TWO COMPETING SPECIES IN AN T-PERIODIC ENVIRONMENT IN N-DIFFERENT LOCATIONS

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ABSTRACT: We consider a model for competition between two species which may be situated in several locations. We give conditions under which the amounts of the species in the various locations tend to become equal with increasing time.

1. INTRODUCTION.

In this work, we consider a model for competition between two species in which the two species are situated in N different locations. We assume that there may be movement of each of the species from a given location to another location. We assume that the rate of such movement is proportional to the difference of the amounts of the species present in the two locations and it is possible for each species to move from any location to another. However, we assume that in each location the growth rate, rate of self destruction and

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where S_L and S_M denote the minimum and the maximum of a function S respectively, it was proved in [1] that such condition imply the existence of a unique T-periodic soltuion of (1.1) with both components positive which is asymptotically stable and attracts all solutions starting in the open first quedrant of the u-v-plane.

In section 2 we fix the conditions under which we work and the notation to be used. In section 3 we stablish and we prove various theorems which lead us to show the existence and uniqueness of a T-periodic solution with the 2N-components positive for our problem which will be globally asymptotically stable.

2. CONDITIONS AND NOTATION.

In this work we used as fundamental refference the paper [1] which consider the problem (1.1) together with condition (1.2) given in the introduction . In [1] was proved that condition (1.2) imply the existence and uniqueness of a solution $(u_o(t), v_o(t))^T$ of (1.1) which is T-periodic, with both components positive and globally asymptotically stable with respect to the solutions of (1) with initial values in the first quadrant of the u-v plane. Here

$$(u_o(t),\ v_o(t))^T = \mathrm{col}(u_o(t),\ v_o(t)),$$

it is colum vector.

Now, we consider two species u and v competing in un T-periodic environment in N locations where each species can move from one location to another.

We, in the next section, will show the existence and uniqueness of a T-periodic solution $(\vec{u}_o, \vec{v}_o)^T$ which is positive in a sense to be determined later, and also we show that this solution is globally asymptotically stable.

In order to do that, we fix the following notation:
$U_i(t)$ denotes the amount of the species u at location $i(i = 1,...,N)$, in analogue form

$V_i(t)$ denotes the amount of the species V at location $i(i = 1,...,n)$. Migration of the species u from location i to location j per unit time at time t will be denoted by $k_{ij}(u_j(t) - u_i(t))$ where $k_{ij} \geq 0$ is a constant. Similarly, $m_{ij}(v_j(t) - v_i(t))$ denotes the migration of v from i to j per unit time at time t where $m_{ij} \geq 0$ is a constant. With this in mind, we have the following system of differential equations:

$$(2.1)\quad \begin{cases} u_i'(t)=\sum_{j=1}^{N} k_{ij}(u_j(t)-u_i(t))+u_i(t)[a(t)-b(t)u_i(t) \\ \qquad\qquad -c(t)v_i(t)] \\ v_i'(t)=\sum_{j=1}^{N} m_{ij}(v_j(t)-v_i(t))+v_i(t)[d(t)-e(t)u_i(t) \\ \qquad\qquad -f(t)v_i(t)]. \end{cases}$$

Where a,b,...,f are as in (1.1), and $u_i(t)\geq 0$, $v_i(t)\geq 0$ $(i=1,...,N)$.

Definition 2.1

Let $\gamma_{ij}=\begin{cases} k_{ij} \text{ if } i\neq j \\ -\sum_{\substack{i\neq 1 \\ j\neq i}}^{N} k_{ij} \text{ if } i=j \end{cases}$ and $\delta_{ij}=\begin{cases} m_{ij} \text{ if } i\neq j \\ -\sum_{\substack{i\neq 1 \\ j\neq i}}^{N} m_{ij} \text{ if } i=j \end{cases}$

Remark.

We have that γ_{ij} and δ_{ij} are nonnegative if $i\neq j$ and also that $\sum_{j=1}^{N}\gamma_{ij}=\sum_{j=1}^{N}\delta_{ij}=0$.

