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ON A PROBLEM OF SAMUEL

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Interest in euclidean rings was revived with the appearance of an excellent and interesting paper of Samuel [3] . Since then more than 20 papers have appeared on this topic. In this paper we consider the following problem of Samuel . Let A be a unique factorization domain. Then every element a of A, $a \neq 0$, is of the form $a = u.\Pi_1^{e_1}\ldots\Pi_r^{e_r}$, where u is a unit of A, Π_i are primes of A and $e_i \geq 0$ are integers for $1 \leq i \leq r$. Set $\phi(a) = e_1 + \ldots\ldots + e_r$. Under what conditions A is euclidean with respect to ϕ?

Before considering this question, we give some examples of domains which are euclidean with respect to a function of the type ϕ.

<u>Examples</u>.

1) $A = k[x]$, the polynomial ring with coefficients in an algebraically closed field k.

2) A is a semilocal principal ideal domain [see Prop.5,3].

3) A is a principal ideal domain such that $A^* \to (\frac{A}{Aa})^*$ is surjective for all a in A, where A^* is the set of all units of A.

4) If A is euclidean for a function θ then localizing A at all primes Π such that $\theta(\Pi) \geq 2$, we find that the localized ring is euclidean for a function of the type ϕ.

In view of these examples we may assume that A is contained in all but a finite number of valuation rings of K, where K is the field of fractions of A. Now we consider the following two cases:

Case 1) <u>A contains a field k</u>.

In this case we suppose that A is a finitely-generated k-algebra. This also includes the case when characteristic(A) = $p \neq 0$. Since A is euclidean we find that transcendental degree of K over k is 0 or 1 i.e. either A is a field or K is an algebraic function field in one variable over K. Thus $A = \bigcap_{P \notin S} v_P$, where S is a finite set of primes of K and v_P is the valuation ring of K at the prime P.

Case 2) <u>A dues not contain a field</u>.

Thus characteristic (A) = 0 and $\mathbb{Z} \subset A$. we now assume that A is a finitely-generated $\mathbb{Z}$-algebra. Since A is euclidean, we find that K, the quotient field of A, is a number field. Thus

$A = \bigcap_{P \notin S} v_P$, where S is a finite set of primes of K containing all the archimedean primes A and v_P is the valuation ring of K at the prime P.

Now we state, without proof, a theorem of Queen and Weinberger. Theorem [p. 68,2] Let $A = \bigcap_{P \notin S} v_P$ be a principal ideal domain, $\#(S) \geq 2$ and that K is a global field. We also assume a certain generalised Riemann hypothesis if K is a number field. Then A is euclidean and the smallest algorithm θ on A is given by

$$\theta(x) = \sum_{P \notin S} \mathrm{ord}_P(x) . n_P \text{ , for } x \neq 0$$

where $n_P = 1$ if $A^* \longrightarrow (\frac{A}{P})^*$ is surjective, $n_P = 2$ otherwise.

In view of this we find that if a subring A of a global field K is euclidean for a function of the type ϕ such that $\phi(\Pi) = 1$ for all primes Π of A, then A is a localization at a large number of primes of K i.e. S is infinite.

We also need the following .

THEOREM [Cunnea, 1]. Let K be an algebraic function field over an algebraically closed field $\mathbf{k}$. Let A be a subring of K such that $k \subset A$, K is a field of fractions of A and A is contained in all but a finite number of valuation rings of K. Then A is a unique factorization domain if and only if genus of K is 0.

Using this result we prove the following.

<u>MAIN THEOREM</u>. Let A be a domain such that

$$k = \{0\} \cup \{\text{units of } A\}$$

is a field and A is not a field. Let K be the quotient field of A. Suppose now that A is a finitely-generated k-algebra which is euclidean for a function ϕ such that $\phi(\Pi) = 1$ for all primes Π of A. Then k is algebraically closed and genus of K is 0. Moreover, $A = k[x]$, the polynomial ring in x with coefficients in k.

<u>PROOF</u>. Since A is euclidean and K is the field of fractions of A, we find that transcendental degree of K over k is less than or equal to 1. Now tr. degree$(K/k) = 0$ implies that A is integral over k and thus a field, a contradiction to our hypothesis. Thus tr.degree$(K/k) = 1$. Choose x in A such that x is transcendental over k. Let $f(x)$ be an irreducible polynomial in $k[x]$ and let

$$f(x) = u.\Pi_1^{e_1} \ldots . \Pi_r^{e_r}$$

be its prime de composition in A. where u is a unit of A and Π_i are primes of A, $1 \leq i \leq r$. Now $k = \{0\} \cup$ $\cup$ {units of A} and $\phi(\Pi_1) = 1$ implies that

$$\left[\frac{A}{\Pi_1 A} : k\right] = 1$$

i.e. $$\left[\frac{k[x]}{(f(x))} : k\right] = 1$$

i.e. degree of $f(x) = 1$

Thus we see that k is algebraically closed. It now follows that k is an algebraic function field over an algebraically closed field k. Since A is euclidean, using the result of Cunnea we find that genus of K is 0 and thus $K = k(x)$, a rational field. Since

$$k = \{0\} \cup \{\text{units of } A\}$$

is a field, we find that $A = k[x]$ and whence the result.

REFERENCES

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3.- P.Samuel "Abount Euclidean Rings" J.Algebra,19(1971),282-301.