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MODULUS OF NONCOMPACT CONVEXITY, ITS PROPERTIES AND APPLICATIONS

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1. INTRODUCTION.

It is well known that the notion of the modulus of convexity plays a very important role in the so-called geometric theory of Banach spaces. It appeared to be the tool allowing to classify Banach spaces with regord to their geometrical structure. Moreover, the modulus of convexity is very useful in the fixed point theory. Many facts concerning this notion and its applications may be found in [4,6,7,11], for example. Recently K. Goebel and T. Sekowski [8] have proposed an interesting generalization of the classical modulus of convexity. Namely, using the concept of Kuratowski measure of non-compactness they defined the so-called modulus of noncompact convexity. With help of this modulus they proved a few interesting facts from the geometric theory of Banach spaces.

This paper is the survey of results concerning the modulus of non - compact convexity which were obtained recently. In the Section 2 results due to Goebel and Sekowski [8] are presented. Sections 3.4 and 8 are the collection of the results stated by the author in [1] . Apart from that Sections 5,6 and 7 give some new results obtained by the

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author.

2. NOTATION, DEFINITIONS AND SOME KNOWN RESULTS.

Let $(E, \| . \|)$ be an infinite dimensional Banach space let $B(x,r)$, $S(x,r)$ denote the ball and the sphere centered at x and of radius r. For simplicity we will write B, S instead of $B(\theta,1)$ and $S(\theta,1)$. If X is a subset of E and $x \varepsilon E$ then $\overline{X}$, conv X, dist(x,X) will denote the clo - sure or the convex closure of X and the distance from a point x to X, respectively. Analogously dist(X,Y) will denote the distance between sets X and Y. By $B(X,r)$ we denote the "ball" centered at a set X and with radius r, i.e. $B(X,r) = \bigcup_{x \varepsilon X} B(x,r)$.

For a bounded set X, the symbol $\alpha(X)$ will denote the Kuratowski measure of noncompactness:

$$\alpha(X) = \inf \left[d >: X \text{ can be convered with a finite number of sets having diameters smaller than } \right].$$

The Hausdorff measure of noncompactness will be denoted by $\chi(X)$:

$$\chi(X) = \inf \left[\varepsilon > 0: X \text{ can be covered with a finite number of balls of radii smaller than } \varepsilon \right].$$

In the sequel we will use first of all the following properties of the function χ:

(1) $\chi(X) = 0 \iff \overline{X}$ is compact.

(2) $X \subset Y \implies \chi(X) \leq \chi(Y)$.

(3) $\chi(\overline{X}) = \chi(\text{Conv } X) = \chi(X)$.

(4) $\chi(\lambda X) = |\lambda|\ \chi(X),\ \lambda \in \mathbb{R}$.

(5) $\chi(X+Y) \leq \chi(X) + \chi(Y)$.

(6) $\chi(x+X) = \chi(X)$

(7) $\chi(B(x,v)) = \chi(S(x,v)) = v$

Let us notice that the function α possesses also the properties (1)-(6) and $\alpha(B(x,r)) = \alpha(S(x,r)) = 2r$. For further properties of these measures we refer to [2].

Recall that the classical Clarkson *modulus of convexity* of the space E [3] is the function δ: <0,2> → <0,1> defined by

$$\delta_E(\varepsilon) = \inf \left[1 - \frac{\|x+y\|}{2}:\ x,y \in \overline{B},\ \|x-y\| \geq \varepsilon\right] .$$

The coefficient of convexity of E is undestood as

$$\varepsilon_o(E) = \sup[\varepsilon\ \ \delta_E(\varepsilon) = 0]$$

The space is called *uniformly convex* if $\varepsilon_o = 0$.

