M_g \in \mathcal{L}(K)$. Let $\mathcal{R} = \{M_g : g \in B(\mathcal{S})\}$. It is easy to verify that $\mathcal{R}$ is a C^*-subalgebra of $\mathcal{L}(K)$.

Given $g \in B(\mathcal{S})$, there exists a bounded sequence (s_n) of $\mathcal{S}$-simple functions on X such that

$$\sup_{t \in X} |s_n(t) - g(t)| \to 0$$

as $n \to \infty$. Then, for each $f \in K$,

$$\|gf - s_n f\|_2 \le \sup_{t \in X} |g(t) - s_n(t)| \, \|f\|_2 \to 0$$

as $n \to \infty$. Thus $M_g = \lim_n M_{s_n}$ in the uniform operator topology.

For an $\mathcal{S}$-simple function $s = \sum_{i=1}^{p} \alpha_i \chi_{\sigma_i}$ we have

$$S(s) = \int_X s dE = \sum_{i=1}^{p} \alpha_i E(\sigma_i).$$

Then by (1) it follows that

$$US(s)U^{-1}f = (\sum_{i=1}^{p} \alpha_i UE(\sigma_i)U^{-1})f = sf = M_s f$$

for $f \in K$. Thus we have

$$US(s_n)U^{-1}f = M_{s_n}f, \ f \in K \tag{2}$$

for all n. On the other hand, as on p. 892 of [4],

$$S(g) = \lim_n S(s_n) \tag{3}$$

in the uniform operator topology. Therefore, by (2) and (3) and by the fact that $\|M_{s_n} - M_g\| \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned}\|(US(g)U^{-1} - M_g)f\|_2 &\le \|(US(g)U^{-1} - US(s_n)U^{-1})f\|_2 + \|(US(s_n)U^{-1} - M_g)f\|_2 \\ &\le \|S(g) - S(s_n)\| \, \|f\|_2 + \|M_{s_n} - M_g\| \, \|f\|_2 \to 0\end{aligned}$$

as $n \to \infty$, for each $f \in K$. This shows that

$$US(g)U^{-1} = M_g.$$

Consequently, $\mathcal{R}$ is spatially isomorphic to $\{S(g) : g \in B(\mathcal{S})\}$. Since $W = \{S(g) : g \in B(\mathcal{S})\}$ by Theorem 4.1, it follows that W is spatially isomorphic to $\mathcal{R}$ and hence (ix) holds.

(ix) $\Rightarrow$ (vi) Taking $\mathcal{R}$ as in the above, it is easy to observe that the constant function 1 is a generating vector for $\mathcal{R}$ and consequently, W,the spatial isomorphic image of $\mathcal{R}$, has a generating vector. Thus (vi) holds.

(ii) $\Leftrightarrow$ (x) by Proposition 2.5.

COROLLARY 5.1. *If T is a normal operator on a separable Hilbert space H, then T has simple spectrum if and only if its resolution of the identity satisfies anyone of the statements in Theorem 3.1 or in Theorem 5.1.*

REMARKS 5.1. *By the equivalence of the statements* (i) *and* (ii) *(resp.* (i) *and* (x)) *of Theorem 5.1, a self-adjoint operator on a separable Hilbert space has simple spectrum if and only if it does so in the sense of Akhiezer and Glazman [1] (resp. in the sense of Stone [16]).*

The following result (Lemma 1.2 of [15]) is deduced as a corollary of the equivalence of (vi) and (ix) in the above theorem.

COROLLARY 5.2. *An abelian von Neumann algebra with generating vector is spatially isomorphic to the algebra of multiplications by bounded measurable functions on L_2 over a finite perfect measure space.*

Proof. Let $\mathcal{A}$ be an abelian von Neumann algebra on H with its maximal ideal space $\mathcal{M}$. Since $\mathcal{M}$ is extremally disconnected, for each $\sigma \in \mathcal{B}(\mathcal{M})$ there is a unique clopen set $\Upsilon(\sigma)$ such that $\Upsilon(\sigma)\Delta\sigma$ is meagre in $\mathcal{M}$ (see p. 159 of [6]). Let $G(\sigma) = \Phi^{-1}(\chi_{\Upsilon(\sigma)})$, where $\Phi : \mathcal{A} \to C(\mathcal{M})$ is the Gelfand isomorphism. Then, as shown on pp. 159-160 of [6], $G(\cdot)$ is a spectral measure on $\mathcal{B}(\mathcal{M})$ and $\mathcal{A}$ is the von Neumann algebra generated by the range of $G(\cdot)$. If $\mathcal{A}$ has a generating vector $x \in H$, then $[G(\sigma)x : \sigma \in \mathcal{B}(\mathcal{M})] = [\mathcal{A}x] = H$ and hence $G(\cdot)$ has the CGS-property in H. Now the equivalence of (vi) and (ix) in Theorem 5.1 establishes the corollary.

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Departamento de Matemáticas, Facultad de Ciencias, Universidad de los Andes, Mérida,Venezuela.
Author's e-mail address: panchapa@ciens.ula.ve