Continuity of invariant measures for families of 1-dimensional maps .

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In this note we shall discuss 1-parameter families of one dimensional maps which experience bifurcations , but nevertheless there exist some natural invariant measures which vary continuously with the paramater . This is , of course ,a kind of stability , in the sense that the outcome of evaluating continuous functions along orbits does not change much under small perturbations of the parameter . Perhaps this is what we observe in computer experiments ,where the picture obtained by plotting an orbit seems to be independent of the starting point and truncation errors .

Let us recall some definitions in order to establish the results contained in this paper :

Let $f : N \to N$ be a C^r map with $N = [0,1]$ or S^1, $r \geq 1$, and let $C(N)$ denote the space of all real valued continuous functions defined on N . If μ is an f-invariant Borel probability ergodic measure , the set of **generic points** is define as

$$G_\mu = \{ x \in N : \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) \to \int \varphi d\mu \text{ when } n \to \infty \text{ for all } \varphi \in C(N) \},$$

and we shall say that an f-invariant Borel probability measure μ is of Bowen-Ruelle-Sinai (**BRS**) if it is ergodic and its set of future generic points has positive Lebesgue measure.

The problem , posed by Bowen [1], we are interested is the following :

Suppose $f_t : N \to N$ is a continuous family of C^r maps, possibly with singularities, with $t \in [-1,1]$, and for each t there exists a unique BRS measure μ_t so that $m(G_{\mu_t}) = m(N)$. Does μ_t vary continuously in the weak star topology ? .

The answer to this question is clearly no . However , it can be reformulated in the following terms : Set

$$\mathcal{M}^+(t) = \{ \Sigma\alpha_i\mu_i : \Sigma\alpha_i = 1 \text{ and for each i the measure } \mu_i \text{ is BRS} \} ,$$

then we can ask if for each t there exists a measure $\mu_t \in \mathcal{M}^+(t)$ so that the function $t \to \mu_t$ is continuous and if we write $\mu_t = \Sigma\alpha_i(t)\mu_i(t)$ then $m(UG_{\mu_{i(t)}}) = m(N)$. We should pointed out that here we are using a version suggested by Wellington De Melo .

A positive answer to this question implies persistence on the output of the system under small variations of the parameter for a given input, that means that our "observations" vary continuously with the parameter although there may be no structural stability.

In this paper we shall prove that for some families of maps experiencing bifurcations, and therefore not being structutally stable [12], their BRS measures , in the above sense, varies continuously . These are families of Lorenz maps of the interval, generic families of immersions of the circle and the 2-dimensional DA family .

The continuity and persistence of the Bowen-Ruelle-Sinai measure is certainly a sort of stability "a la ergodique", and perhaps this is a more realistic definition than the ones demanding metric or topologiacal conjugacy. The geometric models of the Lorenz attractor are not structural stable, see [4], but their Bowen-Ruelle-Sinai measures vary continuously and therefore small perturbations will not alter much the outcome of the system. A recent paper of E.C. Zeeman [14] has introduced a new definition of structural stability which seem to be related to our examples and the role of expansiveness in our proofs suggests a link with Lewowicz´work [6] .

We shall divided our exposition acordingly to the examples , but first we need to recall some definitions and results from ergodic theory ..

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§1. Entropy, Lyapunov exponents and BRS measures .

To prove our results we need some basic facts and results from ergodic theory which we summarize in this section, for more detail information we refer to [15] .

Let us consider the measurable space N with the Borel σ-algebra **B** and let μ be a Borel probability measure on N . The **entropy of a finite measurable partition** $\xi = \{C_1, \ldots, C_k\}$ of N is defined as the number

$$H_\mu(\xi) = \sum_{i=1}^{k} -\mu(C_i) \log \mu(C_i) ,$$

with 0log0 = 0 by convention. If $\xi = \{C_1, \ldots, C_k\}$ and $\eta = \{D_1, \ldots, D_s\}$ are finite measurable partitions their **joint partition** is given by

$$\xi \vee \eta = \{ C_i \cap D_j : 1 \le i \le k, 1 \le j \le s \}.$$

Now let $f : N \to N$ be a continuous map , suppose that μ is a Borel f-invariant probability measure and for $n > 0$ write ξ_n for $\xi v f^{-1}\xi v ... v f^{-n+1}\xi$. Clearly $H(\xi) = H(f^{-1}\xi)$ and $H(\xi_{n+m}) \le H(\xi_n) + H(\xi_m)$, then

$$\lim_{n \to \infty} \frac{1}{n} H(\xi_n)$$

exists , we shall denote it by $h_\mu(f,\xi)$ and call it the **entropy of** f **with respect to** ξ . The **entropy of f with respect to** μ is defined as

$$h_\mu(f) = \sup \{ h_\mu(f,\xi) : \xi \text{ is a finite measurable partition } \} .$$

Sometimes we shall refer to $h_\mu(f)$ as **the entropy of** μ .