Rewriting the equation (2.1) and taking into consideration the definition 2.1 we obtain the equivalent system:

$$(2.2)\begin{cases} u_i'(t)=\sum_{j=1}^{N}\gamma_{ij}u_j(t)+u_i(t)(a(t)-b(t)u_i(t)-c(t)v_i(t))= \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p_i(t,\vec{u},\vec{v}) \\ v_i'(t)=\sum_{j=1}^{N}\delta_{ij}v_j(t)+v_i(d(t)-e(t)u_i(t)-f(t)v_i(t))= \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q_i(t,\vec{u},\vec{v}) \end{cases}$$

with $\vec{u}(t)=(u_1(t),u_2(t),\ldots,u_N(t))^T$ and

$\vec{v}(t)=(v_1(t),v_2(t),\ldots,v_N(t))^T$.

If we write $\vec{p}=(p_1,p_2,\ldots,p_N)^T$ and $\vec{q}=(q_1,\ldots,q_N)^T$, then we have in a compact form the system:

$$(2.3)\begin{cases} \vec{u}'=\vec{p}(t,\vec{u},\vec{v}) \\ \vec{v}'=\vec{q}(t,\vec{u},\vec{v}). \end{cases}$$

Here we assume that given two integers i_1, i_2, $1\le i_1$, $i_2\le N$ and $i_1\ne i_2$, there exists $j_1,\ldots,j_r$ such that $\gamma_{i_1j_1}>0$, $\gamma_{j_1j_2}>0,\ldots,\gamma_{j_ri_2}>0$. Physically this means that species u can get from one location to another. We make the same assumption concerning δ_{ij}.

We use $(\vec{u}(t),\vec{v}(t))^T$ to denote a solution of (2.3).

Also, if $\vec{\xi}=(\xi_1,\ldots,\xi_N)^T$ and $\vec{\eta}=(\eta_1,\ldots,\eta_N)^T$ are two vectors in R^N we say that $\vec{\xi}<\vec{\eta}$ if and only if $\xi_i<\eta_i$ for $i=1,\ldots,N$.

3. SOLUTION OF THE PROBLEM.

Theorem 3.1.

Let $(\vec{u}(t),\vec{v}(t))$ be a solution of (3.1) such that

$\vec{u}(0)>\vec{0}$ and $\vec{v}(0)>\vec{0}$

then

$\vec{u}(t)>\vec{0}$ and $\vec{v}(t)>\vec{0}$ for all $t\geq 0$.

Proof.

If we suppose the contrary, there exists $t^*>0$ such such that $\vec{u}(t)>\vec{0}$ and $\vec{v}(t)>0$ for all $t\varepsilon[0,t^*)$ and $\vec{u}(t^*)\geq\vec{0}$, $\vec{v}(t^*)\geq\vec{0}$ and either $u_m(t^*)=0$ for some $m\varepsilon\{1,\ldots,N\}$ or $v_m(t^*)=0$ for some $m\varepsilon\{1,\ldots,N\}$. Suppose that $u_m(t^*)=0$.

Since $u_m(t)>0$ for $0\leq t<t^*$ and $u_m(t^*)=0$ we have $u_m'(t^*)\leq 0$.

But $u_m'(t^*)=\sum_{j=1}^{N}\gamma_{mj}u_j(t^*)+u_m(t^*)(a(t^*)-b(t^*)u_m(t^*)-c(t^*)v_m(t^*))$.

So $u_m'(t^*)=\sum_{\substack{j=1\\ j\neq m}}^{N}\gamma_{mj}u_j(t^*)\geq 0$, from which we conclude that $u_m'(t^*)=0$.

We want to show that for any $i\neq m$ $u_i(t^*)=0$.

By assumption there exist $j_1,j_2,\ldots,j_r$ such that $\gamma_{mj_1}>0$, $\gamma_{j_1j_2}>0,\ldots,\gamma_{j_ri}>0$.

