NOTAS DE MATEMATICA

Nº 23

REGULAR GENERAL CONTACT MANIFOLDS

POR

JORGE SAENZ

DEPARTAMENTO DE MATEMATICA

FACULTAD DE CIENCIAS

UNIVERSIDAD DE LOS ANDES

MERIDA - VENEZUELA

1978

PREPRINT

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JORGE SAENZ C.

1. INTRODUCCION

It has been proved that a compact connected manifold M^{2n+s} with a regular normal f-structure is the bundle space a principal T^s-bundle over a complex manifold N^{2n}. Moreover, if M^{2n+s} is a K-manifold, then N^{2n} is a Kaehler manifold, [2] . In this work we prove that (theorem 4.1) if the K-structure on M^{2n+s} is an S-structure, then N^{2n} is a Hodge manifold. Conversely (theorem 4.4, given a Hodge manifold N^{2n} and any $s \geq 1$, there exists a principal toroidal bundle $M(N,T^s)$ over N, whose bundle space M^{2n+s} has a regular S-structure.

2. NORMAL f-STRUCTURES

A C^∞-manifold M^{2n+s}, $n \geq 1$, is said to have an f-structure, if the structural group of its tangent bundle is reducible to $U(n) \times O(s)$. This is equivalent to the existence of a tensor field on M of type (1,1), rank 2n, satisfying $f^3 + f = 0$. Almost complex structures (s = 0) and almost contact structures (s = 1) are two ecamples of f-structures. If there exist vector fields E_i and 1-forms, η^i, $1 \leq i \leq s$ such that

$$f(E_i) = 0,\ \eta^i(E_j) = \delta^i_j,\ \eta^i \circ f = 0,\ f^2 = -I + \sum_{i=1}^{s} \eta^i \otimes E_i$$

we say that M^{2n+s} has a framed f-structure, or, simply an (f, E_i, η^i)-structure. A framed f-structure is normal if

$$S = |f,f| + \sum_{i=1}^{s} d\eta^i \otimes E_i$$

vanishes, where $|f,f|$ is the Nijenhuis tensor of f. In this case we have [3] :

1) $L_{E_i}\eta^j = 0$, 2) $[E_i,E_j] = 0$, 3) $L_{E_i} f = 0$,

4) $d\eta^i(fX,Y) = - d\eta^i(X,fY)$

The equality 2) implies that the vertical distribution (the one generated by all the E_i) is integrable.

It is known that for any (f,E_i,η^i)-structure there exists a Riemannian metric g which satisfies

$$g(X,Y) = g(fX,fY) + \sum_{i=1}^{s} \eta^i(X)\,\eta^i(Y)$$

A framed f-structure together with this metric is called a framed metric f-structure, or, simple, an (f,E_i,η^i,g)-structure. The 2-form

$$F(X,Y) = g(X,fY)$$

is called the fundamental 2-form of the (f,E_i,η^i,g)-structure. A K-structure is a normal (f,E_i,η^i,g)-structure whose fundamental 2-form is closed.

Let D be an integrable distribution of dimension h on a manifold N^m. A cubical coordinate neighborhood $(U,(X_j,\ldots\ldots,X^m))$ on N^m is said to be regular with respect to D if $\frac{\partial}{\partial X'},\ldots\ldots,\frac{\partial}{\partial X^h}$ is a basis for $D(p)$, for every $p \in U$, and if each leaf of D intersects U in at most one n-dimensional slice of $(U,(X',\ldots\ldots,X^m))$. We call D regular if each point $p \in N$ has a cubical coordinate neighborhood which is regular with respect to D.

An (f,E_i,n^i)-structure is said to be regular if the vertical distribution is integrable and regular, and if each E_i is regular (the distribution generated by E_i is regular).

Let's state the theorem mentioned at the bigining:

THEOREM 2.1. (Blair, Ludden, Yano). Let M^{2n+s}, $n \geq 1$, be a compact conneted manifold with a regular framed f-structure. Then M^{2n+s} is the bundle space of a principal toroidal bundle over a complex manifold N^{2n}. Moreover if the framed f-structure is a K-structure, then N^{2n} is a Kaehler manifold.

3. TOROIDAL BUNDLES

Let $T' = S'$ and $T^s = \underbrace{S'\times\ldots\times S'}_{s}$ be the one - dimensional and s-dimensional torus respectively. Since these

Lie groups are commutative, by choosing A, a non-zero element of the Lie algebra $L(T')$ of T', we identify $L(T')$ with R, and $L(T^s) = L(T')\times \ldots \times L(T')$ with R^s by means of

$$\underbrace{(0,\ldots,A\ 0,\ldots\ldots,0)}_{i} \longleftrightarrow e_i \quad ,$$

where $e_1,\ldots,e_s$ is the canonical basis of $\mathbb{R}^s$.

