On the general structure of Ricci collineations for type B warped spacetimes

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Abstract

A complete study of the structure of Ricci collineations for type B warped spacetimes is carried out. This study can be used as a method to obtain these symetries in such spacetimes. Special cases as $2+2$ reducible spacetimes, and plane and spherical symmetric spacetimes are considered specifically.

Keywords: symmetries in spacetimes, Killing vector fields, Ricci collineations, type B warped spacetimes.

1 Introduction

In the last years, symmetries in General Relativity have been studied in depth because of their interest from both a mathematical and a physical viewpoint. In fact, symmetries are important not only because of their classical physical significance, but also because they simplify Einstein equations and provide a classification of the spacetimes according to the structure of the corresponding Lie algebra. They are described by vector fields X on the spacetime which satisfy a relation of the form:

$$\pounds_X \Phi = \Lambda,$$

where Φ is any of the quantities g_{ab}, R_{ab}, R^a_{bcd}, etc, Λ is a tensor with the same index symmetries as Φ and $\pounds$ represents the Lie derivative. Depending on Φ and Λ, there are different classes of symmetries (the relation between them was studied in [9]). For example, if $\Phi = g_{ab}$ and $\Lambda = \psi g_{ab}$, with ψ a function, then X is a *Killing* vector field if $\psi = 0$, a *homothetic* vector field if $\psi_{,a} = 0$, a *special conformal* vector field if $\psi_{;ab} = 0$, and a *conformal* vector field if ψ is arbitrary. A symmetry will be called *proper* if it does not belong to any of its subtypes, otherwise it will be said *improper*.

In this article we will concentrate on *Ricci collineations*, that is, the case when $\Phi = R_{ab}$ and $\Lambda = 0$. These symmetries are interesting because, among other things, they provide information about the energy-momentum tensor via the Einstein equations (although Ricci collineations are not usually matter collineations). In order to ensure that Ricci collineations form a Lie algebra with the usual bracket operation, we shall assume that they are smooth vector fields. Recall that this algebra naturally contains all the special conformal vector fields (and thus, all the homothetic and Killing vector fields). Regarding the Ricci tensor, we shall consider (up to Section 5) that it is non-degenerate, i.e. rank 4; in particular, this ensures that the corresponding Lie algebra is finite-dimensional, with maximal dimension being 10. Further information on dimensionality and degenerate Ricci tensors can be found, for example, in references [5], [8].

In [3] the general form and classification of Ricci collineations of Robertson-Walker spacetimes is provided in detail. Afterwards, in [2] the authors compute Ricci collineations of metrics g_{ab} which are conformal to $1 + (n - 1)$ decomposable metrics by using an interesting technique. Roughly speaking, they construct the generic metric G_{ab} defined from the symmetry group of g_{ab}. Then, proper Ricci collineations are the Killing vector fields of G_{ab} which are not Killing vector fields of g_{ab}. This method provides the Ricci collineations of Robertson-Walker spacetimes without any further calculations.

A few years ago, the problem of determining all Ricci collineations of type B warped spacetimes was considered in [6]. This class of spacetimes is important because its structure is satisfied by multiple examples of interest in Physics as Schwarzschild, Robertson-Walker, etc. Unfortunately, the conclusion obtained there cannot be considered the solution of the problem, because it does not give *all* Ricci collineations of such spacetimes. In fact, two simple counterexamples to the main result in [6] were given in [1]. On the other hand, the technique introduced in [2] does not seem to be applicable directly to these spacetimes. In conclusion, this problem remains still open.

Our aim in this article is to describe in a general context the structure of all Ricci collineations of type B warped spacetimes. In fact, after a study of the equations which define these symmetries in such spacetimes, we classify them according to their structure. This classification can be considered a method to obtain all Ricci collineations. In particular, the counterexamples given in [1] are clearly contained in our results, Remark 4.3 (1). This article is organized as follows.

After some preliminaries on type B warped spacetimes, in Section 2 we obtain two conclusions (Propositions 2.1 and 2.3) on the structure of Killing vector fields and Ricci collineations of such spacetimes. In Section 3, these results are applied systematically. Ricci collineations are classified according to having or not mixed variables and, in each case, according to their vertical component. As consequence, an exhaustive description of the structure of these symmetries is obtained. In Section 4, $2+2$ reducible spacetimes (Subsection 4.1) and plane and spherical symmetric spacetimes (Subsection 4.2) are studied specifically. Finally, the case when Ricci tensor is degenerate is briefly considered in Section 5.

2 Preliminaries

Let (M_1, g_1) and (M_2, g_2) be semi-Riemannian manifolds, and $\phi > 0$ a smooth function on M_1. A *warped product* with base (M_1, g_1), fiber (M_2, g_2) and warping function $\phi > 0$ is the product manifold $M = M_1 \times M_2$ endowed with the metric tensor:

$$g^\phi = \pi_1^* g_1 + (\phi \circ \pi_1)^2 \pi_2^* g_2 \equiv g_1 + \phi^2 g_2,$$

where π_1 and π_2 are the natural projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. If, additionally, (M, g^ϕ) is a connected time-oriented fourdimensional Lorentzian manifold, then we say that (M, g^ϕ) is a *warped spacetime*. In this case, a classification can be made according to the respective dimensions of M_1 and M_2 (see [4] and references therein for a general discussion).