From $u'_m(*)=\sum_{\substack{j=1\\ j\neq m}}^{N} \gamma_{mj}u_j(t^*)$ and $\vec{u}(t^*)\geq\vec{0}$, $\gamma_{mj_1}>0$,

and $\gamma_{mj}\geq 0$ for $m\neq j$, we have $u_{j_1}(t^*)=0$.

Replacing m by j_1 in our reasoning we have

$$u'_{j_1}(t^*)=\sum_{\substack{j=1\\ j\neq j_1}}^{N} \gamma_{j_1 j}u_j(t^*)=0.$$

We have $\vec{u}(t^*)\geq\vec{0}$ and $\gamma_{j_1 j}\geq 0$ $(j_1\neq j)$; therefore, since $\gamma_{j_1 j_2}>0$, $u_{j_2}(t^*)=0$. Replacing m by j_2 in our reasoning we have

$0=u'_{j_2}(t^*)=\sum_{\substack{j=1\\ j\neq j_2}}^{N} \gamma_{j_2 j}u_j(t^*)$ and using

$\vec{u}(t^*)\geq 0$, $\gamma_{j_2 j}\geq 0$ for $j\neq j_2$ and $\gamma_{j_2 j_3}>0$ we conclude that $u_{j_3}(t^*)=0$.

Repeating this reasoning a sufficient number of times we get finally that

$$u_{j_1}(t^*)=u_{j_2}(t^*)=\ldots=u_{j_r}(t^*)=0.$$

Replacing m in our reasoning by j_r we have

$u'_{j_r}(t^*)=\sum_{\substack{j=1\\ j\neq j_r}}^{N} \gamma_{j_r j}u_j(t^*)=0$ and taking into consideration,

once again, that $\vec{u}(t^*)\geq 0$, $\gamma_{j_r j}\geq 0$ for $j\neq j_r$, and $\gamma_{j_r i}>0$

we conclude that $u_i(t^*)=0$.

We have shown that if $u_m(t^*)=0$ for some $m\varepsilon\{1,\ldots,N\}$ then $u_i(t^*)=0$ for all i. Thus $\vec{u}(t^*)=0$.

Now let us consider the system:

(**) $\vec{y}'=\vec{q}(t,\vec{0},\vec{y})$ where $\vec{q}=(q_1,\ldots,q_N)^T$, $\vec{y}=(y_1,\ldots,y_N)^T$

and $q_i(t,\vec{0},\vec{y})=\sum_{j=1}^{N}\delta_{ij}y_j(t)+y_i(t)(d(t)-f(t)y_i(t))$.

Let $\vec{y}(t)=\vec{\varphi}(t)$ be the solution of (**) such that $\vec{\varphi}(t^*)=\vec{v}(t^*)$.

Then $(\vec{0},\vec{\varphi}(t))^T$ is solution of (2.2), and since $(\vec{u}(t),\vec{v}(t))^T$ is solution of (2.2) and both satisfy the same initial condition at $t=t^*$, we have from uniqueness that $\vec{u}(t)\equiv\vec{0}$, which is a contradiction. A similar contradiction results if we suppose $v_m(t^*)=0$ for some $m\varepsilon\{1,\ldots,N\}$.

<u>Theorem 3.2</u>.

<u>If</u> $(\vec{u}_1(t),\vec{v}_1(t))^T$ <u>and</u> $(\vec{u}_2(t),\vec{v}_2(t))^T$ <u>are solutions of (3.2) such that</u>

$\vec{0}<\vec{u}_1(0)<\vec{u}_2(0)$ <u>and</u> $\vec{0}<\vec{v}_2(0)<\vec{v}_1(0)$

<u>then</u>

$\vec{0}<\vec{u}_1(t)<\vec{u}_2(t)$ <u>and</u> $\vec{0}<\vec{v}_2(t)<\vec{v}_1(t)$ <u>for all</u> $t\geq 0$.

Proof:

Let $\vec{u}_i(t)=(u_{i1}(t),u_{i2}(t),\ldots,u_{iN}(t))^T$ and $\vec{v}_i(t)=(v_{i1}(t),\ldots,v_{iN}(t))^T$ for $i=1,2$.

