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ON BOWEN-RUELLE-SINAI MEASURES AND HORSESHOES FOR DIFFEOMORPHISMS OF SURFACES

POR

LEONARDO MENDOZA

UNIVERSIDAD DE LOS ANDES

FACULTAD DE CIENCIAS

DEPARTAMENTO DE MATEMATICA

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ABSTRACT

In this paper prove that if μ is a Bowen-Ruellé-Sinai measure for a C^2 diffeomorphism $f: M \to M$ of a surface M then there exists a collection of hyperbolic horseshoes Ω_n satisfying the following conditions:

i) The Hausdorff dimension $\partial(\Omega_n)$ of Ω_n on the unstable manifold $W^u(x)$ of any point $x \varepsilon \Omega_n$ tends to 1 as $n \to \infty$.

ii) The expansion coefficient of each Ω_n is at last B, with $B > 1$. These conditions are sufficient if the function $\mu \to h_\mu(f) - x_\mu(f)$ is upper semicontinuous, here $h_\mu(f)$ and $x_\mu(f)$ denote the entropy and the future Lyapunov exponent of μ.

Escuela de Ciencias
CADIS-UCOLA
Apdo. 400 Barquisimeto
Venezuela

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§0. INTRODUCTION.

An ergodic invariant Borel probability measure μ for a C^2 diffeomorphism $f: M \to M$ of a surface M is called a *Bowen-Ruellê-Sinai* measure if its entropy $h_\mu(f)$ equals its future Lyapunov exponent of $X_\mu(f)$ and they are positive, see [1], [12] and [15]. These measures play an important role in the smooth ergodic theory of diynamical systems since by Ledrappier [5] they are absolutely continuous on unstable leaves, and therefore the set G_μ of future generic points of μ has positive Riemannian measure.

In this paper we prove that if μ is a Bowen-Ruellé-Sinai measure for a C^2 diffeomorphism $f: M \to M$ of a surface M then there exists a collection of hyperbolic horseshoes Ω_n satisfying the following conditions:

(i) The Hausdorff dimension $\partial(\Omega_n)$ of Ω_n on the unstable manifold $W^u(x)$ of any point $x \subset \Omega_n$ tends to 1 as $n \to \infty$.

(ii) The expansion coefficient of each Ω_n is at least B, with $B > 1$.

These conditions are sufficient if the function $\mu \to h_\mu(f)-X_\mu(f)$ is upper semicontinuous.

In [3] Jacobson constructed Bowen-Ruellé-Sinai meausres for certain non-invertible maps of $\mathbb{R}^2$ with one critical point, here the horseshoes Ω_n appeared as the complement of the backward orbit of neighbourhood of the singularity. This pattern of horseshoes of arbitrarily high unstable dimension is generic near tangencies of the stable and unstable manifolds of dissipative periodic points, as follows from [11], therefore this phenomenom is present in the creation of some strange attractors.

To prove the main theorem we shall extend some results of Katok [4] on entropy to pressure of continuous functions. We shall follow closely an umpublished version of S. Newhouse on Katok's work. The idea of studying pressure first came to our attention through McCluskey and Manning [6], in §1. we shall define measure-theoretic pressure for continuous functions in a similar way as Katok did for entropy, [4]. In §2. we revlew some results from smooth ergodic theory and in §3. we pove the result.

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§1. MEASURE-THEORETIC PRESSURE.

Let $T:X \to X$ be a homemorphism of a compact metric space X and μ a T-invariant Borel probability measure on X. If d denotes the metric of X, let

$$d_n(x,y) = \max \{d(T^i(x), T^i(y)) \mid 0 \leq i < n\},$$

for $x,y \in X$, $d_n(.,.)$ is a metric on X and *we shall call it the d_n-metric*. Denote by $B_n(x,\varepsilon)$ the ε-ball centred on x in the d_n-metric.

For $\varepsilon > 0$ a set $E \subset X$ is said to be (n,ε)-*spanning* if $X \subset \bigcup_{x \in E} B_n(x,\varepsilon)$.

Similarly for $\partial > 0$, $\varepsilon > 0$ a set $E \subset X$ is said to μ-(n,ε,∂)- *spanning* if $\mu(\bigcup_{x\varepsilon E} B_n(x,\varepsilon)) \geq 1-\partial$. A set E is said to be$(n,\varepsilon)$-*separated* if for $x=y \varepsilon E$ there exists $i \varepsilon [0,n)$ such that $d(T^i(x),T^i(y)) \geq \varepsilon$.