The notion of *the modulus of noncompact convexity* was defined in [8] in the following way

$$\tilde{\Delta}_E(\varepsilon) = \inf[1-\mathrm{dist}(\theta,X):\ X \subset \overline{B},\ X = \mathrm{Conv}\ X,\ \alpha(X) \geq \varepsilon] .$$

Actually, $\tilde{\Delta}$: <0,2> → <0,1> and is a nondecreasing function. Moreover, $\delta_E(\varepsilon) \leq \tilde{\Delta}_E(\varepsilon)$ for any Banach space E. It was shown in [8] that this inequality may be strong for some spaces.

Similarly the number $\tilde{\varepsilon}_1(E) = \sup[\varepsilon:\ \tilde{\Delta}_E(\varepsilon) = 0]$ was called *the* *coefficient of noncompact convexity* and spaces with $\tilde{\varepsilon}_1 = 0$, $\tilde{\Delta}$-*uniformly convex*. Obviously $\tilde{\varepsilon}_1(E) \leq \varepsilon_o(E)$ and in the case of Day space D, for example, we have $\tilde{\varepsilon}_1(D) = 0$ and $\varepsilon_o(D) = 2$ [8].

The main results proved in [8] may be summarized in the below given Theorem.

THEOREM 1. If $\tilde{\varepsilon}_1(E) < 1$ then E is reflexive and has normal structure.

In what follows we shall use the notion of the modulus of noncompact convexity defined with help of the Hausdorff measure of noncompactness [1].

$$\Delta:\langle 0,1\rangle \to \langle 0,1\rangle \ , \ \Delta_E(\varepsilon)=\inf[1-\mathrm{dist}(\theta,X):X \subset \overline{B},\ X=\mathrm{Conv}\ X,\ \chi(X) \geq \varepsilon].$$

In the similar way as previously by $\varepsilon_1(E)$ we denote the coefficient of noncompact convexity of E (with respect to the modulus Δ). We say that E is Δ-uniformly convex if $\varepsilon_1 = 0$.

Let us notice that the well-known dependence $\chi(X) \leq \alpha(X) \geq 2\,\chi(X)$ (cf. [2]) yields

$$\tilde{\Delta}_E(\varepsilon) \leq \Delta_E(\varepsilon) \leq \tilde{\Delta}_E(2\varepsilon) \ , \quad \varepsilon \in \langle 0,1\rangle$$

for any Banach space E. Hence

$$\varepsilon_1(E) \leq \tilde{\varepsilon}_1(E) \leq 2\,\varepsilon_1 \ .$$

The last inequality permits us to formulate the following.

THEOREM 2. If $\varepsilon_1(E) < 1/2$ then the space E is reflexive and has normal structure.

3. CONTINUITY OF MODULUS OF NONCOMPACT CONVEXITY.

This section is devoted to showing that the modulus of noncompact convexity $\Delta_E(\varepsilon)$ is continuous on the interval <0,1>.

We will need the following result proved by De Blasi in the case of the so-called measure of weak noncompactness [5].

This proof may be adopted withou changes for the measure χ .

LEMMA 1. $\chi(B(X,r)) = \chi(X) + r$, for any $r \geq 0$.

The main result of this section is contained in the following.

THEOREM 3. The function Δ is continuous on the interval <0,1).

PROOF. At first notice that the function Δ is nondecreasing on the interval <0,1>. Further, fix $\varepsilon_1 \in <0,1)$ and take arbitrary $\varepsilon_2 \in (\varepsilon_1, 1)$. For $\eta > 0$ (sufficiently small) we may choose a set X_1 contained in $\overline{B}$ such that $\text{Conv}\, X_1 = X_1$, $\chi(X_1) \geq \varepsilon_1$ and

(1) $$1\text{-dist}\,(\theta, X_1) \leq \Delta(\varepsilon_1) + \eta .$$

Next, putting $k = (1-\varepsilon_2)/(1-\varepsilon_1)$ we see that $k \in (0,1)$. Consider the set $Y = k\, X_1$. Actually $\chi(Y) = k\, \chi(X_1)$ and

$$\text{dist}(\theta, Y) = k\, \text{dist}(\theta, X_1),$$

$$\text{dist}(Y, S) \geq 1-k,$$

so that if we take the set $X_2 = B(X_1, 1-k)$ we can easily verify that $X_2 \subset B$, Conv $X_2 = X_2$ and