In this article we will concentrate on the study of *type B warped spacetimes*, that is, the case when M_1 and M_2 are both of dimension 2. In this case, and whenever we work locally, we can assume:

$$g^\phi = g_{AB}(x^C)dx^A dx^B + \phi^2(x^C) g_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta \qquad \begin{array}{l} A, B, C = 0, 1 \\ \alpha, \beta, \gamma = 2, 3 \end{array}$$

where g_{AB} and $g_{\alpha\beta}$ are the components of g_1 and g_2 in certain charts $(U_1 \subseteq M_1, x^0, x^1)$, $(U_2 \subseteq M_2, x^2, x^3)$, respectively.

Let X be a vector field on M and consider its horizontal and vertical components X_1, X_2; that is,

$$X_1(x^A, x^\alpha) = d\pi_1(X)(x^A, x^\alpha) \qquad X_2(x^A, x^\alpha) = d\pi_2(X)(x^A, x^\alpha).$$

Then, the Lie derivative of g^ϕ with respect to X is:

$$(\pounds_X g^\phi)_{AB} = (\pounds_{X_1} g_1)_{AB} \tag{2.1}$$

$$(\pounds_X g^\phi)_{A\alpha} = g_{AC} X^C_{1,\alpha} + \phi^2 g_{\alpha\beta} X^\beta_{2,A} \tag{2.2}$$

$$(\pounds_X g^\phi)_{\alpha\beta} = \phi^2 (\pounds_{X_2} g_2)_{\alpha\beta} + \phi^2_{,C} X^C_1 g_{\alpha\beta}. \tag{2.3}$$

In order to find the Killing vector fields of (M, g^ϕ), (2.1), (2.2) and (2.3) must be set equal to zero. Condition (2.1) equal to zero is equivalent to: for every $p_2 \in M_2$ the restriction of X_1 to $M_1 \times p_2$ is a Killing vector field (perhaps zero) of $(M_1 \times p_2, g_1)$. On the other hand, (2.3) equal to zero is equivalent to: for every $p_1 \in M_1$ the restriction of X_2 to $p_1 \times M_2$ is a conformal vector field (perhaps zero) of $(p_1 \times M_2, g_2)$ with conformal factor

$$\psi = -\frac{1}{2} \frac{\phi^2_{,C} X^C_1}{\phi^2}.$$

These simple facts are summarized in the following way:

Proposition 2.1 *Let (M, g^ϕ) be a type B warped spacetime with base (M_1, g_1), fiber (M_2, g_2) and warping function $\phi > 0$. A vector field $X \not\equiv 0$ on M is Killing of (M, g^ϕ) if and only if the following statements hold:*

(i) for every $p_1 \in M_1$, X_2 is a conformal vector field (perhaps zero) of $(p_1 \times M_2, g_2)$ with conformal factor ψ,

(ii) for every $p_2 \in M_2$, X_1 is a Killing vector field (perhaps zero) of $(M_1 \times p_2, g_1)$, which satisfies

$$\phi^2_{,C} X_1^C = -2\psi\phi^2 \tag{2.4}$$

and,

(iii) components (2.2) are equal to zero.

A direct computation provides the following components R_{ab} of the Ricci tensor $\mathbf{R}$ of a type B warped spacetime:

$$\begin{array}{l} R_{AB} = \frac{1}{2} R_1 g_{AB} - \frac{2}{\phi}\phi_{A;B} \\ R_{A\alpha} = 0 \\ R_{\alpha\beta} = \frac{1}{2}\left(R_2 - \left(\phi^2\right)^{A}_{;A}\right) g_{\alpha\beta} \ \equiv \ F g_{\alpha\beta}, \end{array} \tag{2.5}$$

where, obviously, $F := \frac{1}{2}\left(R_2 - \left(\phi^2\right)^{A}_{;A}\right)$, R_1 and R_2 are the scalar curvatures of g_1 and g_2, respectively, and the semi-colon indicates the covariant derivative *with respect to* g^ϕ.

Remark 2.2 Although the terms $\phi_{A;B}$ and $(\phi^2)^A_{;A}$ in (2.5) include covariant derivatives with respect to all the metric g^ϕ, a direct computation shows that they are independent of the variables x^γ of M_2. In fact:

$$\begin{array}{l} \phi_{A;B} = \phi_{,AB} - \frac{g^{CD}}{2}(g_{DB,A} + g_{DA,B} - g_{AB,D})\phi_{,C} \\ (\phi^2)^A_{;A} = g^{AB}\phi^2_{,AB} - \frac{g^{AB}g^{CD}}{2}(g_{DB,A} + g_{DA,B} - g_{AB,D})\phi^2_{,C}. \end{array}$$