Let us suppose the contrary; then there exists $t^*>0$ such that

$\vec{0}<\vec{u}_1(t)<\vec{u}_2(t)$ and $\vec{0}<\vec{v}_2(t)<\vec{v}_1(t)$ for all $t\varepsilon[0,t^*)$,

and also

$\vec{u}_1(t^*)\leq\vec{u}_2(t^*)$ and $\vec{v}_2(t^*)\leq\vec{v}_1(t^*)$ and either

$\vec{u}_1(t^*)<\vec{u}_2(t^*)$ or $\vec{v}_2(t^*)<\vec{v}_1(t^*)$ does not hold.

Suppose that $\vec{u}_1(t^*)<\vec{u}_2(t^*)$ does not hold.

Since $\vec{u}_1(t^*)\leq\vec{u}_2(t^*)$, $u_{1j}(t^*)\leq u_{2j}(t^*)$ for $j=1,\ldots,N$, there exists $m\varepsilon\{1,\ldots,N\}$ such that $u_{1m}(t^*)=u_{2m}(t^*)$ and $v_{2m}(t^*)\leq v_{1m}(t^*)$.

Let $h_j(t)=u_{2j}(t)-u_{1j}(t)$ $j=1,\ldots,N$.

Since $h_m(t)>0$ for all $t\varepsilon[0,t^*)$ and $h_m(t^*)=0$ we have $h_m'(t^*)\leq 0$.

Calculating $h_m'(t^*)$ we obtain:

$$h_m'(t^*)=u_{2m}'(t^*)-u_{1m}'(t^*)$$

$$=\sum_{j=1}^{N}\gamma_{mj}(u_{2j}(t^*)-u_{1j}(t^*))+u_{2m}(t^*)c(t^*)(v_{1m}(t^*)-v_{2m}(t^*)).$$

Since $u_{2m}(t^*)c(t^*)(v_{1m}(t^*)-v_{2m}(t^*)\geq 0$ and from hypothesis the terms in the summation are nonnegative, we conclude that $h_m'(t^*)\geq 0$, so we have shown that

$h_m'(t^*)=0$.

Now $h_m'(t^*)=\sum_{\substack{j=1\\ j\neq m}}^{N} \gamma_{mj}(u_{2j}(t^*)-u_{1j}(t^*))+u_{2m}(t^*)c(t^*)(v_{1m}(t^*)$

$-v_{2m}(t^*))$, or,

$$(3.1)\quad h_m'(t^*)=\sum_{\substack{j=1\\ j\neq m}}^{N} \gamma_{mj}h_j(t^*)+u_{2m}(t^*)c(t^*)(v_{1m}(t^*)-v_{2m}(t^*)).$$

Now $\sum_{\substack{j=1\\ j\neq m}}^{N} \gamma_{m_j}h_j(t^*)\geq 0$ and $u_{2m}(t^*)c(t^*)>0$ implies that

$v_{1m}(t^*)-v_{2m}(t^*)=0$.

We will show that for any $i\neq m$ we also have

$h_i(t^*)=u_{2i}(t^*)-u_{1i}(t^*)=0$ and $v_{1i}(t^*)-v_{2i}(t^*)=0$.

By assumption there exist $j_1,j_2,\ldots,j_r$ such that

$\gamma_{mj_1}>0$, $\gamma_{j_1j_2}>0,\ldots,\gamma_{j_ri}>0$.

From (3.1) since $j_1\neq m$, $\gamma_{mj_1}>0$ and $\gamma_{mj}\geq 0$ $(j\neq m)$

we conclude that $h_{j_1}(t^*)=0$.