(2) $$\mathrm{dist}(\theta, X_2) = k\,\mathrm{dist}(\theta, X_1) - 1+k.$$

Moreover, in view of Lemma 1 we infer

$$\chi(X_2) = k\,\chi(X_1) + 1-k \geq k\,\varepsilon_1 + 1-k = \varepsilon_2 .$$

Now by (1) and (2) we get

$$1-\mathrm{dist}(\theta, X_2) = 1-k\,\mathrm{dist}(\theta, X_1) + 1-k=$$

$$= k(1-\mathrm{dist}(\theta, X_1)) + 2(1-k) \leq k(\Delta(\varepsilon_1)+\eta)+2(1-k).$$

Hence

$$\Delta(\varepsilon_2) \leq k(\Delta(\varepsilon_1) +\eta) + 2\,(1-k).$$

Finally, keeping in mind that η was chosen arbitrarily, we have

$$\Delta(\varepsilon_2) \leq k\,\Delta(\varepsilon_1) + 2(1-k)$$

what implies

$$\Delta(\varepsilon_2) - \Delta(\varepsilon_1) \leq k\,\Delta(\varepsilon_1) - \Delta(\varepsilon_1)+2(1-k)=$$

$$= (1-k)(2-\Delta(\varepsilon_1)) \leq 2(1-k) = 2(\varepsilon_2-\varepsilon_1)/(1-\varepsilon_i).$$

Thus the proof is complete.

Let us mention that our method of proving depends mostly on the result of Lemma 1. Because we do not know if the equality

$\alpha(B(X,t)) = \alpha(X) + 2t$ is true, we are not able to tell something about the continuity of the function $\tilde{\Delta}$.

4. THE CASE OF REFLEXIVE SPACE.

Throughout this section we will always assume that E is a reflexive Banach space. This assumption permits us to deduce that for nonempty, closed and convex subset X of E and for any $y \in E$ there is at least one $x \in X$ with the property $dis(y,X)=\|y-x\|$ [10]. We show below that this fact has some significance in order to obtain additional properties of a modulus of noncompact convexity.

Let us suppose that a number $\varepsilon \in (0,1)$ is fixed. Take an arbitrary $\eta > 0$ and a set $X \subset \overline{B}$, $X = \text{Conv}\ X$, $\chi(X) \geq \varepsilon$ such that

$$(3) \qquad 1-\text{dist}(\theta,X) \leq \Delta(\varepsilon) + \eta .$$

Next, let k be on arbitrary number in the interval (0,1). Choose $x \in X$ with the property $\text{dist}(\theta,X) = \|X\|$ and consider the set $X_1 = kX + ((1-k)/\|X\|)\, x$. Then $\chi(X_1) \geq k\,\varepsilon$ and

$$\text{dist}(\theta,X_1) = k\ \text{dist}(\theta,X) + 1-k .$$

Moreover, $X_1 \subset \overline{B}$. Further we have

$$\text{dist}(\theta,X) = \frac{1}{k}\ (\text{dist}(\theta,X_1) + k-1)$$

what by (3) allows us to infer

$$1- \Delta(\varepsilon) \leq \text{dist}(\theta,X) + \eta = \frac{1}{k}\ (\text{dist}(\theta,X_1) + k-1) + \eta .$$

and finally

$$\Delta(k\varepsilon) \leq k \quad \Delta(\varepsilon).$$

Thus we can state our next result.

THEOREM 4. If E is a reflexive Banach space then $\Delta_E(\varepsilon)$ is a subhomogeneous function i.e.

$$\Delta(k\varepsilon) \leq k \quad \Delta\ (\varepsilon)$$

for any k, $\varepsilon \in <0,1>$.