Let us denote by C(X) the set of continuous real valued functions $L:X \to \mathbb{R}$. If $L \varepsilon C(X)$, write $S_n L(x)$ for $\sum_{i=0}^{n-1} L(T^i(x))$. Define

$$Q(T,L,n,\varepsilon) = \inf\{ \sum_{x\varepsilon E} \exp S_n L(x) \mid E \text{ is } (n,\varepsilon)\text{-spanning}\}$$

and for $\partial > 0$

$$Q_,(T,n,\varepsilon,\partial) = \inf\{ \sum_{x\varepsilon E} \exp S_n L(x) \mid E \text{ is } \mu\text{-}(n,\varepsilon,\partial)\text{-spanning}\}.$$

The *topological pressure of* T is defined as the map $P(T,):C(X)\to\mathbb{R}$, where

$$P(T,L) = \lim_{c\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Q(T,L,n,\varepsilon).$$

Similarly the *measure theoretic pressure of* T *with respect to* μ is defined by

$$P_\mu(T,L) = \lim_{\partial\to 0} \lim_{c\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Q_\mu(T,L,n,\varepsilon,\partial).$$

THEOREM 1.1.

Let $T:X \to X$ *be a homeomorphism of a compact metric space* X, *then for* $L \varepsilon C(X)$ *and* μ an ergodic T-invariant Borel probability measure

$$P_\mu(T,L) = h_\mu(T) + \int L \, d\mu .$$

PROOF:

We shall prove that $P_\mu(T,L) \geq h_\mu(T) + \int L\, d\mu$. The converse inequality follows from applications of the Shannon-MacMillan-Breiman and the Ergodic theorems. The proof uses similar arguments to those of Misiurewicz's proof of the Variational Principal [8].

If B is a finite measurable partition of X, say $B=\{A_1,\ldots\ldots,A_k)$, choose $B_i \subset A_i$ compact such that $=\{B_o,B_1,\ldots,B_k\}$, where $B_o=X\setminus\bigcup_{i=1}^{k} B_i$, has conditional entropy, see [16] for definition, $H_\mu(B|\text{¥}) < 1$.

For $r > 0$, set

$$Y_N = \{y \in X | -1/n \log \mu(\text{¥}_n(y)) \geq h_\mu(T,\text{¥})-r \quad \forall n \geq N \text{ and}$$
$$1/n\ S_nL(y) \geq \int L\, d\mu - r \quad \forall n \geq N\},$$

where $\text{¥}_n(y)$ denotes the element of $\bigvee_{i=0}^{n-1} T^{-i}\text{¥}$ *containing* y *and* $h_\mu(T,\text{¥})$ he entropy of μ with respect to ¥.

A combination of the Shannon-MacMillan-Breiman theorem, Egorov's theorem and Birkhoff's Ergodic theorem implies that for large N, $\mu(Y_n) > 0$.

Choose $\varepsilon > 0$ such that:

(i) $2\varepsilon < b - \min\{d(B_i,B_j)|\ i=j\}$,

(ii) $d(x,y) < \varepsilon$ implies $|l(x) - L(y)| < r$.

Since $B_n(x,\varepsilon)\ Y_N$ can be covered by at most 2^n elements of ¥, then

$\mu(B_n(x,\varepsilon) \cap Y_N) \leq \exp n(\log 2 - h_\mu(T,\gamma) + r)$. Now let E be a μ-(n,ε,∂) -spanning set for $n \geq N$ and $0 < \partial << \mu(Y_n)$ and consider the set $E' = \{x \subset E \mid B_n(x,\varepsilon) \cap Y_N = \emptyset\}$. By continuity if $y(x) \in B_n(x,\varepsilon) \cap Y_N$ then $S_n L(x) - S_n L(y(x)) \geq nr$. Therefore it follows that

$$\sum_{x\varepsilon E} \exp S_n L(x) \exp -n\left(\int L d\mu - 3r - \log 2 + h_\mu(T,\gamma)\right) \geq$$