Replacing m by j_1 in our reasoning we have

$$0=h_{j_1}'(t^*)=\sum_{\substack{j=1\\ j\neq j_1}}^{N}\gamma_{j_1j}h_j(t^*)+u_{2j_1}(t^*)c(t^*)(v_{1j_1}(t^*)-v_{2j_1}(t^*)),$$

and taking into consideration that

$u_{2j_1}(t^*)c(t^*)>0$, $\gamma_{j_1 j}\geq 0$ for $j\neq j_1$, $\gamma_{j_1 j_2}>0$ and

$h_j(t^*)\geq 0$ we obtain $h_{j_2}(t^*)=0$ and $v_{1j_1}(t^*)-v_{2j_1}(t^*)=0$.

Replacing m by j_2 in our reasoning we have:

$$0=h'_{j_2}(t^*)=\sum_{\substack{j=1\\ j\neq j_2}}^{N} \gamma_{j_2 j}h_j(t^*)+u_{2j_2}(t^*)c(t^*)(v_{1j_2}(t^*)-v_{2j_2}(t^*)),$$

and using that $\gamma_{j_2 j_3}>0$, $\gamma_{j_2 j}\geq 0$ for $j\neq j_2$,

$u_{2j_2}(t^*)c(t^*)>0$ and $h_j(t^*)\geq 0$ we conclude that $h_{j_3}(t^*)=0$

and $v_{1j_2}(t^*)-v_{2j_2}(t^*)=0$.

Repeating this reasoning a sufficient number of times we finally get

$h_{j_1}(t^*)=h_{j_2}(t^*)=\ldots=h_{j_r}(t^*)=0$.

Replacing m by j_r in our reasoning we have

$$0=h'_{j_r}(t^*)=\sum_{\substack{j=1\\ j\neq j_r}}^{N} \gamma_{j_r j}h_j(t^*)+u_{2j_r}(t^*)c(t^*)(v_{1j_r}(t^*)-v_{2j_r}(t^*)).$$

Since $\gamma_{j_r j}\geq 0$ for $j\neq j_r$, $\gamma_{j_r i}>0$, $h_j(t^*)\geq 0$, $u_{2j_r}(t^*)>0$

and $c(t^*)>0$, we obtain that

$h_i(t^*)=0$ and $v_{1j_r}(t^*)-v_{2j_r}(t^*)=0$,

so $u_{2i}(t^*)=u_{1i}(t^*)$.

Finally replacing m by i in our reasoning we obtain

$v_{1i}(t^*) = v_{2i}(t^*)$.

Since i is arbitrary, $u_1(t^*) = u_2(t^*)$ and $v_1(t^*)=v_2(t^*)$ which contradicts the uniqueness theorem because $\vec{u}_1(0) < \vec{u}_2(0)$ and $\vec{v}_2(0) < \vec{v}_1(0)$.

A similar contradiction is reached if it is assumed that $\vec{v}_2(t^*) < \vec{v}_1(t^*)$ is not true.

Remark. By continuity of the solution with respect to initial conditions, it follows that if $(\vec{u}_1(t), \vec{v}_1(t))^T$ and $(\vec{u}_2(t), \vec{v}_2(t))^T$ are solutions of (2.2) with $\vec{0} < \vec{u}_1(0) \le \vec{u}_2(0)$ and $\vec{0} < \vec{v}_2(0) \le \vec{v}_1(0)$, then $\vec{0} < \vec{u}_1(t) \le \vec{u}_2(t)$ and $\vec{0} < \vec{v}_2(t) \le \vec{v}_1(t)$ for $t \ge 0$.

Definition 3.3.

Let ε, K_1 and K_2 such that

(3.2) $\quad \varepsilon < a_M/b_L < K_1$ and $\varepsilon < d_M/f_L < K_2$ and

(3.3) $\quad a_L - b_M \varepsilon - c_M k_L > 0$ and $d_L - e_M k_1 - f_M \varepsilon > 0$.

holds

So, we define the set $\hat{A}$ by:

$$\hat{A} = \{(\vec{\xi},\vec{\eta})^T : \varepsilon \le \xi_i \le k_1 \text{ and } \varepsilon \le \eta_i \le k_2 \text{ for } i=1,\dots,N\}.$$

Lemma 3.4.

