Comment on Ricci Collineations of Static Spherically Symmetric Spacetimes.

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April 11, 2001

Abstract

We present a counter example to a theorem given by Amir *et al.* J. Math. Phys. **35**, 3005 (1994). We also comment on a misleading statement of the same reference.

In a recent paper, M. Jamil Amir *et al.* [1] have presented a detailed analysis of Ricci collineations (RC) for static, spherically symmetric spacetimes, with a special focus on the relationship between RC and isometries (KV). This has led the authors to the following theorem:
Theorem 1: *Spherically symmetric static spacetimes with nonzero (and non*

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infinite) determinant of the Ricci tensor have RCs identical with the Killing Vector, but when the determinant is zero there may be additional degrees of freedom, giving infinitely many RCs for each degree of freedom.

We have found a counter example to this result. The metric for spherically symmetric static spacetimes can be written in the form[2]

$$ds^2 = -\, e^{\nu(r)}\, dt^2 + \, e^{\lambda(r)}\, dr^2 + \, r^2(d\vartheta^2 + \, \sin^2\vartheta\, d\phi^2) \tag{1}$$

The Ricci tensor for this metric is diagonal and can be written as: $R_{00} = A(r)$, $R_{11} = B(r)$, $R_{22} = C(r)$ and $R_{33} = C(r)\sin^2\vartheta$. Let us consider the particular case in which $A(r) = C(r) = 1$, which leads to the metric

$$\nu(r) = \frac{r^4}{8r_0^2} + h\,\ln\frac{r}{r_0} + k \tag{2}$$

$$\lambda(r) = \nu(r) + 2\,\ln\frac{r}{r_0} \tag{3}$$

with

$$B(r) = 2\frac{h+1}{r^2} \tag{4}$$

where r_0, h and k are constants.

From the equation for RC

$$\pounds_\xi\, R_{ab} = 0 \tag{5}$$

we obtain,

$$\xi^t_{,t} = 0 \tag{6}$$

$$\xi^t_{,r} + B(r)\xi^r_{,t} = 0 \tag{7}$$

$$\xi^t_{,\vartheta} + \xi^\vartheta_{,t} = 0 \tag{8}$$

$$\xi^t_{,\phi} + \sin^2\vartheta\;\xi^\phi_{,t} = 0 \tag{9}$$

$$B'(r)\xi^r + 2B(r)\;\xi^r_{,r} = 0 \tag{10}$$

$$B(r)\xi^r_{,\vartheta} + \xi^\vartheta_{,r} = 0 \tag{11}$$

$$B(r)\xi^r_{,\phi} + \sin^2\vartheta\xi^\phi_{,r} = 0 \tag{12}$$

$$\xi^{\vartheta}_{,\vartheta} = 0 \tag{13}$$

$$\xi^{\vartheta}_{,\phi} + \sin^2\vartheta\ \xi^{\phi}_{,\vartheta} = 0 \tag{14}$$

$$\cot\vartheta\ \xi^{\vartheta} + \xi^{\phi}_{,\phi} = 0 \tag{15}$$

Equations (6) and (10) can be integrated, giving $\xi^t = \Sigma(r,\vartheta,\phi)$ and $\xi^r = K(t,\vartheta,\phi)\, B^{-1/2}$ respectively. Substituting these expressions into the ϑ- derivative of eq. (7) we find that $K(t,\vartheta,\phi) = S_1(\vartheta,\phi)\ t + S_2(\vartheta,\phi)$. Using these results into (8) and (9), we obtain an expression for ξ:

$$\xi^t = \Sigma(r,\vartheta,\phi), \quad \xi^r = \frac{S_1(\vartheta,\phi)\ t + S_2(\vartheta,\phi)}{B^{1/2}} \tag{16}$$

$$\xi^{\vartheta} = -\Sigma_{,\vartheta}\ t + \Gamma(r,\vartheta,\phi), \quad \mathit{and} \quad \xi^{\phi} = -\Sigma_{,\phi}\ t + \Psi(r,\vartheta,\phi) \tag{17}$$

Substitution of ξ into (7) and (11)-(15) enables us to completely determine the functions $\Sigma(r,\vartheta,\phi)$, $S_1(\vartheta,\phi)$, $S_2(\vartheta,\phi)$, $\Gamma(r,\vartheta,\phi)$ and $\Psi(r,\vartheta,\phi)$. Then,

$$\begin{aligned}
\xi^0 &= -c_4\sqrt{2\ (h+1)}\ \ln r + c_0 \\
\xi^1 &= \frac{c_4\, t + c_5}{\sqrt{2\ (h+1)}} r \\
\xi^2 &= c_1\sin\phi - c_2\cos\phi \\
\xi^3 &= (c_1\cos\phi + c_2\sin\phi)\cot\vartheta + c_3
\end{aligned} \tag{18}$$

According to Theorem 1, (18) should represent an isometry; however it is easy to see that ξ does not reduce to a KV unless $c_4 = c_5 = 0$. This result invalidates the theorem stated above. Moreover, it is straightforward, but tedious, to show that the same condition is necessary for ξ to reduce to a Riemann collineation ($\pounds_\xi\, R^a_{bcd} = 0$). Therefore (18) is a proper RC.

Next, we should like to make a remark on a misleading statement that appears in Amir *et al.* [1]: "... there is no reason, *a priori*, why a RC should be a KV or *vice versa*..." . It is easy to show that any KV is a RC. Indeed, Katzin *et al.*[3] have proved that a necessary an sufficient condition for a spacetime to admit a curvature collineation (CC) is

$$\begin{aligned}
&((\pounds_\xi\, g_{im})_{;j} + (\pounds_\xi\, g_{mj})_{;i} - (\pounds_\xi\, g_{ij})_{;m})_{;k} - \\
&\quad ((\pounds_\xi\, g_{km})_{;j} + (\pounds_\xi\, g_{mj})_{;k} - (\pounds_\xi\, g_{kj}\,)_{;m})_{;i} = 0
\end{aligned} \tag{19}$$

Then, it is evident that a KV satisfies the above condition and, as it is well known, every CC is a RC.

The analysis of RC for the general case of static, spherically symmetric metrics, i.e. $A(r)$ and $C(r)$ arbitrary, is obviously more involved and its complete solution will be reported elsewhere. The study of RC for *non-static* spherically symmetric metrics gives interesting results which will be the subject of a subsequent publication.

This work has been partially supported by the Consejo de Desarrollo Científico Humanístico y Tecnológico de la Universidad de Los Andes; and the Programa de Formación e Intercambio Científico (Plan III). Algebraic calculations of the present work has been checked with MAPLE V. The authors wish to thank the staff of the *SUMA*, the computational facility of the Faculty of Science (Universidad de Los Andes), for making this work possible.

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