The Lie derivative of $\mathbf{R}$ with respect to X is:

$$(\pounds_X \mathbf{R})_{AB} = R_{AB,C} X_1^C + R_{AC} X^C_{1,B} + R_{CB} X^C_{1,A} \tag{2.6}$$

$$(\pounds_X \mathbf{R})_{A\alpha} = R_{AC} X^C_{1,\alpha} + R_{\alpha\beta} X^\beta_{2,A} \tag{2.7}$$

$$(\pounds_X \mathbf{R})_{\alpha\beta} = F(\pounds_{X_2} g_2)_{\alpha\beta} + F_{,C} X_1^C g_{\alpha\beta} + F_{,\gamma} X_2^\gamma g_{\alpha\beta}. \tag{2.8}$$

In the following, our aim will be to find the Ricci collineations of (M, g^ϕ); that is, the vector fields $X \not\equiv 0$ on M such that (2.6), (2.7) and (2.8) are equal to zero.

As commented in the Introduction, we will assume that $\mathbf{R}$ is non-degenerate. Therefore, $F \neq 0$ everywhere. Moreover, from (2.5) and Remark 2.2, R_{AB}

can be seen as the components of a metric tensor g_R defined on M_1. Then, reasoning as in Proposition 2.1, condition (2.6) equal to zero is equivalent to: for every $p_2 \in M_2$ the restriction of X_1 to $M_1 \times p_2$ is a Killing vector field (perhaps zero) of $(M_1 \times p_2, g_R)$. On the other hand, (2.8) equal to zero is equivalent to: for every $p_1 \in M_1$ the restriction of X_2 to $p_1 \times M_2$ is a conformal vector field (perhaps zero) of $(p_1 \times M_2, g_2)$ with conformal factor

$$\psi = -\frac{1}{2}\frac{F_{,C}X_1^C + F_{,\gamma}X_2^\gamma}{F}. \tag{2.9}$$

Equation (2.9) can be simplified by using the classical expression of the Lie derivative of the Ricci $\mathbf{R}_h$ of a semi-Riemannian manifold (N, h) with respect to a conformal vector field Y of conformal factor ξ (see [7]); that is,

$$(\pounds_Y \mathbf{R}_h)_{ab} = -(n-2)\xi_{|ab} - (\Delta_h \xi)h_{ab}, \tag{2.10}$$

where $n = dim\, N$ and $\Delta_h \xi = \xi_{|cd}h^{cd}$ is the Laplacian of ξ with respect to h (obviously, the stroke denotes the covariant derivative with respect to h). In fact, assume that X_2 is a conformal vector field of $(p_1 \times M_2, g_2)$ with conformal factor ψ. Then, from (2.10) we obtain:

$$(\pounds_{X_2}\mathbf{R}_{g_2})_{\alpha\beta} = -(\Delta_{g_2}\psi)g_{\alpha\beta}.$$

But, obviously,

$$\pounds_{X_2}(\mathbf{R}_{g_2})_{\alpha\beta} = \pounds_{X_2}\left(\frac{1}{2}R_2 g_2\right)_{\alpha\beta} = \frac{1}{2}(R_{2,\gamma}X_2^\gamma + 2\psi R_2)g_{\alpha\beta};$$

thus

$$R_{2,\gamma}X_2^\gamma + 2\psi R_2 = -2\Delta_{g_2}\psi. \tag{2.11}$$

On the other hand, by replacing in (2.9) the expression of F we have:

$$2\psi(R_2 - (\phi^2)^{A}_{;A}) = (\phi^2)^{A}_{;A,C}X_1^C - R_{2,\gamma}X_2^\gamma. \tag{2.12}$$

Therefore, from (2.11) and (2.12) we obtain:

$$(\phi^2)^{A}_{;A,C}X_1^C = -2\psi(\phi^2)^{A}_{;A} - 2\Delta_{g_2}\psi. \tag{2.13}$$

These facts are summarized in the following result:

Proposition 2.3 *Let (M, g^ϕ) be a type B warped spacetime with base (M_1, g_1), fiber (M_2, g_2) and warping function $\phi > 0$. A vector field $X \not\equiv 0$ on M is a Ricci collineation of (M, g^ϕ) if and only if the following statements hold:*

(i) for every $p_1 \in M_1$, X_2 is a conformal vector field (perhaps zero) of $(p_1 \times M_2, g_2)$ with conformal factor ψ,

(ii) for every $p_2 \in M_2$, X_1 is a Killing vector field (perhaps zero) of $(M_1 \times p_2, g_R)$, which satisfies (2.13), and

(iii) components (2.7) are equal to zero.

In the next section, Propositions 2.1 and 2.3 will be exploited in order to describe the general structure of Ricci collineations of (M, g^ϕ).

3 Ricci collineations of type B warped space-times

For simplicity, firstly we will classify these symmetries in two families. In the first family, we will include Ricci collineations whose variables are not mixed, that is, when the corresponding vector field X can be written as

$$X(x^A, x^\alpha) = X_1(x^A) + X_2(x^\alpha).$$

Our study is completed by including in a second family Ricci collineations such that either $\partial X_1/\partial x^\alpha \neq 0$ or $\partial X_2/\partial x^A \neq 0$.

FAMILY 1. Ricci collineations with non-mixed variables.