From the above theorem we may deduce some simple corollaries.

COROLLARY 1. $\Delta(\varepsilon) \leq \varepsilon$ for any $\varepsilon \in <0,1>$.

COROLLARY 2. The function Δ is strictly increasing on the interval $< \varepsilon_1(E),1 >$.

Indeed, for $t_1 < t_2 \leq 1$, $t_1 > \varepsilon_1(E)$, if we put in Theorem 4 $\varepsilon = t_2$, $k = t_1/t_2$ we have

$$\Delta(t_1) \leq (t_1/t_2) \quad \Delta(t_2)$$

what implies $\Delta(t_2)/\ \Delta(t_1) \geq t_2/t_1 > 1$. Thus $\Delta(t_1) < \Delta(t_2)$.

COROLLARY 3. $\Delta(t_2) - \Delta(t_1) \geq (t_2-t_1)/\ \Delta(t_1)$ for any $t_1,t_2\ \varepsilon(\varepsilon_1(E),1>$, $t_1 \leq t_2$.

COROLLARY 4. The function $\varepsilon \to \Delta(\varepsilon)/\ \varepsilon$ is nondecreasing on the interval $<0,1>$ and $\Delta(\varepsilon_1 + \varepsilon_2) \geq \Delta(\varepsilon_1) + \Delta(\varepsilon_2)$ provided $\varepsilon_1 + \varepsilon_2 \leq 1$.

We omit simple proofs of the two last corollaries.

5. A RESULT IN REFLEXIVE AND SMOOTH SPACE.

Throughout this section we will always assume that E is a reflexive and smooth Banach space.

It is well known that if E is a reflexive space then every linear and continuous functional $f \in E^*$ attains its norm on the unit sphere S (cf. [10]; the reverse assertion is also true according to the famous theorem due to James).

Furthermore, let us take $f \in E^*$, $\|f\| = 1$. For an arbitrary $d > 0$ consider the hyperplane $X_d = [x \in E: f(x) = d]$. Then for $x \in X_d$

$$\|x\| \geq f(x) = d$$

so that

$$\text{dist}(\theta, X_d) \geq d.$$

On the other hand there exists $x_o \in S$ such that $f(x_o) = 1$. Consider $y = d \cdot x_o$. Obviously $f(y) = df(x_o) = d$ so that $y \in X_d$. Apart from that $\|y\| = d$ and finally

$$\text{dist}(\theta, X_d) = d.$$

The similar statement is also valid for the half-space

$$X_d^+ = [x \in E: f(x) \geq d]$$

so we can write

$$\mathrm{dist}(\theta, X_d^+) = d.$$

Further let us consider the function $\tilde{\Delta}_1$: $\langle 0,2\rangle \to \langle 0,1\rangle$ defined by the formula

$$\tilde{\Delta}_1(\varepsilon) = \inf\left[1-\mathrm{dist}(\theta,X) : X = X_d^+ \cap \overline{B},\ \alpha(X) \geq \varepsilon\right]$$

where the infimum is taken also over all $d > 0$ and $f \in E^*$, $\|f\| = 1$.

Obviously we have

$$\tilde{\Delta}(\varepsilon) \leq \tilde{\Delta}_1(\varepsilon) .$$

Suppose now that for some $\varepsilon \in (0,2)$

$$\tilde{\Delta}(\varepsilon) < \tilde{\Delta}_1(\varepsilon).$$

Next take $\eta > 0$ such that $\tilde{\Delta}(\varepsilon) + \eta < \tilde{\Delta}_1(\varepsilon)$ and consider the set $A \subset \overline{B}$ being convex and closed, $\alpha(A) \geq \varepsilon$ and such that

$$1 - \mathrm{dist}(\theta,A) \leq \tilde{\Delta}(\varepsilon) + \eta .$$

Then for any set $A_1 = X_d^+ \cap B$, $\alpha(A_1) \geq \varepsilon$ we have

$$1-\mathrm{dist}(\theta,A) \leq \tilde{\Delta}(\varepsilon) + \eta < \tilde{\Delta}_1(\varepsilon) \leq 1 - \mathrm{dist}(\theta,A_1). \tag{4}$$