$$\sum_{x\varepsilon E} \exp\left(S_n L(x) - n \int L d\varepsilon\right) \exp -n(-3r - \log 2 + h_\mu(T,\gamma)) \geq$$

$$\sum_{x\varepsilon E} \exp\left(S_n L(x) - S_n L(y(x) + S_n L(y(x)) - n\int L d\mu\right) \exp -n(-3r - \log 2 + h_\mu(T,\gamma)) \geq$$

$$\sum_{x\varepsilon E} \exp -nr \exp -nr \exp 2nr \exp -n(-r - \log 2 + h_\mu(T,\gamma)) \geq$$

$$\sum_{x\varepsilon E} \exp -n(-r - \log 2 + h_\mu(T,\gamma)) \geq$$

$$\sum_{x\varepsilon E} \mu(B_n(x,\varepsilon) \cap Y_n) \geq \mu\left(\bigcup_{x\varepsilon E} B_n(x,\varepsilon)\right) > 0,$$

which implies that

$$Q_\mu(T,L,n,\varepsilon,\partial) \geq \int L d\mu - 3r - \log 2 + h_\mu(T,\gamma).$$

Since r and B are arbitrary, and $h_\mu(T,B) \leq h\ (T,\gamma) + h_\mu(B|\gamma)$

$$P_\mu(T,L) \geq \int L d\mu - 1 - \log 2 + h_\mu(T).$$

Now apply the above procedure to T^n and $S_m L$, to obtain

$$P_\mu(T,L) \geq 1/m(S_m L d\mu - 1 - \log 2 + h_\mu(T^n))$$

so letting $m \to \infty$

$$p_\mu(T,L) \geq \int L d\mu + h_\mu(T).$$

The converse inequality follows from the Shannon-MacMillan-Breiman theorem and the Ergodic theorem. #

REMARK. The above proof is different from the one given by Katok [4] for entropy since we do not use the Hamming metrics.

COROLLARY 1.2.

Let $T:X \to X$ *and* $L:X \to \mathbb{R}$ *be as above, then*

$$P(T,L) = \sup \{p_\mu(T,L) \mid \mu \text{ is T-invariant}\}.$$

PROOF.

Apply Walters' Variational Principal, [16]. #

§2. SOME DEFINITIONS AND FACTS FROM SMOOTH ERGODIC THEORY.

For an invariant Borel probability measure μ of a diffeomorphism $f: M \to M$ of a surface M we define its *future Lyapunov exponent* as

$$X_\mu(f) = \lim_{n\to\infty} 1/n \int \log \| D_x f^n \| \, d\mu \ .$$

Similarly one can define the *past Lyapunov exponent* $X_\mu(f^{-1})$ of μ by considering f^{-1}. A measure μ is said to have *non-zero exponent* if both its future and past Lyapunov exponent are different from zero, so

$$X_\mu-\{\min X_\mu(f),\ X_\mu(f^{-1})\} > 0.$$

Ruellé proved in [13] that $h_\mu(f) \leq \max\{0, X_\mu\}$ and in [7] we proved that if μ is ergodic with positive entropy and f is C^2 then $h_\mu(f) / X_\mu(f)$ equals the Hausdorff dimension $\partial(\mu)$ of the quotient measure defined by the family of stable manifolds.

McCluskey and Manning [6] have shown that if Ω is hyperbolic basic set for f, then the Hausdorff dimension of the intersection of the unstable manifold of any point $x \subset \Omega$ with the set Ω equals the unique zero of the function $t \to p(f|\Omega, -tL^u)$, with $L^u(x)=\log\| D_x f|E^u\|$ and E^u denotes the expanding subspace of the tangent space at x. Unfortunately their methods do not extend to then non-uniform hyperbolic sets, since the function $L^u(x)$ is not neccesarilly continuous and the existence of Markov pertitions [1] is not guaranteed.

We say that $\Omega \subset M$ is a *horseshoe* for f if there exists $n > 0$ such that $\Omega = \Omega^0 \cup \ldots \cup \Omega^{n-1}$ and $f^n|\Omega$ is conjugate a full shift in k symbols, see [9]. See [11] for general definitions and standard results on smooth dynamical systems.

The formulation of the following definitions and statements are due to S. Newhouse. Let $f: M \to M$ be as above.