Notice that, in this case, statements (iii) in Propositions 2.1 and 2.3 always hold. On the other hand, from Proposition 2.3, we can distinguish four types in this family attending to the vertical component X_2 of X.

TYPE 1.1: X_2 is a Killing vector field (perhaps zero) of (M_2, g_2).

¿From Proposition 2.3 (ii), $X \not\equiv 0$ will be a Ricci collineation if, additionally, X_1 is a Killing vector field (perhaps zero) of (M_1, g_R) with $(\phi^2)^A_{;A,C} X_1^C = 0$. Therefore, Ricci collineations $X \not\equiv 0$ of type 1.1 are:

$$X = X_1 + X_2 = \sum_{i=1}^{k_R} a_i^1 X_1^i + \sum_{j=1}^{k_2} a_j^2 X_2^j$$

where

(i) $\{X_1^i\}_{i=1}^{k_R}$ is the algebra of Killing vector fields of (M_1, g_R),

(ii) $\{X_2^j\}_{j=1}^{k_2}$ is the algebra of Killing vector fields of (M_2, g_2), and

(iii) coefficients $\{a_i^1\}_{i=1}^{k_R}$ satisfy

$$\sum_{i=1}^{k_R} a_i^1 (\phi^2)^A_{;A,C} X_1^{i\,C} = 0. \tag{3.1}$$

Additionally, from Proposition 2.1 X is not a Killing vector field of (M, g^ϕ) if,

(iv) either $\phi^2_{,C} X_1^C \neq 0$ or X_1 is not a Killing vector field of (M_1, g_1) (in particular, $X_1 \not\equiv 0$).

Remark 3.1 As $dim\, M_i = 2$, $i = 1, 2$, necessarily $k_R, k_2 = 0, 1, 3$. But, from (iv), (M, g^ϕ) admits proper Ricci collineations of type 1.1 only if $k_R = 1, 3$. Therefore, in this case, if the curvature of (M_1, g_R) is not constant, necessarily $k_R = 1$, and thus, equation (3.1) reduces to $(\phi^2)^A_{;A,C} X_1^{1\,C} = 0$.

TYPE 1.2: X_2 is a proper homothetic vector field of (M_2, g_2).

Obviously, this type of collineations only exists if the curvature of (M_2, g_2) is not a constant different from zero. In this case, X will be a Ricci collineation if, additionally, X_1 is a Killing vector field (perhaps zero) of (M_1, g_R) with

$$(\phi^2)^A_{;A,C} X_1^C = -2\lambda (\phi^2)^A_{;A}, \tag{3.2}$$

where $\lambda \neq 0$ is the homothetic factor of X_2. From (3.2), recall that if $X_1 = 0$ then, necessarily $(\phi^2)^A_{;A} = 0$.

In conclusion, Ricci collineations $X \not\equiv 0$ of type 1.2 are:

$$X = X_1 + X_2 = \sum_{i=1}^{k_R} a_i^1 X_1^i + \sum_{j=1}^{k_2} a_j^2 X_2^j + \lambda Y,$$

where,

(i) as before, $\{X_1^i\}_{i=1}^{k_R}$, $\{X_2^j\}_{j=1}^{k_2}$ are the algebras of Killing vector fields of (M_1, g_R), (M_2, g_2), respectively,

(ii) Y is the homothetic vector field of (M_2, g_2) with homothetic factor 1, and

(iii) coefficients $\{a_i^1\}_{i=1}^{k_R}$ and $\lambda \neq 0$ satisfy

$$\sum_{i=1}^{k_R} a_i^1 (\phi^2)^A_{;A,C} X_1^{iC} = -2\lambda (\phi^2)^A_{;A}.$$

Additionally, X is not a Killing vector field of (M, g^ϕ) if,

(iv) either $\phi^2_{,C} X_1^C \neq -2\lambda \phi^2$ or X_1 is not a Killing vector field of (M_1, g_1).

TYPE 1.3: X_2 is a proper special conformal vector field of (M_2, g_2).

This type of collineations only exists if $(\phi^2)^A_{;A} = 0$. In fact, now the conformal factor ψ associated to X_2 is a non-constant function of x^α with $\Delta_{g_2}\psi = 0$. Therefore, if we assume that (2.13) holds, and derive it with respect to x^γ, we deduce that $(\phi^2)^A_{;A} = 0$.

Under this restriction, all Ricci collineations $X \not\equiv 0$ of type 1.3 are given by:

$$X = X_1 + X_2 = \sum_{i=1}^{k_R} a_i^1 X_1^i + \sum_{j=1}^{s_2} a_j^2 X_2^j,$$

where,

(i) $\{X_1^i\}_{i=1}^{k_R}$ is the algebra of Killing vector fields of (M_1, g_R),

(ii) $\{X_2^j\}_{j=1}^{s_2}$ is the algebra of special conformal vector fields of (M_2, g_2) and,

(iii) some of the coefficients $\{a_j^2\}_{j=h_2+1}^{s_2}$ are different from zero, being h_2 the dimension of the algebra of homothetic vector fields of (M_2, g_2).