Next let $a \in A$ be such that $\mathrm{dist}(\theta,A) = \|\theta\|$. Consider the ball $B_{\|\theta\|} = B(\theta, \|\theta\|)$. In view of the assumption that E is smooth we infer that there exists exactly one hyperplane $X_{\|\theta\|}$ tangent to $B_{\|\theta\|}$

at the point a [10]. Let $X^+_{\|\theta\|}$ be the half-space associated with $X_{\|\theta\|}$. Of course $X^+_{\|\theta\|}$ contains the set A and for the set $A_1 = X^+_{\|\theta\|} \cap \overline{B} \supset A$ we get

$$\text{dist}(\theta, X^+_{\|\theta\|}) = \text{dist}(\theta, A) = \text{dist}(\theta, A_1).$$

Moreover, $\alpha(A_1) \geq \alpha(A) \geq \varepsilon$. But the last facts contradict to (4) and the proof is complete.

Now we can st-te the folllowing theorem.

THEOREM 5. if E is a reflexive and smooth Banach space then

$$\tilde{\Delta}(\varepsilon) = \tilde{\Delta}_1(\varepsilon).$$

Let us remark that similar result is also true if we make use of the Hausdorff measure χ, so we can write

$$\Delta(\varepsilon) = \Delta_1(\varepsilon) \ ,$$

where $\Delta_1(\varepsilon)$ is defined in the same way as $\tilde{\Delta}_1(\varepsilon)$ (only with α replaced by χ).

6. COMPUTATION FOR A HILBERT SPACE.

In this section we apply the result of the previous one in order to calculate the modulus of noncompact convexity in a Hilbert space.

Let us assume that H is a real Hilbert space with the scalar product (.,.). Actually H is reflexive and even uniformly smooth [10]. Next, let us fix $d \in (0,1)$, $Z \in S$ and consider the hyperplane and the half-space

$$X_d = \left[x \in H: (z,x-dz) = 0\right] = \left[x \in H: (z,x) = d\right]$$

$$X_d^+ = \left[x \in H: (z,x-dz) \geq 0\right] = \left[x \in H: (z,x) \geq d\right] .$$

Denote $A_d = X_d \cap \overline{B}$, $A_d^+ = X_d^+ \cap \overline{B}$.

In order to calculate the diameter of the A_d let us take $x \in A_d$ such that $x \in S$ and consider the vector $y = x - dz$. Then y is orthogonal to dz so that according to Phytagorean rule we have $\|y\| = \sqrt{1-d^2}$. On the other hand $A_d \subset A_d^+ \subset \overline{B}\,(dz,\sqrt{1-d^2})$ so that we obtain

$$\text{diam } A_d = \text{diam } A_d^+ = 2\sqrt{1-d^2} .$$

Let us note that the set $A_o = A_d - dz$ has the same diameter as A_d. Moreover, this set lies in the hyperplane $F = [X:(x,z) = 0]$. Hence A_o may be treated as the unit ball in F. On the other hand it is known that codim F = 1 what allows us to deduce that

$$(5) \qquad \alpha(A_o) = \alpha(A_d) = \alpha(A_d^+) = 2\sqrt{1-d^2}$$

and similarly

$$(6) \qquad \chi(A_o) = \chi(A_d) = \sqrt{1-d^2} .$$

In what follows let us take a number $\varepsilon > 0$ and put $d = \sqrt{1-\varepsilon^2}$. Consider the sets A_d^+ and A_d defined as above. Then

$$(7) \qquad \text{dist}(\theta,A_d^+) = \text{dist}(\theta,A_d) = d = \sqrt{1-\varepsilon^2}.$$