Fix $0 \leq r < 1$ and let $I = \{-1,1\}$ for $u: I \to I$ a C^1 map with $|Du| < r$

we say that $\{(u(y),y)\}$ $(\{x,u(x)\})$ is a *u-curve* (*s-curve*). Given $u_1 \leq u_2$ u-curves (s-curves) we shall call the set $V=\{(x,y)\varepsilon I^2 | U_1(y) \leq x \leq u_2(y)\}$ $(H =\{(x,y) \ \varepsilon \ I^2 | u_1(x) \leq y \leq u_2(x)\})$ a *u-rectangle* (*s-rectangle*). We shall say that $R_x \subset M$ is a *rectangle* in M if there exists a C^1 embedding G such that $G(I^2) = R_x$ and $G(0,0) = x$, if U is a u-rectangle in I^2 we shall call G(U) a u-rectangle in R_x.

A (r,λ)-*rectangle cover* of a set $\Omega \subset M$ for $r > 0, \lambda > 1$ is a finite collection of rectangles $\{R_{x1}, R_{x2}, \ldots, R_{xt}\}$ on M satisfying:

(1) $\Omega \subset \bigcup_{i=1}^{t} B(x_i,r)$, $B(x_i,r) \subset \text{int } R_{xi}$ and $x_i \varepsilon \ \Omega$.

(ii) If $x \ \varepsilon \ \Omega$, $f^n(x) \ \varepsilon \ \Omega$ for some $n > 0$, $x \ \varepsilon \ B(x_i,r)$ and $f^n(x) \varepsilon \ B(x_j,r)$, then the connected component of $R_{xi} \cap f^{-n} R_{xj}$ containing x, that we denote by $C(x,R_{xi} \cap f^{-n} R_{xj})$, is an s-rectangle in R_{xi} and $f^m C(x,R_{xi} \cap f^{-1} R_{xj})$ is a u-rectangle in R_{xj}.

(iii) $\text{diam } f^m C(x,R_{xi} \cap f^{-n} R_{xj}) \leq 3 \text{ diam } R_{xi} \max\{\lambda^{-m},\lambda^{-(n-m)}\}$ for $0 \leq m \leq n$. #

THEOREM 2.1. [4]

Let $f: M \to M$ be a C^2 *diffeomorphism of a surface* M *preserving an ergodic Borel probability measure* μ *with non-zero exponents, then for any* $r > 0$ *there exists a compact set* Ω *with measure arbitrarilly near* 1 *which admits a* (r,λ)-*rectangle cover of small diameters and* $\lambda=\lambda(x_\mu)$. #

§3. THE MAIN RESULT.

THEOREM 3.1.

If μ is a Bowen-Rullé-Sinai measure for a C^2 diffeomorphism $f:M\to M$ of a surface M then there exists a collection of hyperbolic horseshoes Ω_n satisfying the following conditions:

(*i*) *The Hausdorff dimension $\partial(\Omega_n)$ of Ω_n on the unstable manifold $W^u(x)$ of any point $x \in \Omega_n$ tends to 1 as $n \to \infty$.*

(*ii*) *The expansion coefficient of each Ω_n is at least B, with $B > 1$.*

Before giving the proof of the theorem we shall establish some conditions to have the converse of the theorem true.

COROLLARY 3.2.

These conditions are sufficient if the function $\mu \to h_\mu(f) - x_\mu(f)$ is upper semicontinuous.

PROOF.

Since the horseshoes Ω_n are hyperbolic then for each n there exist a measure μ_n supported on Ω_n such that $\partial(\mu_n) = \partial(\Omega_n)$, by [7] $\partial(\mu_n) = h_{\mu n}(f)/x_{\mu n}(f)$ and by (ii) $x_{\mu n}(f) \geq B > 1$ for all n. Thus if $\mu_n \to \mu$ weakly then $h_\mu(f) = x_\mu(f)$ and since the function $\mu \to x\ (f)$ is always upper semicontinuous then $h_\mu(f) = x_\mu(f) > 0$. #

PROOF OF THE THEOREM.