Let $P[N,T^s]$ be the set of all T^s-bundles over the manifold N. If $P(N,T^s,\pi)$ and $Q(N,T^s,\pi)$ are two elements in this set, on $\Delta(P\times Q) = \{(u,v) \in P\times Q \ / \ \pi(u) = \pi'(v)\}$ we define the equivalent relation:

$(u_1,v_1) \sim (u_2,v_2) \iff \exists\ t \in T^s$ such that $(u_1t,v_1t^{-1}) = (u_2,v_2)$.

The action of T^s on $\Delta(P\times Q)$ given by $((u,v),t) \to (ut,v)$, induces an action of T^s on

$$P + Q = \frac{\Delta(P\times Q)}{\sim}$$

obteining, in this way, the new T^s- bundle $P + Q$. It is known that $P[N,T^s]$ with this operation, "+", is an abelian group whose identity element is the trivial bundle $N \times T^s$. [4] .

If ω is a connection form with curvature form Ω of a bundle $P(N,T^s)$, then.

$$\omega = \sum_{i=1}^{s} \omega_i \otimes e_i \quad \text{and} \quad \Omega = \sum_{i=1}^{s} d\omega_i \otimes e_i$$

Each real 2-form $d\omega_i$ is horizontal and right invariant, therefore there exists a unique real 2-form Ω_i^* on N satisfying $d\omega_i = \pi^* \Omega_i^*$. Since the forms Ω_i^* are closed, they determine s cohomology classes $[\Omega_i^*]$, $1 \leq i \leq s$ in $H^2(N,R)$. These cohomology classes are independent from the conection. In this way we get the function

$$\Psi: P[N,T^s] \to \bigoplus_{i=1}^{s} H^2(N,R) \text{ given by } P \to ([\Omega_1^*],\ldots,[\Omega_s^*]).$$

Our intention now is to show that Ψ is a group homomorphism.

Suppose that $\{\phi_{\beta\alpha}\}$ are the transition function of $P(N,T^s)$ corresponding to some covering $\{U_\alpha\}$. Each function $\phi_{\beta\alpha} : U_\beta \cap U_\alpha \to T^s$ can be written as $\left(\phi^1_{\beta\alpha},\ldots,\phi^s_{\beta\alpha}\right)$. Now $\{\phi^i_{\beta\alpha}\}$ are the transition functions of a 1-dimensional toroidal bundle P_i over N. If we construct the whitney sum $P_1 \oplus \ldots \oplus P_s$, it happens that a set of transition functions of this sum is precisely $\{\phi_{\beta\alpha}\}$. In other words, P and $P_1 \oplus \ldots \oplus P_s$ have the same transition function. Therefore we may assume that

$$P = P_1 \oplus \ldots \oplus P_s \quad \text{and} \quad P[N,T^s] = \bigoplus_{i=1}^{s} P[N,T^1]$$

Let h_i be the proyection $h_i: P_1 \oplus \ldots \oplus P_s \to P_i$. If Ω_i is a curvature form on P_i, there is a conection on P whose curvature form Ω satisfies:

$$\Omega = \sum_{i=1}^{s} h_i^* \Omega_i \otimes e_i$$

Therefore we can assume that the function

$$\Psi : P[N,T^s] = \bigoplus_{i=1}^{s} P[N,T'] \to \bigoplus_{i=1}^{s} H^2(N,R)$$

is given by $\Psi = \psi x \underset{s}{\ldots} x\psi$ where ψ is the function

$$\psi : P[N,T'] \to H^2(N,R) \text{ such that } \psi(P_i) = [\Omega_i^*]$$

But this ψ is precisely the function defined by S. Kobayashi in page 32 of [4] . Furthermore, he proves that $\psi : P[N,T'] \to H^2(N,R)$ is a group homomorphism wich sends $P[N,T']$ onto $H^2(N,Z)_b$, where $H^2(N,Z)_b$ is the subgroup of $H^2(N,R)$ formed by all the elements which contain an integral closed from. Therefore

THEOREM 3.1. The function

$$\Psi : P[N,T^s] \to \bigoplus_{i=1}^{s} H^2(N,R)$$

$$P \to ([\Omega_1^*], \ldots, [\Omega_s^*])$$

is a group homomorphism, which sends $P[N,T^s]$ onto $\bigoplus_{i=1}^{s} H^2(N,Z)_b$