Finally combining (5), (6), (7) and Theorem 5 we state that the moduli of noncompact convexity in a Hilbert space H have the form

$$\Delta_H(\varepsilon) = \Delta_1(\varepsilon) = 1- \sqrt{1-\varepsilon^2} \quad \varepsilon \in <0,1>,$$

$$\tilde{\Delta}_H(\varepsilon) = \tilde{\Delta}_1(\varepsilon) = 1-\sqrt{1-(\frac{\varepsilon}{2})^2} \;, \; \varepsilon \in <0,2> .$$

The last result agrees with the formula for the modulus of noncompact convexity in the Hilbert space ℓ^2 which was obtained in [8]. Moreover, let us remark that

$$\Delta_H(\varepsilon) = \tilde{\Delta}_H(2\varepsilon) = \delta_H(2\varepsilon)$$

for $\varepsilon \in <0,1>$ (cf. [7]).

7. SOME APPLICATION.

Now we will assume that E is Δ-uniformly convex Banach space.

Let us denote by S^* the unit sphere in the dual space E^*. By virtue of Theorem 2 the space E is reflexive so that for every $f \in S^*$ there exists $x \in S$ such that $f(x) = \|x\| = \|f\| = 1$. Further, fix $f \in S^*$. Take an arbitrary $f_1 \in S^*$ and a number $\varepsilon \in (0,1>$ and assume that $\|f-f_1\| \leq \varepsilon$. Let $x_1 \in S$ be such that $f(x_1) = 1$. Next consider the set

$$A_\varepsilon = X^+_{1-\varepsilon} \cap \overline{B} = [x \in \overline{B}: f(x) \geq 1 - \varepsilon].$$

Let us notice that in view of inequality

$$1-f(x_1) = |f_1(x_1)-f((x_1)| \leq \|f_1-f\| \leq \varepsilon$$

we obtain that $x_1 \in A_\varepsilon$.

Taking into account that Δ is increasing and continuous on the interval <0,1> we con infer that

$$(8) \qquad \chi(A\varepsilon) \leq \Delta^{-1}(\varepsilon).$$

In what follows let us take an arbitrary sequence $(f_n) \varepsilon S^*$ which converges (in the sense of the norm in E^*) to the functional $f \varepsilon S^*$. Let (x_n) be a sequence contained in S such that x_n denotes an arbitrary element of S in which the norm ot f_n is attained $(n=1,2,\ldots)$. Then we have.

THEOREM 6. The sequence (x_n) is relatively compact.

PROOF. Denote $\varepsilon_u = \| f_n - f \|$, $n=1,2,\ldots$ Let us construct the sequence of sets A_{ε_n} according to the previously described method. Without loss of generality we may assume that (ε_n) is nonincreasing.

Then (A_{ε_n}) is a sequence of sets being nonempty, closed and convex. Moreover, it can be casily proved that $A_{\varepsilon_n} \supset A_{\varepsilon_{n+1}}$ for $n=1,2,\ldots.$ Apart from that applying (8) we get

$$\chi(A_{\varepsilon_n}) \leq \Delta^{-1}(\varepsilon_n).$$

Thus taking into account the continuity of the function A and $\Delta(0) = 0$ we can deduce

$$\lim_{n\to\infty} \chi(A_{\varepsilon_n}) = 0.$$

Hence the set $A_\infty = \bigcap_{n=1}^{\infty} A_{\varepsilon_n}$ is nonempty, convex and compact. Moreover

$$\chi(\{x_1,x_2,\ldots,\}) = \chi(\{x_n,x_{n+1},\ldots\}) \leq \chi(A_n)$$

for every n, what implies that $\chi(\{x_1,x_2,\ldots\}) = 0$. This statement finishes the proof.

Let us remark that denoting by x an arbitrary element of $\dot{S}$ in which f attains its norm we cannot deduce that the sequence (x_n) converges to x.

Nevertheless we have the following assertion.