Moreover, these collineations are not Killing vector fields of (M, g^ϕ) because they do not satisfy (2.4).

Remark 3.2 Notice that condition $(\phi^2)^A_{;A} = 0$ implies that proper homothetic and special conformal vector fields of (M_2, g_2) are also Ricci collineations of (M, g^ϕ) of types 1.2 and 1.3, respectively. Moreover, they are not Killing vector fields of (M, g^ϕ) (since they do not satisfy (2.4)).

TYPE 1.4: X_2 is a proper conformal vector field of (M_2, g_2).

This type of collineations only exists if $(\phi^2)_{;A}^{A}$ remains constant wherever ψ is not constant. In fact, if we assume that (2.13) holds, and derive it with respect to x^γ, we deduce that

$$\Delta_{g_2}\psi = -(\phi^2)_{;A}^{A} \cdot \psi = -\text{const} \cdot \psi \tag{3.3}$$

on such a domain.

In conclusion, Ricci collineations $X \not\equiv 0$ of type 1.4 satisfy the expression:

$$X = X_1 + X_2 = \sum_{i=1}^{k_R} a_i^1 X_1^i + \sum_{j=1}^{c_2} a_j^2 X_2^j,$$

where

(i) $\{X_1^i\}_{i=1}^{k_R}$ is the algebra of Killing vector fields of (M_1, g_R),

(ii) $\{X_2^j\}_{j=1}^{c_2}$ is the conformal algebra of (M_2, g_2) and,

(iii) coefficients $\{a_j^2\}_{j=k_2+1}^{c_2}$ are such that the conformal factor of X_2

$$\psi = \sum_{j=k_2+1}^{c_2} a_j^2 \psi_2^j$$

satisfies (2.13) (in particular, satisfies (3.3) wherever ψ is not constant), where $\{\psi_2^j\}_{j=k_2+1}^{c_2}$ are the corresponding conformal factors of $\{X_2^j\}_{j=k_2+1}^{c_2}$, and some of the coefficients $\{a_j^2\}_{j=s_2+1}^{c_2}$ must be different from zero.

Again, these collineations are not Killing vector fields of (M, g^ϕ) because they do not satisfy (2.4).

FAMILY 2. Ricci collineations with mixed variables.

In this family, the dependence of X_1 and X_2 is not restricted to x^A and x^α, respectively. Therefore, *(iii)* in Propositions 2.1 and 2.3 must be also taken into account in order to find these symmetries. Summarizing, Ricci collineations $X \not\equiv 0$ are given now by:

$$X = X_1 + X_2 = \sum_{i=1}^{k_R} a_i^1(x^\alpha) X_1^i + \sum_{j=1}^{c_2} a_j^2(x^A) X_2^j$$

where

(i) $\{X_1^i\}_{i=1}^{k_R}$ is the algebra of Killing vector fields of (M_1, g_R),

(ii) $\{X_2^j\}_{j=1}^{c_2}$ is the conformal algebra of (M_2, g_2), and

(iii) functions $\{a_i^1(x^\alpha)\}_{i=1}^{k_R}$, $\{a_j^2(x^A)\}_{j=1}^{c_2}$ satisfy

$$\sum_{i=1}^{k_R} a_i^1(x^\alpha)(\phi^2)^A_{;A,C} X_1^{iC} = -2\left(\sum_{j=k_2+1}^{c_2} a_j^2(x^A)\psi_2^j\right)(\phi^2)^A_{;A} - 2\sum_{j=s_2+1}^{c_2} a_j^2(x^A)\Delta_{g_2}\psi_2^j, \tag{3.4}$$

$$\sum_{i=1}^{k_R} \frac{da_i^1(x^\gamma)}{dx^\alpha} R_{AC} X_1^{iC} + \sum_{j=1}^{c_2} \frac{da_j^2(x^C)}{dx^A} R_{\alpha\beta} X_2^{j\beta} = 0, \quad A = 0,1,\ \alpha = 2,3 \tag{3.5}$$

(the indexes k_2, s_2 are again the dimensions of the homothetic and special conformal algebras).
Additionally, X is not a Killing vector field of (M, g^ϕ) if

(iv) any of the statements of Proposition 2.1 do not hold.

Analogously to Family 1, we classify these collineations in four types.

TYPE 2.1: For every $p_1 \in M_1$, X_2 is a Killing vector field (perhaps zero) of $(p_1 \times M_2, g_2)$.

In this case the only functions which can be different from zero are $\{a_i^1(x^\alpha)\}_{i=1}^{k_R}$, $\{a_j^2(x^A)\}_{j=1}^{k_2}$. As a consequence, equation (3.4) reduces to

$$\sum_{i=1}^{k_R} a_i^1(x^\alpha)(\phi^2)^A_{;A,C} X_1^{iC} = 0.$$

TYPE 2.2: For every $p_1 \in M_1$, X_2 is a homothetic vector field (perhaps zero) of $(p_1 \times M_2, g_2)$ which is not always Killing.