4. REGULAR S-STRUCTURES

DEFINICION. A manifold M^{2n+s} is said to have an s-contact structure if there exist on M s global, linearly independent 1-forms $\eta^1, \ldots, \eta^s$ such that $d\eta' = \ldots = d\eta^s$, $d\eta^i$ has rank $2n$ and, at every point of M,

$$\eta' \wedge \ldots \wedge \eta^s \wedge (d\eta^i)^n \neq 0$$

It is known [1] that if M^{2n+s} has s-contact structure, then it has an (f, E_i, η^i, g)-structure, which we call associated to the s-contact structure, such that $F = d\eta^i$, where F is the fundamental 2-form. A normal (f, E_i, η^i, g)-structure asociated to an s-contact structure is called an S-structure. Notice that an S-structure is a K-structure.

THEOREM 4.1. Let M^{2n+s} be a compact connected manifold with a regular S-structure (f, E_i, η^i, g), $i = 1, \ldots, s$. Then M^{2n+s} is the bundle space of a principal toroidal bundle over a Hodge manifold N^{2n} .

PROOF. By theorem 2.1 and its proof we have that M^{2n+s} is the bundle space of a principal T^s bundle over a Kaehler manifold N^{2n}, and that the group action is given by the one-parameter groups of transformations of the vector fields $E_1, \ldots, E_s$.

Now we claim that the form

$$\omega = \sum_{i=1}^{s} \eta^i \otimes e_i$$

is a conection form. This is, ω satisfies:

a) $R_t^* \omega = \omega$, for $t \in T^s$.

b) $\omega(X^*) = X$, where X^* is the fundamental vector fields of X, with X in the Lie algebra of T^s.

Part a) follows from the fact $L_{F_i} \eta^j = 0, i, j = 1, \dots s_j$, which is a consecuence of the normality of the S-structure . For part b) it suffices to prove it for the vector e_i , $i = 1, \dots, s$. But this follows immediately from $e_i^* = F_i$.

On the other hand from the proof of theorem 2.1. We also have that the fundamental form of the f-structure, F, and the fundamental for of the Kaehlerian structure, Ω^* , are related by

$$F = \pi^* \Omega^*$$

Where π is bundle proyection. But, in the particular case of an S-structure we have $F = d\eta^i$, $i = 1, \dots, s$. Therefore $d\eta^i = \pi^* \Omega^*$. Hence, by theorem 3.1. $[\Omega^*]$ is in $H(N, Z)_b$, which says that N^{2n} is a Hogde manifold.

THEOREM 4.2. Let $M(N,T^s, \pi)$ be a principal toroidal bundle whose base space N^{2n} has an almost Hermitian structure. Then M has a regular (f,E_i, η^i, σ)-structure, $i = 1, \ldots, s$.

PROOF: Fix a conection form $\omega = \sum_{i=1}^{s} \eta^i \otimes e_i$ on M and let E_i be the fundamental vector of e_i, $1 \leq i \leq s$. Then we have

$$\eta^i (E_j) = \delta^i_j$$

Let (J,g') be the almost Hermitian structure of N. If $u \in M$, $\pi(u) = v$ and $\bar{\pi}_v: T_v(N) \to T_u(M)$ is the lifting with respect to the fixed connection, define f by

$$f(X) = (\bar{\pi}_v \circ j \circ \pi_u) (X), \; X \in T_u (M)$$

Then we have $f(E_i) = 0$ and $\eta^i \circ f = 0$ $i = 1, \ldots, s$. We also have

$$f^2(X) = (\bar{\pi} \circ j \circ \pi)^2(X) = -(\bar{\pi} \circ \pi)(X) = -X + \sum_{i=1}^{s} \eta^i(X)E_i$$

This is, $f^2 = -I + \sum_{i=1}^{s} \eta^i \otimes E_i$. Thus we have an (f,E_i,η^i)-structure, $1 \leq i \leq s$, on M. Furthermore, the Riemannian metric g on M defined by

$$g(X,Y) = g'(\pi X,\pi Y) + \sum_{i=1}^{s} \eta^i(X)\eta^i(Y)$$

is associated to this (f, E_i, η^i)-structure, since

$$g(fX,fY) = g'(\pi fX, \pi fY) + \sum_{i=1}^{s} \eta^i(fX)\eta^i(fY) = g'(J\pi X, J\pi Y) =$$

$$= g'(\pi X, \pi Y) = g(X,Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y)$$

It is clear from the definition of E_i that each one of these is regular. The regularity of the distribution determined by all the E_i's (vertical distribution) follows form the theorem XIV of [5], which says that if the leaf space of an integral distribution is a manifold and if the proyection mapping takes the tangent space of any point onto the tangent space of its proyection, then the distribution must be regular.