Let $(u(t),v(t))^T$ be any solution of (1.1); then the pair $(\vec{u}(t),\vec{v}(t))^T$ defined by:

$\vec{u}(t)=(u_1(t),\ldots,u_N(t))^T$ <u>with</u> $u_j(t)=u(t)$ <u>for</u> $j=1,\ldots,N$

<u>and</u> $\vec{v}(t)=(v_1(t),\ldots,v_N(t))^T$ <u>with</u> $v_j(t)=v(t)$ <u>for</u>

$j=1,\ldots,N$, <u>is</u> <u>solution</u> <u>of</u> (2.2).

<u>Proof</u>:

This proof follows easily taking into consideration that $\sum_{j=1}^{N} \gamma_{mj}=\sum_{j=1}^{N} \delta_{mj}=0$ for each $m\varepsilon\{1,\ldots,N\}$.

<u>Remark</u>:

From this lemma we have that the pair $(\vec{u}_o(t),\vec{v}_o(t))^T$ with $\vec{u}_o(t)=(u_o(t),\ldots,u_o(t))^T$ and

$\vec{v}_o(t)=(v_o(t),\ldots,\vec{v}_o(t))^T$ where $(u_o(t),v_o(t))^T$ is the unique T-periodic solution of (1.1), is a T-periodic solution of (2.2).

<u>Theorem 3.5</u>.

If $(\vec{u}(t),\vec{v}(t))^T$ is a solution of (2.2) with $(\vec{u}(0),\vec{v}(0)\varepsilon\hat{A}$, then $(\vec{u}(t),\vec{v}(t))\varepsilon\hat{A}$ for $t\geq 0$ and $\|\vec{u}(t)-\vec{u}_o(t)\|\to 0$, $\|\vec{v}(t)-\vec{v}_o(t)\|\to 0$ as $t\to\infty$ where $(\vec{u}_o(t),\vec{v}_o(t))^T$ is the T-periodic solution of (2.2) defined above.

<u>Proof</u>:

Let $(u_1(t),\ v_1(t))^T$ nad $(u_2(t),\ v_2(t))^T$ be the solutions of (1.1) such that $(u_1(0),\ v_1(0))^T = (\varepsilon, k_2)^T$ and $(u_2(0),\ v_2(0))^T = (k_1, \varepsilon)^T$.

In [1] it was proved the following if $0 < u_1(0) < u_2(0) \leq k_1$ and $0 < v_2(0) < v_1(0) \leq k_2$ then $0 < u_1(t) < u_2(t) < k_1$ and $0 < v_2(t) < v_1(t) < k_2$ for all $t > 0$.

Also, it was proved that

$$|| u_i(t) - u_o(t) || \to 0 \quad \text{and} \quad || v_i(t) - v_o(t) || \to 0$$

when $t \to \infty$ for $i = 1,2$. This implies that if $(u_1(t),\ v_1(t))^T$ and $(u_2(t),\ v_2(t))^T$ are the solutions of (2.2) defined by

$$(u_1(t),\ v_1(t))^T = ((u_i(t),\ldots,u_i(t))^T,\ (v_i(t),\ldots,v_i(t))^T)^T$$

then $\vec{u}_1(0) < \vec{u}_1(t) < \vec{u}_2(t) < \vec{u}_2(0)$ and $\vec{v}_2(0) < \vec{v}_2(t) < \vec{v}_1(t) < \vec{v}_1(0)$ for $t > 0$ and $|| \vec{u}_i(t) - \vec{u}_\circ(t) || \to 0,\ || \vec{v}_i(t) - \vec{v}_\circ(t) || \to 0$ as $t \to \infty$ for i-1,2. If $(\vec{u}(t), \vec{v}(t))^T$ is a solution of (3.2) with $(\vec{u}(0),\ \vec{v}(0)) \ \varepsilon\ \hat{A}$ then $u_1(0) \leq \vec{u}(0) \leq \vec{u}_2(0)$ and $\vec{v}_2(0) \leq \vec{v}(0) \leq \vec{v}_1(0)$ and so by the remark following the proof of theorem 3.2 we have $\vec{u}_1(0) < \vec{u}_1(t) \leq \vec{u}(t) \leq \vec{u}_2(t) < \vec{u}_2(0)$ and $\vec{v}_2(0) < v_2(t) \leq \vec{v}(t) \leq \vec{v}_1(t) < \vec{v}_1(0)$. This implies taht $(\vec{u}(t),\ \vec{v}(t)) \ \varepsilon\ \hat{A}$ for $t \geq 0$ and $|| \vec{u}(t) - \vec{u}_\circ(t) || \to 0$, $|| \vec{v}(t) - \vec{v}_\circ(t) || \to 0$ as $t \to \infty$.