This type of collineations only exists if the curvature of (M_2, g_2) is not a constant different from zero. In this case, only the functions $\{a_i^1(x^\alpha)\}_{i=1}^{k_R}$, $\{a_j^2(x^A)\}_{j=1}^{k_2}$, $a_{k_2+1}^2(x^A) \equiv \lambda(x^A)$ can be different from zero, and they must satisfy:

$$\sum_{i=1}^{k_R} a_i^1(x^\alpha)(\phi^2)^A_{;A,C} X_1^{iC} = -2\lambda(x^A)(\phi^2)^A_{;A},$$

where we have assumed the homothetic factor $\psi_2^{k_2+1}$ normalized to 1.

TYPE 2.3: For every $p_1 \in M_1$, X_2 is a special conformal vector field (perhaps zero) of $(p_1 \times M_2, g_2)$ which is not always homothetic.

In this case, only the functions $\{a_i^1(x^\alpha)\}_{i=1}^{k_R}$, $\{a_j^2(x^A)\}_{j=1}^{s_2}$ can be different from zero. As a consequence, equation (3.4) reduces now to:

$$\sum_{i=1}^{k_R} a_i^1(x^\alpha)(\phi^2)^A_{;A,C} X_1^{i\,C} = -2\left(\sum_{j=k_2+1}^{s_2} a_j^2(x^A)\psi_2^j\right)(\phi^2)^A_{;A}.$$

TYPE 2.4: For every $p_1 \in M_1$, X_2 is a conformal vector field (perhaps zero) of $(p_1 \times M_2, g_2)$ which is not always special. In general, we cannot simplify the structure of these Ricci collineations.

In the following section a brief application of our study to some examples of type B warped spacetimes is carried out. Without any further calculations, we obtain an interesting information about the particular structure of their symmetries.

4 Examples

In this section we will apply our point of view to the following families of type B warped spacetimes: $2+2$ reducible spacetimes, and plane and spherical symmetric spacetimes.

4.1 $2+2$ reducible spacetimes

In this case the product manifold $M = M_1 \times M_2$ is endowed with the metric tensor:

$$g = \pi_1^* g_1 + \pi_2^* g_2 \equiv g_1 + g_2.$$

Therefore, these spacetimes are type B warped spacetimes with $\phi^2 = 1$ and, thus, we can apply our previous study. Firstly, take into account that now $g_R = 1/2 R_1 g_1 = \mathbf{R}_{g_1}$. Thus, Killing vector fields X_1 of (M_1, g_R) are just the conformal vector fields of (M_1, g_1) with conformal factors satisfying $\Delta_{g_1}\psi_1 = 0$ (recall (2.10)). Therefore, if we apply Proposition 2.3 to these spacetimes, we obtain the following consequences:

Corollary 4.1 *Ricci collineations $X \not\equiv 0$, with non-mixed variables, of a $2+2$ reducible spacetime (M, g) are the vector fields $X = X_1 + X_2$ such that, X_l are conformal vector fields of (M_l, g_l) with conformal factors ψ_l satisfying*

$$\Delta_{g_l}\psi_l = 0, \quad l = 1, 2.$$

Corollary 4.2 *Ricci collineations $X \not\equiv 0$, with mixed variables, of a $2+2$ reducible spacetime (M,g) are the vector fields*

$$X = X_1 + X_2 = \sum_{i=1}^{c_1} a_i^1(x^\alpha) X_1^i + \sum_{j=1}^{c_2} a_j^2(x^A) X_2^j,$$

where $\{X_1^i\}_{i=1}^{c_1}$, $\{X_2^j\}_{j=1}^{c_2}$ are the conformal algebras of (M_1,g_1), (M_2,g_2), respectively, and functions $\{a_i^1(x^\alpha)\}_{i=1}^{c_1}$, $\{a_j^2(x^A)\}_{j=1}^{c_2}$ satisfy

$$\begin{array}{l} \sum_{i=s_1+1}^{c_1} a_i^1(x^\alpha)\Delta_{g_1}\psi_1^i = \sum_{j=s_2+1}^{c_2} a_j^2(x^A)\Delta_{g_2}\psi_2^j = 0, \\ \sum_{i=1}^{c_1} \frac{da_i^1(x^\gamma)}{dx^\alpha} R_{AC} X_1^{iC} + \sum_{j=1}^{c_2} \frac{da_j^2(x^C)}{dx^A} R_{\alpha\beta} X_2^{j\beta} = 0, \quad A = 0,1, \ \alpha = 2,3, \end{array}$$

being $\{\psi_1^i\}_{i=1}^{c_1}$, $\{\psi_2^j\}_{j=1}^{c_2}$ the corresponding conformal factors.

Remark 4.3 (1) Counterexamples given in [1] are clearly contained in these results. In fact, $M = \mathbb{R}^2 \times \mathbb{R}^2$ endowed with

$$g = e^{t^2/2}(-dt^2 + dx^2) + e^{-y^2/2}(dy^2 + dz^2)$$

is a $2+2$ reducible spacetime and, both, $X = \partial_t + \partial_y$, $Y = z\partial_t + \partial_y + t\partial_z$ are Ricci collineations which satisfy hypotheses of Corollaries 4.1 and 4.2, respectively.

