

Power expansions in terms of shifted Chebyshev-Lanczos polynomials

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Abstract

We exhibit two procedures to express x^n in terms of the shifted Chebyshev polynomials, which is useful to reduce the degree of a polynomial in the interval $[0, 1]$.

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1 Introduction

In numerical analysis may be necessary to reduce, with small error, the degree of a polynomial in the interval $[0,1]$, which is possible employing the Modified Chebyshev polynomials $\bar{T}_r(x)$ defined by [1]:

$$\bar{T}_0(x) = \frac{1}{2}, \quad \bar{T}_k(x) = T_k(2x - 1), \quad k = 1, 2, \dots \quad (1)$$

where the first-kind Chebyshev polynomials $T_r(x)$ are given by the recurrence relation [2-6]:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad k = 1, 2, \dots \quad (2)$$

therefore

$$\bar{T}_0 = \frac{1}{2}, \quad \bar{T}_1 = 2x - 1, \quad \bar{T}_2 = 8x^2 - 8x + 1,$$

$$\bar{T}_3 = 32x^3 - 48x^2 + 18x - 1, \quad \bar{T}_4 = 128x^4 - 256x^3 + 160x^2 - 32x + 1, \text{ etc.}$$

In the mentioned reduction process we need the powers x^n in terms of \bar{T}_r , then from (3):

$$x^0 = 2\bar{T}_0, \quad x = \frac{1}{2}(2\bar{T}_0 + \bar{T}_1), \quad x^2 = \frac{1}{8}(6\bar{T}_0 + 4\bar{T}_1 + \bar{T}_2),$$

$$x^3 = \frac{1}{32}(20\bar{T}_0 + 15\bar{T}_1 + 6\bar{T}_2 + \bar{T}_3), \tag{3}$$

$$x^4 = \frac{1}{128}(70\bar{T}_0 + 56\bar{T}_1 + 28\bar{T}_2 + 8\bar{T}_3 + \bar{T}_4), \quad \text{etc.}$$

that is [1]:

$$\frac{1}{2}(4x)^n = \sum_{r=0}^n \binom{2n}{n-r} \bar{T}_r, \quad n = 0, 1, \dots \tag{4}$$

The next Section exhibits an algorithm to obtain x^j in function of \bar{T}_r if we know the corresponding expansion of x^{j-1} , and also another procedure which employs to (5) as a Newton's binomial expression.

2 x^n in terms of \bar{T}_r .

We may write (5) in the form:

	\bar{T}_0	\bar{T}_1	\bar{T}_2	\bar{T}_3	\bar{T}_4	...
$\frac{1}{2}(4x)^0$	1	0	0	0	0	...
$\frac{1}{2}(4x)^1$	2	1	0	0	0	
$\frac{1}{2}(4x)^2$	6	4	1	0	0	
$\frac{1}{2}(4x)^3$	20	15	6	1	0	...
$\frac{1}{2}(4x)^4$	70	56	28	8	1	
\vdots	\vdots			\vdots		\ddots

(5)

or in function of the columns vectors $(\frac{1}{2}(4x)^j)$ and (\overline{T}_r) for a given n :

$$\begin{pmatrix} \frac{1}{2}(4x)^0 \\ \frac{1}{2}(4x)^1 \\ \vdots \\ \frac{1}{2}(4x)^n \end{pmatrix} = \underset{\sim}{A} \cdot \begin{pmatrix} \overline{T}_0 \\ \overline{T}_1 \\ \vdots \\ \overline{T}_n \end{pmatrix}, \quad (6)$$

where $\underset{\sim}{A}$ is the $(n+1) \times (n+1)$ triangular matrix of coefficients appearing in (6):

$$\underset{\sim}{A} = (a_{jr}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & \\ 6 & 4 & 1 & 0 & 0 & \\ 20 & 15 & 6 & 1 & 0 & \dots \\ 70 & 56 & 28 & 8 & 1 & \\ \vdots & & & \vdots & & \ddots \end{pmatrix}, \quad j, r = 0, 1, \dots, n \quad (7)$$

then $(\overline{T}_k) = \underset{\sim}{A}^{-1} \cdot (\frac{1}{2}(4x)^r)$ reproduces (3).

The relations (5) and (7) imply that:

$$a_{jr} = \binom{2j}{j-r}, \quad j, r = 0, 1, \dots \quad (8)$$

thus

$$a_{jj} = 1, \quad a_{jr} = 0, \quad r > j, \quad (9)$$

and we can prove the following properties not found explicitly in the literature:

$$a_{j+1,0} = 2(a_{j0} + a_{j1}), \quad j = 0, 1, 2, \dots \quad (10)$$

$$a_{jr} = a_{j-1,r-1} + 2a_{j-1,r} + a_{j-1,r+1}, \quad r, j = 1, 2, 3, \dots$$

The formulae (11) permit to construct the row j of \tilde{A} if we know its row $j - 1$, and they represent an algorithm to express x^n in terms of \bar{T}_r whose systematic use minimize the amount of arithmetical computations involved in (5).

On the other hand, the expansion (5) can be written as:

$$\frac{1}{2}(4x)^n = \sum_{k=0}^n \binom{2n}{k} \bar{T}_{n-k} = \sum_{k=0}^{2n} \binom{2n}{k} \bar{T}^{n-k}, \quad (11)$$

where we use the notation:

$$\bar{T}^{-j} = 0, \quad j = 1, 2, \dots, \quad \bar{T}^r \equiv T_r, \quad r = 0, 1, 2, \dots \quad (12)$$

very employed in Gregory-Newton and Stirling interpolations [7].

Thus (12) adopts the form of a Newton's binomial expression:

$$\frac{1}{2}(4x)^n = \frac{1}{\bar{T}^n} \sum_{k=0}^{2n} \binom{2n}{k} \bar{T}^{2n-k} = \frac{1}{\bar{T}^n} (1 + \bar{T})^{2n}, \quad (13)$$

which is a procedure alternative to (11) to obtain x^n in function of \bar{T}_r . For example:

$$\begin{aligned} \frac{1}{2}(4x)^2 &= \frac{1}{\bar{T}^2} (1 + \bar{T})^4 = \frac{1}{\bar{T}^2} (1 + 4\bar{T} + 6\bar{T}^2 + 4\bar{T}^3 + \bar{T}^4), \\ &= \bar{T}^{-2} + 4\bar{T}^{-1} + 6\bar{T}^0 + 4\bar{T} + \bar{T}^2 \stackrel{(13)}{=} 6\bar{T}_0 + 4\bar{T}_1 + \bar{T}_2, \quad etc. \end{aligned}$$

in according with (6). The relation (14) may be easily manipulated by a computer via some symbolic language as MAPLE.

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