Remark.

This tells us that $(\vec{u}_o(t), \vec{v}_o(t))^T$ is asymptotically stable in $\hat{A}$, and this implies that this T-periodic solution of (2.2) is unique in $\hat{A}$.

Theorem 3.6. (Global Stability of $(\vec{u}_o(t), \vec{v}_o(t))^T$).

Let k_1 and k_2 satisfy $k_1 > a_M/b_L$, $k_2 > d_M/f_L$ and $a_L - c_M k_2 > 0$, $d_L - e_M k_1 > 0$.

Let $(\vec{u}(t), \vec{v}(t))^T$ be any solution of (2.2) such that $\vec{u}(0) > \vec{0}$ and $\vec{v}(0) > \vec{0}$. Then there exists $t^* > 0$ such that $u_i(t) \le k_1$, $v_i(t) \le k_2$ for $i=1,\ldots,N$ and $t \ge t^*$, and

$$\lim_{t\to\infty} \| \vec{u}(t) - \vec{u}_o(t) \| = 0, \quad \lim_{t\to\infty} \| \vec{v}(t) - \vec{v}_o(t) \| = 0.$$

Proof.

First, consider the uncoupled system of self competing problem

$$(3.4) \qquad U' = U\,[a(t) - b(t)U]$$

$$(3.5) \qquad V' = V\,[d(t) - f(t)V]$$

where a,b,d and f are as in (1.1) and $U(t), V(t) > 0$.

It is well known that the equations (3.4) and (3.5) have an unique T-periodic, positive solution denoted by $U_o(t)$

and $V_0(t)$ respectively, which are globally asymptotically stable (see 2).

Ou the other hand, it is easy to see that if $(U_0, V_0)^T$ is the T-periodic solution of (3.4) and (3.5) then

(3.6) $a_L/b_M \leq U_0(t) \leq a_M/b_L$ and

(3.7) $d_L/f_M \leq V_0(t) \leq d_M/f_L$ for all $t \geq 0$.

Now, let $(U(t), V(t))^T$ be any solution of (3.4)-(3.5) such that

$$\max \{u_i(0): i=1,\ldots,N\} < U(0) \quad \text{and}$$

$$\max \{v_i(0): i=1,\ldots,N\} < V(0).$$

<u>CLAIM:</u>

$u_i(t) < U(t)$ and $v_i(t) < V(t)$ for all $t \geq 0$ and $i=1,\ldots,N$. Suppose, on the contrary, that our claim is not true. Then there exists $t_0 > 0$ such that for all $i=1,\ldots,N$, $u_i(t) < U(t)$ and $v_i(t) < V(t)$ for all $t \in [0,t_0)$, but at $t = t_0$, $u_m(t_0) = U(t_0)$ for some $m \in \{1,\ldots,N\}$, or $v_m(t_0) = V(t_0)$ for some $m \in \{1,\ldots,N\}$. Suppose $u_m(t_0)=U(t_0)$ holds.