If we write $F_k(x)$ for $-1/k \log\| D_x f^k \|$, then $x_\mu(f) = \inf_k \int -F_k(x)d\mu$

and $0=h_\mu(f)-x_\mu(f) = \sup_k\{h_\mu(f) + \int F_k(x)d\mu\}$, so for $r > 0$ there exists $k > 0$ such that $0 \geq h_\mu(f) + \int F_k(x)d\mu > -r$. The function $x \to F_k(x)$ is continuous and therefore $p_\mu(f,F_k) = h_\mu(f) + \int F_k(x)d\mu$. Now let $\partial > 0$, $\varepsilon > 0$ be such that

$$\limsup_{n\to\infty} 1/n \log Q(n,\varepsilon,\partial) \geq -r$$

and if $d(x,y) < \varepsilon$ then $|F_k(x) - F_k(y)| < r$.

By Theorem 2.1 we can choose $\Omega = \Omega_x \subset M$ such that $\mu(\Omega) > 1-\partial/2$, for $r > 0$ small and $\lambda = \lambda(x_\mu) > 1$ the set Ω admits a (r,λ)-rectangle cover $\{Rx_1, Rx_2,\ldots Rx_i\}$ such that diam $Rx_i < \varepsilon/3$. Now let γ be a partition of M with diam $\gamma < r/2$ and

$$\Omega_n = \{x \in M \mid f^q(x) \in \gamma(x) \text{ for some } q \in [n,(1+r)n]\}.$$

LEMMA 3.3. [4]

$$\mu(\Omega_n) \to \mu(\Omega) \text{ as } n \to \infty. \ \#$$

So for n large $\mu(\Omega_n) > 1-\partial$. Let $E_n \subset \Omega_n$ be an (n,ε)-separated set of maximal cardinality, clearly $\Omega_n \subset \bigcup_{x\in E_n} B_n(x,\varepsilon)$ and therefore there exist infinitely many n's such that

$$\sum_{x\in E_n} \exp S_n(F_k(x)) \geq \exp -2nr.$$

For each $q \in [n,(1+r)n]$ let $V_q = \{x \in E_n \mid f^q(x) \in \gamma(x)\}$, now let m be the value of q that maximises $\sum_{x\in V_q} \exp S_n(F_k(x))$, since $\exp nr \geq nr$

$$\sum_{x\varepsilon V_m} \exp S_n(F_k(x)) \geq \exp -3nr.$$

Consider $V_m \cap Rx_j$ for $1 \leq j \leq t$ and choose the value i of j that maximizes $\sum_{x\varepsilon V_m \cap Rx_j} \exp S_n(F_k(x))$. Thus if we write D_m for $V_m \cap Rx_j$

$$\sum_{x\varepsilon D_m} \exp S_n(F_k(x)) \geq 1/t \sum_{x\varepsilon V_m} \exp S_n(F_k(x)) \geq 1/t \exp -4nr.$$

So consider Rx_j and D_m. Each $x \varepsilon D_m$ returms to Rx_i in m iterations, thus $C(f^m(x), Rx_i \cap f^m Rx_i)$ is a u-rectangle in Rx_i and $f^{-m}C(f^m(x), Rx_i \cap f^m Rx_i)$ an s-rectangle. This follows from the facts that $d(x_i,x) < r$ and $d(f^m(x),x_i) < r$, and (ii) of the definition of a(r,λ)-rectangle cover.

If $y \subset C(x, Rx_i \cap f^{-m} Rx_i)$ then by (iii) of the definition of a $(r, \;)$-rectangle cover

$$d(f'(x),f'(y)) \leq \text{diam } f'(C(x,Rx_i \;\; f^{-m}Rx_i)) \leq 3 \text{ diam } Rx_i \leq \varepsilon \text{ for } 1 \varepsilon [0,m),$$

wich implies: (i) $|S_m F_k(y) - S_m F_k(x)| \leq mr$ and (ii) that if y=x and $y \subset C(x, Rx_i \cap f^{-m} Rx_i)$ then $y \varepsilon V_m$, otherwise it would contradict the separability of V_m.