THEOREM 4.3. The framed f-structure defined in the previous theorem is normal if and only if the following two conditions hold:

1) J is a complex structure, 2) $d\omega(fX,Y) = -d\omega(X,fY)$, for any X,Y.

PROOF: Since 2) es equivalent to 3) $d\omega(fX,fY) = d\omega(X,Y)$, the theorem will follow as soon as we prove the two equalities:

a) $\pi(S(X,Y)) = [J,J](\pi X, \pi Y)$; X,Y right invariant vector fields.

b) $\omega(S(X,Y)) = d\omega(X,Y) - d\omega(fX,fY)$, for any X,Y.

a) If X,Y are right invariant vector fields on M, so are $[X,Y]$, $f(X)$ and $f(Y)$. (f is right invariant). Besides, we have the relations: $\pi[X,Y] = [\pi X, \pi Y]$ and $\pi\circ f = J\circ\pi$. Therefore

$$\pi(S(X,Y)) = \pi([f,f]\ (X,Y) + \sum d\eta^i\ (X,Y)E_i) = [J,J]\ (\pi X, \pi Y)$$

b) Since f is horizontal we have $d\omega(fX,fY) = -\,\omega([fX,fY])$. Hence $\omega(S(X,Y)) = \omega([fX,fY]) + d\omega(X,Y) = -\,d\omega(fX,fY) + d\omega(X,Y)$.

THEOREM 4.4. Let N^{2n} be a Hodge manifold. Then for each $s \geq 1$ there exists a principal toroidal bundle $M(N,T^s,\pi)$, whose bundle space M^{2n+s} has a regular S-structure.

PROOF: Let (J,g') be the Hodge structure on N, and Ω^* its fundamental 2-form. Since $[\Omega^*] \in H^2(N,Z)_b$, then

$$(\underbrace{[\Omega^*],\ldots\ldots,[\Omega^*]}_{s}) \in \bigoplus_{i=1}^{s} H^2(N,Z)_b$$

By theorem 3.1., there exists a toroidal bundle $M = M(N,T^s,\pi)$ such that $\Psi(M) = ([\Omega^*],\ldots\ldots,[\Omega^*])$. We can find a connection form $\omega = \sum_{i=1}^{s} \eta^i \otimes e_i$ whose curvature form $d\omega$ satisfies

$$d\omega = \sum_{i=1}^{s} d\eta^i \otimes e_i = \sum_{i=1}^{s} \pi^*\ \Omega^* \otimes e_i$$

The forms $\eta',\ldots\ldots,\eta^s$ define an s-contact structure on M^{2n+s}. In fact, since $d\eta^i = \pi^*\ \Omega^*$, the ranh of $d\eta^i$ is $2n$.

On the other hand, if $E_1, \ldots, E_s$ are the fundamental vector fields of $e_1, \ldots, e_s$, we have $\eta^i(E_j) = \delta^i_j$. Now, taking $E_1, \ldots, E_s$ and $X_1, \ldots, X_{2n}$ horizontal and linearly independent vectors, we get

$$\eta' \wedge \ldots \wedge \eta^s \wedge (d\eta^i)^n (E_1, \ldots E_s, X_1, \ldots X_{2n}) = (d\eta^i)^n (X_1, \ldots, X_{2n}) =$$

$$= \Omega^*(\pi X, \ldots \pi X_{2n}) \neq 0$$

which proves that $\eta' \wedge \ldots \eta^s \wedge (d\eta^i)^n \neq 0$ at every point of M.

If (f, E_i, η^i, g) is the framed f-structure on M constructed in the theorem 4.2 using the Hodge structure (J, g') on N , we have

$$F(X,Y) = g(X,fY) = g'(\pi X, \pi fY) = g'(\pi X, J\pi Y) = \Omega^*(\pi X, \pi Y) = d\eta^i (X,Y)$$

Therefore this $(f, E_i \eta^i, g)$-structure is associated to this s-contact structure defined by $\eta', \ldots, \eta^s$. By theorem 4.2. and its proof, $(f, E_i \eta^i, g)$ is regular. On the other hand, since J is a complex structure and $d\omega(fX, fY) = d\omega(X,Y)$, (f, E_i, η^i, g) is normal and therefore a regular S-structure on M.

R E F E R E N C E S

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