A partition is called an **entropy-generator** for μ if $\vee f^{-i}\xi = \mathbf{B}$ mod 0 . The Kolmogorov-Sinai Theorem establishes that if ξ is an entropy-generator for μ , then $h_\mu(f) = h_\mu(f,\xi)$.

We shall need a lemma cocerning the upper semicontinuity of entropy for families of maps , although we deal with non-invertible maps, for simplicity we shall prove the lemma for homeomorphisms of compact metric spaces .

Lemma 1.1.

Let $f_t: X \to X$ be a continuous family of homeomorphisms of a compact metric space X, and for each $-1 \le t \le 1$ the transformation f_t preserves a measure μ_t . Suppose there exists a finite measurable partition ξ which is an entropy generator for each μ_t and $\mu_0(\partial\xi) = 0$. Then if $\mu_t \to \mu_0$ as $t \to 0$, we have that

$$\limsup_{t \to 0} h_{\mu_t}(f_t) \le h_{\mu_0}(f_0) .$$

Proof :

Since ξ is a common generator for all t's, we have that

$$h_{\mu_t}(f_t) = h_{\mu_t}(f,\xi) = \inf \frac{1}{n} H_{\mu_t}(\xi_n(f_t)) ,$$

where $\xi_n(f_t) = \xi \vee f_t^{-1}\xi v .... v f_t^{-n+1}\xi$.

Fix $\alpha > 0$ and choose n suffcientely large so that

$$\frac{1}{n} H_{\mu_0}(\xi_n(f_0)) \leq h_{\mu_0}(f_0) + \alpha .$$

Since f_t is a continuous family , $\mu_t \to \mu_0$ as $t \to 0$ and n is fixed

$$| H_{\mu_0}(\xi_n(f_0)) - H_{\mu_t}(\xi_n(f_t)) | \to 0$$

as $t \to 0$.

Therefore for small t we have that

$$h_{\mu_0}(f_0) \geq \frac{1}{n} H_{\mu_0}(\xi_n(f_0)) - \alpha \geq \frac{1}{n} H_{\mu_t}(\xi_n(f_t)) - 2\alpha$$

$$\geq h_{\mu_t}(f_t, \xi) - 2\alpha = h_{\mu_t}(f_t) - 2\alpha .$$

Which proves the lemma .

If $f: N \to N$ is a differentiable map we can define the **upper Lyapunov exponent of f at x** as

$$\bar{\chi}(x) = \limsup_{n \to \infty} \frac{1}{n} \log | D_x f^n | .$$

When the limit in the above formula exists we say that the **Lyapunov exponent at x** exists and shall be denoted by $\chi(x)$. The chain rule implies that

$$\chi(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} \log | f'(f^i(x)) | ,$$

and by the Ergodic Theorem if μ is an f-invariant Borel probability measure for μ-almost all points the Lyapunov exponet exists, furthermore if μ is ergodic it is constant almost everywhere, say $\chi(x) = \chi_\mu$, and equals $\int \log | f' | \, d\mu$. Thus we can talk about the Lyapunov exponent of an ergodic measure .

Several results link the entropy of a measure with its Lyapunov exponent, see [5],[9] for references, in particular Pesin's formula says that if μ is absolutely continuous then the entropy is equal to the exponent . Ledrappier [5] was the first one to obtain a converse of Pesin's entropy formula , see also [9] for an alternative proof .

Theorem [5] .

Let $f: N \to N$ be a C^2 map, possibly with a finite number of singularities . If μ is an f-invariant ergodic probability measure with $h_\mu(f) = \chi_\mu > 0$, then μ is absolutely continuous with respect to the Lebesgue measure and therefore a BRS measure .

§2. The Poincaré maps of geometric models of the Lorenz attractor .