THEOREM 7. If we additionally assume that E is strictly convex then $\lim_{n\to\infty} x_n = x$.

PROOF. Because of the fact that $\mathrm{dist}(\theta, A_{\varepsilon_n}) = 1-\varepsilon_n$ we deduce that $A_\infty \subset S$. On the other hand $x \in A_\infty$. Thus in view of strict convexity of E we obtain that $A_\infty = \{x\}$ (cf. [10]). Hence $\lim_{n\to\infty} x_n = x$ what gives the thesis of our theorem.

8. STABILITY.

As we have established in Theorem 2 every Banach space E for which $\varepsilon_1(E) < 1/2$ has normal structure. Thus, according to the well known Kirk's fixed point theorem the space E has the fixed point property what means that every nonempty, closed, convex and bounded subset of E possesses the fixed point property with respect to nonexpansive self-mappings [7,9]. We show now that this property is stable with regard to the slight change of the norm.

Assume that $(E, \| \cdot \|_1)$ is a Banach space for which $\varepsilon_1 < \frac{1}{2}$. Let $\| \cdot \|_2$ be the equivalent norm on the space E i.e. there exist positive constants M and M such that

$$m \| x \|_1 \leq \| x \|_2 \leq M \| x \|_1$$

for every $x \in E$. Let χ_1 and χ_2 denote the Hausdorff measures of non-compactness in the spaces $(E, \| \cdot \|_1)$, $(E, \| \cdot \|_2)$, respectively. Then we can casily show that

$$m\, \chi_1(x) \leq \chi_2(x) \leq M\, \chi_1(X)$$

for any bounded subset X of the space E.

Further, let Δ_1, Δ_2 be moduli of noncompact convexity with respect the suitable norms.

Let us fix $\varepsilon > 0$ and $\eta\, \varepsilon (\theta, 1)$. Next, let us take $X \subset \overline{B}_2$, $X = \operatorname{Conv} X$, $\chi_2(X) \geq \varepsilon$ and such that

$$\operatorname{dist}_2(\theta, X) \geq 1 - \Delta_2(\varepsilon) - \eta$$

(here the indexes denote that we consider the ball or the distance with respect to the suitable norm). Then we have $\chi_1(X) \geq \varepsilon / M$ and

$$\operatorname{dist}_1(\theta, X) \geq (1/M) \operatorname{dist}_2(\theta, X).$$

Moreover, $X \subset \overline{B}_1(\theta, 1/m)$. Hence we get

$$(1/M)\ \mathrm{dist}_2(\theta,X) \leq \mathrm{dist}_1(\theta,X) \leq (1-\Delta_1(m\,\varepsilon/M))\ (1/m)$$

what implies

$$1-\Delta_2(\varepsilon) - \eta \leq (M/m)\ (1-\Delta_1(m\,\varepsilon/M)).$$

Finally the last inequality yields

$$(9) \qquad \Delta_2(\varepsilon) \geq 1 - k(1-\Delta_1(\varepsilon/k))$$

where $k = M/m \geq 1$.

Let $D > 1$ be a unique solution of the equation

$$(10) \qquad 1-(1/D) = \Delta_1(1/2D),$$

which exists in view of continuity of the function Δ_1 (Theorem 3). Now if $1 \leq k < D$ then $k(1-\Delta_1(1/2\,k)) < 1$ so that (9) allows us to infer that $\Delta_2(1/2) > 0$. This assertion means that the coefficient of noncompact convexity for the norm $\|\cdot\|_2$ is smaller than 1/2 and in view of Theorem 2 the space $(E,\|\cdot\|_2)$ has normal structure. Thus we have.

THEOREM 8. Let E be a Banach space with $\varepsilon_1 < 1/2$ and let $D > 1$ satisfy (10). If F is another Banach space having the Banach-Marzur distance less than D then its coefficient of noncompact convexity is also smaller than 1/2.

Let us remark that similar result for the coefficient of convexity was obtained in [7].

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