Hence there exists $\#V_m$ disjoint s-rectangles mapped by f^m onto $\#V_m$ u-rectangles. Using Propositions 2.4 and 2.5 of [4] it can be shown that the definition of a shoe given in [9] is satisfied. So let

$$\Omega^* = \bigcap_{i=-\infty}^{\infty} f^{m1}\Big(\bigcup_{x\varepsilon D_m} C(x, Rx_i \cap f^{-m} Rx_i)\Big)$$

by Theorem 3.1 of [9] $f^m|\Omega^*$ is conjugate to the full shift in $\#V_m$ symbols. The same arguments used by Katok [4] (pag. 163-165) show that Ω^* is a hyperbolic set for f^m, thus $\Omega \bigcup_{i=0}^{m-1} f^i \Omega^*$ is a hyperbolic set for f. Moreover there exists a constant $c = c(\Omega)$ such that if $x \in \Omega$ and $T_x M = E^u(x) \oplus E^s(x)$ then

$$\| D_x f^j(v) \| \leq c \lambda^{-j} \| v \| \quad \text{for} \quad v \in E^s(x),$$

and

$$\| D_x f^{-j}(c) \| \leq c \lambda^{-j} \| v \| \quad \text{for} \quad v \in E^u(x).$$

We shall estimate $P(f|\Omega, F_k|\Omega)$. It is known [14] that

$$P(f|\Omega, F_k|\Omega) = \limsup_{j\to\infty} 1/j \log \sum_{x\in A_j} \exp S_j F_k(x),$$

where $A_j = \{x \in \Omega | f^j(x) = x\}$.

So if $y \subset \Omega^*$ is a periodic point of period $N = jm$, then there exists a unique j-tuple $x=x(y) = (x^1, x^2, \ldots, x^j)$, $x^i \in V_m$, such that

$$d(f^i(y), f^i(x^i)) < \varepsilon \quad \text{for } i \in [0,m)$$

$$d(f^i(y), f^i(x^2)) < \varepsilon \quad \text{for } i \in [m,2m)$$

$$\vdots$$

$$d(f^i(y), f^i(x^i)) < \varepsilon \quad \text{for } i \in [(j-1)m, jm).$$

Therefore $S_n(F_k(y)+r) \geq S_m F_k(x^1) + S_m F_k(x^2) + \ldots + S_m F_k(x^j)$, and

$$\sum_{y\in A_n} S_N(F_k(y)+r) \geq \sum_{x(y):y\in A_n} \prod_{i=1}^{j} \exp S_m F_k(x^i) = \left(\sum_{x\in V_m} \exp S_m F_k(x)\right)^i.$$

Since $S_m F_k(x) \geq S_n F_k(x)+(M-n-1)\inf F_k$ and $N = jm$

$$1/j \log \sum_{y\varepsilon A_{jm}} S_{jm}(F_k(y)+r) \geq \log \sum_{x\varepsilon V_m} \exp S_m F_k(x) \geq$$

$$\log \sum_{x\varepsilon V_m} \exp S_n F_k(x)+(m-n-1) \inf F_k.$$

Thus letting $j \to \infty$ we obtain that

$$P(f|\Omega,F_k+r|\Omega) \geq 1/m \log \sum_{x\varepsilon V_m} \exp S_n F_k(x)+(m-n-1) \inf F_k \geq$$

$$n/m \ (P_\mu(f,F_k) - 34)+(m-n-1) \inf F_k.$$

Now using the inequalities

$$1/(1+r)n \leq (m-n-1)/m \leq r-1/n, \quad 1/(1+r)\leq n/m,$$

so when $n \to \infty$, we obtain that

$$P(f|\Omega,F_k+r|\Omega) \geq (P_\mu(f,F_k)-3r)/(1+r)$$

Since $f|\Omega$ is expansive there exists an equilibrium state [14] μ^r for $F_k + r$ so that

$$P(f|\Omega,F_k+r|\Omega) = h_\mu r(f)+ \int(F_k+r)d\mu^r \geq -4r/(1+r)-r,$$

hence

$$0 \geq h_\mu r(f) - x_\mu r \geq h_\mu r(f)- 1/k \int \log\|D_k f^k\| \, d\mu^r \geq -4r/(1+r)-2r.$$

Finally letting $r \to 0$ we can choose a sequence of ergodic measures μ'

such that $x_{\mu^i} \geq \lambda$ and

$$\sup_i \{h_{\mu^i}(f) - x_{\mu^i}\} = 0.$$

By [7] $h_{\mu^i}(f) = \partial(\mu')x_{\mu^i}$, so $\sup_i \{(\partial(\mu')-1)x_{\mu^i}\} = 0$ and thus $\sup_1 \partial(\mu') = 1$ and since each μ' is supported on a Ω_i the result follows.

#

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