The Lorenz attractor [7] is an attracting set of a flow associate to a system of ordinary differential equations in R^3 . The dynamics of this flow still is not riguruous understood, but the computer simulatios have given rise to some geometric models for which a complete description have been obtained by Gukenheimer and Williams, [3], [4] and also see Sparrow [13], by studying a semiflow on a branched 2-manifold which turns out to be the suspension of map of the interval with a discontinuity where the derivative goes to infinity. The topological class of the attractor is determined by the kneading sequence of the end points of the one dimensional maps and the singularity, which implies that the geometric models of the Lorenz attractor are not structually stable [4] . Robinson [11] has shown that if a model flow is C^{20} the stable manifold is differentaible and the one dimensional map satisfies the conditions listed below for a C^2 open neighbourhood of this flow .The "Poincaré maps" of the geometric Lorenz attractors are decribe by the following set of conditions :

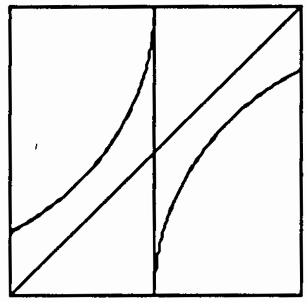
i) g has a single discontinuity at some $x=c$;

ii) the limit of g from the left side of c is 1 , and the one from the right side is 0, $g(0) < c < g(1)$;

iii) g is non-uniformily continuously differentiable on $[0,1]\backslash\{c\}$ and there is a $\lambda > 1$ such that $|g'(x)| > \lambda$ for all $x \neq c$;

iv) the limit of $g'(x)$ is infinity as x appoaches c from either side .

v) the inverse branches of f are C^{1+r} for some $r > 0$.



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Picture 1

Robinson shows in [10] that for a map satisfying the above conditions with $\lambda \geq \sqrt{2}$, there exists a unique BRS measure.

We can define a C^1 metric on the set of Lorenz maps of the interval having the same singularity at a point c , by saying that the distance between two Lorenz maps equals the supremum of the distances of their restriction to the complements of all ε-neighbourhoods of c. To be more precise , let us denote by $d_1(,)$ the standar C^1 metric of maps well defined on subintervals and write f_ε for the restriction of f to the interval $[0,1] \setminus (c-\varepsilon,c+\varepsilon)$, then define the C^1 metric on the set of Lorenz maps as

$$d_1^{\infty}(f,g) = \sup_{\varepsilon > 0} \{ d_1(f_\varepsilon,g_\varepsilon) \mid \varepsilon > 0 \} .$$

In the set of all Lorenz maps, the metric can be defined as follows : In the case that we have no common singularity for f and g , we first fixed f and make a change of coordinates on g, through a conjugacy h(g), so that they have the singularity on the same point, similarly we procced fixing g to obtain h(f) . The C^1 metric is then defined as the maximum of the distance of the conjugacies to the identity, and the C^1 distance of maps after conjugacies .

Theorem 2.1.

There exists an open set O of C^2 flows in R^3 containing a C^{20} geometric model of the Lorenz attractor so that for each flow $\varphi \in O$, there exists a Lorenz map $f = f(\varphi)$ of the interval admiting a unique ergodic measure μ_f absolutely contnuous with respect to the Lebesgue measure so that the function $f \rightarrow \mu_f$ is continuous .

Proof :

By Robinson [10] there exists an open set O of C^2 flows in R^3 containing a C^{20} geometric model of the Lorenz attractor so that for each flow $\varphi \in O$, there exists a Lorenz map $f = f(\varphi)$ of the interval admiting a unique ergodic measure μ_f absolutely contnuous with respect to the Lebesgue measure. Furthermore, the proof of this statement and [11] yields that the interval maps vary "continuously" in the C^1 metric.

To simplify our exposition let us consider a continuous ,in the C^1 metric , family of Lorenz maps $\{f_t\}$ having a common singularity at c and admitting a BRS measure μ_t. Since they are continuous in the C^1 metric their inverse branches vary continuously as well, and we can apply Lemma 1.1 . Assume that $0 \leq t \leq 1$ ant let $\mu_{t_k} = \mu_k$ be a convergent sequence of BRS measures with $t_k \to 0$ as $k \to \infty$, say $\mu_k \to \mu$. Since each μ_k is absolutely continuous with respect to the Lebesgue measure we have that for n sufficientely large

$$\begin{aligned} 0 &= \lim_{k\to\infty} \{ h_{\mu_k}(f_{t_k}) - \int \log |Df_{t_k}| \, d\mu_k \} \\ &\leq \limsup_{k\to\infty} \{ h_{\mu_k}(f_{t_k}) - \int \min \{ \log |Df_{t_k}| , n \} \, d\mu_k \} \\ &\leq h_\mu(f) - \int \min \{\log |Df| , n\} \, d\mu . \end{aligned}$$

Which obviously implies that

$$h_\mu(f) = \int \log |Df| \, d\mu .$$

Then applying a Theorem of Leddrappier [5], we have that by uniqueness that $\mu = \mu_0$, and the proof is completed .

§3. Inmersions of the circle and DA-families .