If $h_j(t) = U(t) - u_j(t)$, then $h_j(t) > 0$ for all $t \in [0,t_0)$ $j=1,\ldots,N$ and $h_m(t_0) = 0$. This implies that

$h_m'(t_\circ) \leq 0.$

We have $h_m'(t_\circ) = U'(t_\circ) = U'(t_\circ) - u_m'(t_\circ)$ where

$$U'(t_\circ) = U(t_\circ)\ (a(t_\circ) - b(t_\circ)U(t_\circ)) = \sum_{j=1}^{N} \gamma_{mj}\ U(t_\circ) + U(t_\circ)(a(t_\circ) - b(t_\circ)U(t_\circ))$$

because $\sum_{j=1}^{N} \gamma_{mj} = 0$, and

$$u_m'(t_\circ) = \sum \gamma_{mj}\ u_j(t_\circ) + u_m(t_\circ)\ (a(t_\circ) - b(t_\circ)\ u_m(t_\circ) - c(t_\circ)\ v_m(t_\circ)).$$

Now $u_m(t_\circ) = U(t_\circ)$ and so

$$u_m'(t_\circ) = \sum_{j=1}^{N} \gamma_{mj}\ u_j(t_\circ) + U(t_\circ)\ (a(t_\circ) - b(t_\circ)\ U(t_\circ) - c(t_\circ)\ v_m(t_\circ)).$$

From this we obtain

$$h_m'(t_\circ) = \sum_{\substack{j=1 \\ j \neq m}}^{N} \gamma_{mj}\ h_j(t_\circ) + c(t_\circ)\ U(t_\circ)\ v_m(t_\circ)$$ and since

$\gamma_{mj} \geq 0$ for $j \neq m$, $h_j(t_\circ) \geq 0$, $c(t_\circ) > 0$, $U(t_\circ) > 0$ and $v_m(t_\circ) > 0$, we have $h_m'(t_\circ) > 0$, which is a contradiction. This proves our claim, because a similar contradiction is reached if we suppose that $v_m(t_\circ)=V(t_\circ)$ holds.

On the other hand, we have $(U(t),\ V(t))^T$ is solution of (3.4) - (3.5) with $U(0)>0$ and $V(0)>0$, so $U(t)\to U_\circ(t)$ and $V(t)\to V_\circ(t)$ as $t\to\infty$ where $a_L/b_M \le U_\circ(t) \le a_M/b_L < k_1$ and $d_M/f_L \le V_\circ(t) \le d_L/f_M < K_2$ for all $t \ge 0$. Therefore, for t large, ie, for $t \ge t^*$ for some $t^* > 0$, we have $0 < U(t) < t_1$ and $0 < V(t) < k_2$.

Hence there exists $m>0$ such that $0<U(mT)<k_1$ and $0<V(mT)<k_2$ and therefore $0<u_i(mT)<U(mT)$ and $0<v_i(mT)<V(mT)$ for $i=1,\ldots,N$. Choose $\varepsilon>0$ such that $\varepsilon<u_i(mT)$ and $\varepsilon<v_i(mT)$ for $i=1,\ldots,N$, and such that inequalities (3.2) and (3.3) are satisfied. Let $\hat{A}$ be defined as in definition 3.3.

Now we have $(\vec{u}(mT),\ \vec{v}(mT))\varepsilon\mathring{\hat{A}}$.
If $\vec{u}_3(t)=\vec{u}(t+mT)$ and $\vec{v}_3(t)=\vec{v}(t+mT)$, then we have that $(\vec{u}_3(t),\ \vec{v}_3(t))^T$ is solution of (2.2) because of T-periodicity.

Since $(\vec{u}_3(0),\ \vec{v}_3(0))\varepsilon\hat{A}$, it follows from Theorem 3.5 that $(\vec{u}_3(t),\ \vec{v}_3(t))\varepsilon\hat{A}$ for all $t\ge0$ and $\|\vec{u}_3(t)-\vec{u}_{\circ}(t)\|\to0$, $\|\vec{v}_3(t)-\vec{v}_\circ(t)\|\to0$ as $t\to\infty$.
This means:

$\|\vec{u}(t)-\vec{u}_o(t)\|\to 0$, $\|\vec{v}(t)-\vec{v}_o(t)\|\to 0$ as $t\to\infty$ and the proof is complete.

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