(2) From equations (2.5), it is clear that both corollaries also hold for spacetimes not necessarily $2+2$ reducible, but satisfying $\phi_{A;B} = (\phi^2)_{;A}^{A} \equiv 0$.

4.2 Plane and spherical symmetric spacetimes

Consider now the family of spacetimes $M = \mathbb{R}^2 \times M_2$ endowed with the metric tensor:

$$g^\phi = -e^{2v}dt^2 + e^{2w}dx^2 + \phi^2 g_2$$

where v, w and ϕ are each functions of t and x, and

$$(M_2, g_2) = \left\{ \begin{array}{l} \mathbb{R}^2 \\ \mathbb{S}^2 \end{array} \right.$$

endowed with their corresponding usual metrics (if we also include the hyperbolic space, g^ϕ can be characterized by admitting a group G_3 acting multiply-transitively on spacelike orbits V_2, see [10]). To avoid the vanishing of F, and thus, the degeneracy of Ricci tensor $\mathbf{R}$ (recall (2.5)), we will also assume

$$(\phi^2)_{;A}^{A} \neq R_2 \qquad \text{for all } (t,x) \in \mathbb{R}^2. \tag{4.1}$$

¿From Section 3, the component X_1 of a Ricci collineation $X \not\equiv 0$ of these spacetimes is different from zero only if (M_1, g_R) admits some Killing vector fields. In this case, the dimension k_R of the corresponding algebra must be 3 or 1, depending on if v, w and ϕ makes $(\mathbb{R}^2, g_R)$ being maximally symmetric or not. To study the structure of X_2, we must consider the two cases separately.

4.2.1. Plane symmetry:

The conformal algebra of the plane $(\mathbb{R}^2, dy^2{+}dz^2)$ is the (infinite-dimensional) *Virasoro algebra*, which has the following special conformal vector fields:

$$\begin{array}{ll} X_2^1 = \partial_y & \psi_2^1 = 0 \\ X_2^2 = \partial_z & \psi_2^2 = 0 \\ X_2^3 = z\partial_y - y\partial_z & \psi_2^3 = 0 \\ X_2^4 = y\partial_y + z\partial_z & \psi_2^4 = 1 \\ X_2^5 = (y^2 - z^2)\partial_y + 2yz\partial_z & \psi_2^5 = 2y \\ X_2^6 = 2yz\partial_y + (z^2 - y^2)\partial_z & \psi_2^6 = 2z. \end{array}$$

Therefore, $k_2 = 3$, $h_2 = 4$, $s_2 = 6$ (and $c_2 = \infty$). In this case, we can establish the following:

(i) The vertical component X_2 of a Ricci collineation $X \not\equiv 0$ of type 1.1 is a linear combination of the $k_2 = 3$ Killing vector fields of the plane. On the other hand, the horizontal component X_1 satisfies the equation in k_R variables (3.1).

(ii) The component X_2 of a Ricci collineation of type 1.2 is a linear combination of the $h_2 = 4$ homothetic vector fields of the plane. On the other hand, the horizontal component X_1 satisfies the equation (3.2). If $k_R = 0$, there are not Ricci collineations of this type since, in this case, (3.2) reduces to $(\phi^2)_{;A}^{A} = 0$, which contradicts (4.1).

(iii) There are not Ricci collineations of type 1.3. Moreover, there are collineations of type 1.4 only if $(\phi^2)_{;A}^{A}$ =const$\neq 0$.

(iv) Ricci collineations in Family 2 must satisfy equations (3.4) and (3.5), which are in general complicated. If $k_R = 0$ and we consider collineations of type 2.3, these equations reduce to:

$$a_4^2 + 2y\, a_5^2 + 2z\, a_6^2 = 0$$

and

$$\begin{array}{l}
a^2_{1,t} + z\, a^2_{3,t} + y\, a^2_{4,t} + (y^2 - z^2)a^2_{5,t} + 2yz\, a^2_{6,t} = 0 \\
a^2_{2,t} - y\, a^2_{3,t} + z\, a^2_{4,t} + 2yz\, a^2_{5,t} + (z^2 - y^2)a^2_{6,t} = 0 \\
a^2_{1,x} + z\, a^2_{3,x} + y\, a^2_{4,x} + (y^2 - z^2)a^2_{5,x} + 2yz\, a^2_{6,x} = 0 \\
a^2_{2,x} - y\, a^2_{3,x} + z\, a^2_{4,x} + 2yz\, a^2_{5,x} + (z^2 - y^2)a^2_{6,x} = 0.
\end{array}$$