By Mañé [8] in the generic families of C^2 inmersions of the circle, bifuracations only occur by the appearance of a nonhyperbolic peridic points . Thus in a generic family the bifurcation set is discrete and the only interesting cases are the one concerning the DA-type bifurcations, otherwise one can prescribe the BRS make their changes continuous. So we shall just describe the continuity throug the DA-type bifurcation . The standard DA-bifurcation was introduced by Smale [12], here we shall use its 1-dimensional version which starts with an expanding map of the circle and bifurcates into a map whose non-wandering set consists of a repellor and a sink .

Let us consider a continuous family $\{f_t\}$ of C^2 inmersions of the circle with $-1 \leq t \leq 1$, f_{-1} an expanding map, at t=0 a fixed point is created through a generic saddle-node bifurcation and for $t > 0$ the maps are Axiom A with non-wandering set consisting of a sink and a repellor . For $t < 0$ there exists a unique invariant ergodic measure μ_t which is absolutely continuous with respect to the Lebesgue measure. At $t = 0$ the non-wandering set $\Omega(f_0)$ is transitive, has measure zero and all its periodic points are hyperbolic except the saddle-node . For $t > 0$ the stable manifold of the sink has measure 1 .

Theorem 3.1.

Let $\{f_t\}$ be a continuous family of C^2 inmersions of the circle as above, then for each t there exists a BRS measure μ_t which vary continuously with t .

Proof .

For each $t < 0$ there exists a unique BRS measure absolutely continuous with respect to the Lebesgue measure, and they vary continuously by construction . For $t > 0$, the BRS measures are the Dirac measures supported on the the unique sinks . For t=0 the BRS maesure is given by the Dirac measure in the saddle-node point, the nonwandering set , following Díaz and Viana, has Hausdorf dimension less than 1, and therefore Lebesgue measure zero . Now let $\mu_{t_k} = \mu_k$ be a convergent sequence of BRS measures with $t_k \to 0^+$ as $k \to \infty$, say $\mu_k \to \mu$. Obviusly μ is an f-invariant measure . Since each μ_k is absolutely continuous with respect to the Lebesgue measure, then by Lemma 1.1 we have that

$$0 = \lim_{k\to\infty} \{ h_{\mu_k}(f_{t_k}) - \int \log |Df_{t_k}| \, d\mu_k \}$$

$$\leq h_{\mu}(f_0) - \int \log |Df_0| \, d\mu$$

Which obviously implies that for μ its entropy equals its Lyapunov exponent, by Díaz and Viana [2] the nonwandering set has Hausdorff dimension less than 1 and therefore μ has zero entropy and has to be concentrated in the saddle-node fixed point . For $t > 0$

the BRS measure is just the Dirac measure on the sinks .

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Corollary 3.2.

Let $\{f_t\}$ be a generic 1-parameter family of C^2 inmersions of the circle . Then for each t there exists a measure μ_t which is a convex combination of BRS measures with the the union of generic points having Lebesgue measure 1 , so that the function $t \to \mu_t$ is continuous .

Proof :

The only case left to dicuss is when family bifurcates from an Axiom A situation where all the BRS measures are sinks to a new Axiom A case with one extra sink . This happens generically through the creation of a nonhyperbolic periodic point and we have either a saddle-node or flip type bifurcation . So let $\{f_t\}$ be a family of C^2 inmersions of the circle where only one bifurcation occurs, say at $t = 0$. For all t we define μ_t as a convex combination of the Dirac measures define on the periodic sinks according to thier periods and Lyapunov exponents.This definition claerly makes our theorem work .

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Corollary 3.3 .

The Hausdorff dimension of BRS measures is not a continuous function .

Proof :

Let us recall that the Hausdorff dimension of a measure μ is defined as

$$HD(\mu) = \inf \{ HD(Y) \mid \mu(Y) = 1 \} .$$

Then clearly for $t < 0$ we have $HD(\mu_t) = 1$ and $HD(\mu_t) = 0$ for $t \geq 0$.

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The 2-dimensional DA-family $\{f_t\}$, can be studied with the same arguments used above . we recall that in surface case we start at t=-1 with a linear Anosov diffeomorphism, at t=0 a generic saddle-node bifurcation occurs by the creation of a new non-hyperbolic fixed point , afterwards we have an Axiom A diffeomorhism whose nonwandering set consists of an attractor and a source , see [2] for a complete desription of the DA-family .

Theorem 3.4 .

Let $\{f_t\}$ be a continuous family of C^2 diffeomorphisms of the 2-torus as above, then for each t there exists a BRS measure μ_t which vary continuously with t .

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