4.2.2. Spherical symmetry:

In this case, the second space (M_2, g_2) is the unitary bidimensional sphere $\mathbb{S}^2$. The *local* conformal algebra of $\mathbb{S}^2$, like that of the plane, is the Virasoro algebra. In order to single out a finite-dimensional subalgebra from it, we will impose that conformal vectors must be *globally* defined on $\mathbb{S}^2$. Then, a well-known computation shows that the only *global* conformal vector fields of $\mathbb{S}^2$ expressed in spherical coordinates are:

$$\begin{array}{ll}
X^1_2 = \cos\varphi\partial_\theta - \sin\varphi\cot\theta\partial_\varphi & \psi^1_2 = 0 \\
X^2_2 = \sin\varphi\partial_\theta + \cos\varphi\cot\theta\partial_\varphi & \psi^2_2 = 0 \\
X^3_2 = \partial_\varphi & \psi^3_2 = 0 \\
X^4_2 = \sin\theta\partial_\theta & \psi^4_2 = \cos\theta \\
X^5_2 = \cos\theta\cos\varphi\partial_\theta - \frac{\sin\varphi}{\sin\theta}\partial_\varphi & \psi^5_2 = -\sin\theta\cos\varphi \\
X^6_2 = \cos\theta\sin\varphi\partial_\theta - \frac{\cos\varphi}{\sin\theta}\partial_\varphi & \psi^6_2 = -\sin\theta\sin\varphi.
\end{array}$$

(Nevertheless, recall that other conformal vectors—necessarily *locally* defined—can appear as vertical component of a Ricci collineation of a spherically symmetric spacetime.)

In conclusion, $k_2 = h_2 = s_2 = 3$ and $c_2 = 6$. Therefore, we obtain the following:

(i) The vertical component X_2 of a Ricci collineation $X \not\equiv 0$ of type 1.1 is a linear combination of the $k_2 = 3$ Killing vector fields of $\mathbb{S}^2$. On the other hand, the horizontal component X_1 satisfies the equation in k_R variables (3.1).

(ii) As the curvature of $\mathbb{S}^2$ is a constant different from zero, there are not Ricci collineations of types 1.2, 2.2. Even more, as $s_2 - h_2 = 0$, there are not Ricci collineations of types 1.3, 2.3 either.

(iii) A simple computation shows that $\Delta_{g_2}\psi^j_2 = -2\psi^j_2$, $j = 4, 5, 6$. But then, (3.3) implies $(\phi^2)^{;A}_{;A} = R_2 = 2$, which contradicts (4.1). Therefore, there are not Ricci collineations of type 1.4.

(iv) Ricci collineations in Family 2 must satisfy equations (3.4), (3.5), which are in general complicated. If $k_R = 0$, there are not Ricci collineations in this family. In fact, in this case (3.4) (or, equivalently, (2.13)) implies again $(\phi^2)^{;A}_{;A} = R_2 = 2$, in contradiction with (4.1).

5 The degenerate case

For completeness, we briefly analyse here the cases when Ricci tensor is degenerate. From (2.5), the Ricci tensor of a type B warped spacetime is degenerate if $F \equiv 0$ or $\phi_{A;B} \equiv \frac{\phi}{4} R_1 g_{AB}$ (if both identities hold, the Ricci tensor is zero and any vector field is a Ricci collineation).

Consider the case $F \equiv 0$ (or, equivalently, $(\phi^2)^{;A}_{;A} \equiv R_2 = 0$). Then, equations (2.6)–(2.8) show that a vector field $X = X_1 + X_2 \not\equiv 0$ is a Ricci collineation of the spacetime if and only if X_1 is a Killing vector field (perhaps zero) of $(M_1 \times p_2, g_R)$ satisfying $R_{AC} X^C_{1,\alpha} = 0$ for every $p_2 \in M_2$. In particular, any Killing vector field of (M_1, g_R) is always a Ricci collineation of the spacetime. Moreover, the group of Ricci collineations becomes infinity, since every vector field $X \not\equiv 0$ with horizontal component $X_1 \equiv 0$ generates a Ricci collineation. That is, the vertical component (which is just the component where Ricci tensor degenerates) of these Ricci collineations is largely arbitrary (see [3, Section 2] for a similar property in Robertson-Walker spacetimes).

The situation is more complicated when the source of degeneracy is the identity $\phi_{A;B} \equiv \frac{\phi}{4} R_1 g_{AB}$. In this case, Proposition 2.3 shows that a vector field $X = X_1 + X_2 \not\equiv 0$ is a Ricci collineation of the spacetime if and only if X_2 is a conformal vector field (perhaps zero) of $(p_1 \times M_2, g_2)$ satisfying $R_{\alpha\beta} X^\beta_{2,A} = 0$ for every $p_1 \in M_1$ and, additionally, X_1 satisfies (2.13) for every $p_2 \in M_2$. So, in this case we have restrictions on both, the vertical and horizontal components of X. This is due to the dependence of F on both components, and breaks the similarities with respect to the Robertson-Walker case.

6 Conclusion

By analyzing the equations which characterize Ricci collineations of type B warped spacetimes, we have determined the structure of these symmetries. They have been classified in eight types according to having or not mixed variables, and according to their vertical component. As a consequence, several examples of interest have been considered, and new information about their collineations has been provided. This study must be understood as an initial point to begin a systematic computation of Ricci collineations for a wide family of spacetimes of this class.

As a final remark we would like to point out that it would be very useful to use computer algebra packages to automate the search of symmetries. Nevertheless, as far as we know, the available algorithms can only check if a given vector field is a symmetry or not (at least this is the case of GRTensor with which we have some familiarity).

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